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The Markov Moment Problem and Extremal Problems

M. G. Kreĭn

A. A. Nudel′man

American Mathematical Society

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ПРОБЛЕМА МОМЕНТОВ МАРКОВА И ЭКСТРЕМАЛЬНЫЕ ЗАДАЧИ

М. Г. Крейн и А. А. Нудельман

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ABSTRACT. In this book an extensive circle of questions originating in the classical work of P. L Čebyšev and A. A. Markov is considered from the contemporary standpoint. It is shown how results and methods of the generalized moment problem are interlaced with various questions of the geometry of convex bodies, algebra and function theory. From this standpoint the structure of convex and conical hulls of curves is studied in detail and isoperimetric inequalities for convex hulls are established; a theory of orthogonal and quasiorthogonal polynomials is constructed; problems of the St. Petersburg school on limiting magnitudes of integrals and on least deviating functions (in various metrics) are generalized and solved; problems in approximation theory and interpolation and extrapolation in various function classes (analytic, absolutely monotone, almost periodic, etc.) are solved, as well as certain problems in optimal control of linear objects.

The last chapter establishes a duality principle between problems of best approximation in a normed space and the abstract L-problem of moments. Various illustrations of the principle are given.

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APPENDIX

INTEGRAL REPRESENTATIONS OF ANALYTIC FUNCTIONS IN CERTAIN SPECIAL CLASSES

In this Appendix we assemble information on certain important classes of analytic functions which we have used throughout the book to illustrate the applications of the general methods of various problems of constructive function theory and the classical moment problem.

In particular, we shall deal with integral representations of functions in these classes. Some of the representations (for the classes $\mathcal{C}$ and $\mathfrak{R}$) are quite well known, and the proofs (Theorems A.1 and A.2) will only be outlined.

1. *Classes* $\mathcal{C}$ *and* $\mathfrak{R}$. A function $f(z)$ is in class $\mathcal{C}$ if

1) $f(z)$ *is holomorphic in* $|z| < 1$, *and*

2) $\operatorname{Re} f(z) \geqslant 0$ *for* $|z| < 1$.

THEOREM A.1 (F. RIESZ AND HERGLOTZ). *A function* $f(z)$ *is in class* $\mathcal{C}$ *if and only if it admits a representation*

$$f(z) = i \operatorname{Im} f(0) + \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\tau(\theta), \tag{A.1}$$

where $\tau(\theta)$ *is a nondecreasing function.*

PROOF. Sufficiency is proved directly. The proof of necessity is simplest when the harmonic function $u(z) = \operatorname{Re} f(z)$ is continuous in the closed disk $|z| \leqslant 1$.

In that case, using the Poisson kernel ($z = re^{i\varphi}$),

$$\frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} = \operatorname{Re}\left\{\frac{e^{i\theta} + z}{e^{i\theta} - z}\right\},$$

we see that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left\{\frac{e^{i\theta} + z}{e^{i\theta} - z}\right\} u(e^{i\theta})\, d\theta,$$

whence we get (A.1) with $\tau(\theta) = (1/2\pi) \int_0^\theta u(e^{i\theta})d\theta$. In the general case one introduces functions $f_r(z) = f(rz)$ $(0 < r < 1)$ which, as just proved, admit representations of type (A.1). The proof is completed by letting $r \longrightarrow 1 - 0$ and using Helly's theorem.

P.A.1. A function $f(z) \in \mathcal{C}$ takes real values on the interval $(-1, 1)$ of the real axis if and only if it admits the representation

$$f(z) = \int_0^{2\pi} \frac{1 - z^2}{1 - 2z\cos\theta + z^2}\, d\tau(\theta),$$

where $\tau(\theta)$ is a bounded nondecreasing function.

A function $F(z)$ is in class $\mathfrak{R}$ if

1) *$F(z)$ is holomorphic in the upper half-plane, and*
2) $\operatorname{Im} F(z) \geqslant 0$ *for* $\operatorname{Im} z > 0$.

THEOREM A.2 (R. NEVANLINNA). *A function $F(z)$ is in class $\mathfrak{R}$ if and only if it admits an additive representation*

$$F(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sigma(t), \tag{A.2}$$

where α is a real number, $\beta \geqslant 0$ and $\sigma(t)$ is a nondecreasing function such that the integral $\int_{-\infty}^{\infty} (1 + t^2)^{-1}\, d\sigma(t)$ is convergent.

The proof is based on the relationship between the classes $\mathcal{C}$ and $\mathfrak{R}$: $F(z) \in \mathfrak{R}$ if and only if $f(\zeta) \in \mathcal{C}$, where $\zeta = (z - i)/(z + i)$ and $f(\zeta) = -iF(z)$.

The representation (A.2) is derived from (A.1) via the substitution $z \longrightarrow (z - i)/(z + i), -\cot(\theta/2) \longrightarrow t$, using the identity

$$\frac{1 + tz}{t - z} = \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right)(1 + t^2).$$

We mention without proof that the representation (A.2) is unique if the function $\sigma(t)$ is normalized in some way, say,

$$\sigma(0) = 0, \quad \sigma(t) = \frac{1}{2}[\sigma(t + 0) + \sigma(t - 0)].$$

Thus normalized, $\sigma(t)$ is determined in terms of $F(z)$ by the Stieltjes inversion formula:

$$\sigma(t_2) - \sigma(t_1) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \int_{t_1}^{t_2} \operatorname{Im} F(x + i\varepsilon)\, dx.$$

It is easily checked that

$$\beta = \lim_{y \to +\infty} \operatorname{Im} F(iy)/y.$$

It can also be shown that β is the sum of jumps of the function τ at the points 0 and 2π; the function $\sigma(t)$ in (A.2) is related to $\tau(\theta)$ in (A.1) by $(1 + t^2)^{-1} d\sigma(t) = d\tau(\theta)$, where $t = -\cot(\theta/2)$ and, suitably normalized, is given by the Stieltjes inversion formula:

$$\tau(\theta_2) - \tau(\theta_1) = \frac{1}{\pi} \lim_{\rho \uparrow 1} \int_{\theta_1}^{\theta_2} \operatorname{Re} f(\rho e^{i\varphi})\, d\varphi \quad (0 < \theta_1 < \theta_2 < 2\pi).$$

THEOREM A.3. *Any function of class $\mathfrak{R}$ not identically zero admits a unique multiplicative representation*

$$F(z) = C \exp \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) f(t)\, dt, \tag{A.3}$$

where $C > 0$, $f(t)$ is a summable function such that $0 \leqslant f(t) \leqslant 1$ almost everywhere and $\int_{-\infty}^{\infty} (1 + t^2)^{-1} f(t)\, dt < \infty$.

PROOF. The function

$$\ln F(z) = \ln |F(z)| + i \arg F(z)$$

is in class $\mathfrak{R}$, since it is holomorphic in the upper half-plane[1] and its imaginary part is nonnegative there because, by definition, $0 \leqslant \arg F(z) < \pi$. Therefore,

$$\ln F(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\tau(t).$$

By the inversion formula ($t_1 < t_2$),

$$0 \leqslant \tau(t_2) - \tau(t_1) = \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{t_1}^{t_2} \arg F(x + i\varepsilon)\, dx \leqslant t_2 - t_1.$$

Thus $\tau(t)$ is absolutely continuous, and almost everywhere

$$0 \leqslant f(t) = \frac{d\tau(t)}{dt} \leqslant 1.$$

Now, since $0 \leqslant \arg F(z) < \pi$, we have

$$\beta = \lim_{y \to +\infty} \arg F(iy)/y = 0.$$

Thus,

$$\ln F(z) = \alpha + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) f(t)\, dt,$$

where $0 \leqslant f(t) \leqslant 1$ almost everywhere. This implies (A.3).

(!)P.A.2. $F(z) \in \mathfrak{R}$ if and only if $-1/F(z) \in \mathfrak{R}$.

2. *Class* $\mathfrak{S}$. A function $F(z)$ is in class $\mathfrak{S}$ if

1) *$F(z)$ is in class $\mathfrak{R}$, and*

2) *$F(z)$ is holomorphic and nonnegative on the negative real axis* $(-\infty, 0)$.

[1] It is easily seen from (A.2) that $F(z) \neq 0$ for $\operatorname{Im} z > 0$, except in the trivial case $F(z) \equiv 0$.

Theorem A.4. *A function $F(z)$ is in class $\mathcal{S}$ if and only if it admits a representation*

$$F(z)=\gamma+\int_0^\infty \frac{d\sigma(t)}{t-z}, \tag{A.4}$$

where $\gamma \geqslant 0$ and $\int_0^\infty (1+t)^{-1}\,d\sigma(t)<\infty$.

Proof. Sufficiency is obvious. To prove necessity, let $F(z)\in\mathcal{S}$. Since $F(z)\in\mathfrak{R}$, we have a representation of type (A.2).

Since $F(z)$ is holomorphic and real on the negative real axis, it follows via the inversion formula that $\sigma(t)$ is constant there. Since $\sigma(-0)=\sigma(0)=0$, we have

$$F(z)=\alpha+\beta z+\int_0^\infty\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\sigma(t).$$

Set $z=-a$, where $a>1$. By assumption,

$$F(-a)=\alpha-\beta a+\int_0^\infty \frac{1-at}{(t+a)(1+t^2)}\,d\sigma(t)\geqslant 0.$$

Therefore,

$$C_a=\alpha+\int_0^1 \frac{1-at}{(t+a)(1+t^2)}\,d\sigma(t)\geqslant \beta a+\int_1^\infty \frac{at-1}{(t+a)(1+t^2)}\,d\sigma(t),$$

so that $\beta a\leqslant C_a$. Since $\beta\geqslant 0$ and C_a is bounded as $a\to\infty$, it follows that $\beta=0$. Now, for all $N>1$,

$$C_a\geqslant\int_1^N \frac{at-1}{(t+a)(t^2+1)}\,d\sigma(t),$$

whence

$$C_\infty\geqslant\int_1^N \frac{t\,d\sigma(t)}{t^2+1}.$$

Consequently, the integral $\int_0^\infty t(1+t^2)^{-1}\,d\sigma(t)$ is convergent, and hence so is the integral $\int_0^\infty (1+t)^{-1}\,d\sigma(t)$.

Thus,

$$F(z)=\gamma+\int_0^\infty \frac{d\sigma(t)}{t-z},$$

where $\gamma=\alpha-\int_0^\infty t(1+t^2)^{-1}\,d\sigma(t)$. Since $\gamma=\lim_{x\to-\infty}F(x)$, we have $\gamma\geqslant 0$, and the proof is complete.

(!)P.A.3. A function $G(z)$ has the properties $G(z)\in\mathfrak{R}$ and $G(z)=-\overline{G(-\bar{z})}$ if and only if $G(z)=zF(z^2)$, where $F(z)\in\mathcal{S}$.

THEOREM A.5. *A function $F(z)$ is in class $\mathfrak{S}$ if and only if both $F(z)$ and $zF(z)$ are in class $\mathfrak{R}$.*

PROOF. Let $F(z)$ be in $\mathfrak{S}$. By definition, it is also in $\mathfrak{R}$. In addition, it follows from (3.4) ($z = x + iy, y > 0$) that

$$\operatorname{Im} zF(z) = \gamma y + \int_0^\infty \frac{y}{|t-z|^2}\, t\,d\sigma(t) \geqslant 0,$$

so that $zF(z) \in \mathfrak{R}$.

Now suppose that $F(z)$ and $zF(z)$ are in $\mathfrak{R}$. We must show that $F(z) \in \mathfrak{S}$. Since $F(z)$ has a representation (A.2), it follows that for $z = x + iy$ ($y > 0$)

$$\operatorname{Im} F(z) = y\left(\beta + \int_{-\infty}^{\infty} \frac{d\sigma(t)}{|t-z|^2}\right) \geqslant 0 \tag{A.5}$$

and

$$\operatorname{Im}(zF(z)) = y\left(\alpha + 2\beta x + \int_{-\infty}^{\infty}\left(\frac{1}{|t-z|^2} - \frac{1}{1+t^2}\right) t\,d\sigma(t)\right) \geqslant 0. \tag{A.6}$$

We claim that $\int_{-\infty}^{\infty} (t-x)^{-2}\, d\sigma(t)$ is convergent for all $x < 0$.

Let $\int_{-\infty}^{\infty} (t-x_0)^{-2}\, d\sigma(t) = +\infty$, where $x_0 < 0$. Then, for sufficiently small $\epsilon > 0$,

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} t\,(t-x_0)^{-2} d\sigma(t) = -\infty.$$

Since

$$\operatorname{Im}(zF(z)) = y\left(\alpha + 2\beta x + I(y) + \int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{t\,d\sigma(t)}{(t-x_0)^2 + y^2}\right),$$

where $I(y)$ remains bounded as $y \longrightarrow +0$, it follows that $\operatorname{Im}(zF(z)) < 0$ for sufficiently small $y > 0$, and this is impossible. Thus $\int_{-\infty}^{\infty} (t-x)^{-2}\, d\sigma(t)$ is indeed convergent for all $x < 0$. Using this, we at once infer from (A.5) that

$$\lim \operatorname{Im} F(x + iy) = 0$$

as $y \longrightarrow +0$, for all $x < 0$. This in turn means that the function $\sigma(t)$ is constant for $t < 0$ (see the inversion formula). Letting $x \longrightarrow -\infty$ in (A.6), we now find that $\beta = 0$. Now, as in the proof of Theorem A.4, we observe that the integral $\int_0^\infty t(1+t^2)^{-1}\, d\sigma(t)$ is convergent, and it then follows from (A.6) that

$$\gamma = \alpha - \int_0^\infty \frac{t\,d\sigma(t)}{1+t^2} \geqslant 0.$$

Consequently, the function $F(z)$ admits a representation (A.4), Q.E.D.

(!)P.A.4. A function $F(z)$ admits a representation

$$F(z) = \delta + \int_{-\infty}^{0} \frac{d\sigma(t)}{t-z} \quad \left(\delta \leqslant 0,\ d\sigma(t) \geqslant 0,\ \int_{-\infty}^{0} \frac{d\sigma(t)}{1-t} < \infty\right)$$

if and only if both $F(z)$ and $(-z)F(z)$ are in class $\mathfrak{R}$.

3. *Class* $\mathfrak{R}[a, b]$. A function $F(z)$ is in class $\mathfrak{R}[a, b]$ if

1) *$F(z)$ is in class $\mathfrak{R}$, and*

2) *$F(z)$ is holomorphic and positive in the interval $(-\infty, a)$, and holomorphic and negative in the interval $(b, +\infty)$.*

THEOREM A.6. *A function $F(z)$ is in class $\mathfrak{R}[a, b]$ if and only if it admits a representation*

$$F(z) = \int_{a}^{b} \frac{d\sigma(t)}{t-z}, \tag{A.7}$$

where $\sigma(t)$ is a bounded nondecreasing function.

PROOF. Sufficiency is verified directly. We prove necessity. Since $F(z)$ is holomorphic and real for $z = x < a$ and for $z = x > b$, it follows from the inversion formula that the function $\sigma(t)$ in (A.2) is constant for $t < a$ and $t > b$, so that it is indeed bounded. Thus

$$F(z) = \gamma + \beta z + \int_{a}^{b} \frac{d\sigma(t)}{t-z},$$

where $\beta \geqslant 0$. It is easy to see that if $\beta > 0$, then $F(z)$ is positive for sufficiently large $z = x > a$, which is impossible. Thus $\beta = 0$. Now, if $\gamma \neq 0$, then for sufficiently large $x > 0$ the values of $F(x)$ and $F(-x)$ must have opposite signs, which is also impossible. Thus $\gamma = 0$.

(!)P.A.5. A function $F(z)$ is in class $\mathfrak{R}[a, b]$ if and only if both $(z-a)F(z)$ and $(b-z)F(z)$ are in $\mathfrak{R}$.

Hint. To prove sufficiency, set $z' = z - a$, $f_1(z') = F(z)$, $z'' = z - b$ and $f_2(z'') = F(z)$, and use the fact that $f_1(z')$, $z'f_1(z')$, $f_2(z'')$ and $(-z'')f_2(z'')$ all belong to $\mathfrak{R}$.

(!)P.A.6. The values of all the functions $F(z) \in \mathfrak{R}[a, b]$ at a fixed point z ($\operatorname{Im} z > 0$) fill out the angle $\Phi(z)$ between the rays

$$w = \frac{\tau}{b-z} \quad (0 \leqslant \tau < +\infty)$$

and

$$w = \frac{\tau}{a-z} \quad (0 \leqslant \tau < +\infty).$$

The magnitude of this angle is

$$\varphi(z) = \arg\frac{b-z}{a-z} \quad (0 < \varphi(z) < \pi)^{1}).(^2)$$

Hint. It follows from P.A.4 that

$$0 \leqslant \arg[(z-a)F(z)] < \pi, \quad 0 \leqslant \arg[(b-z)F(z)] < \pi.$$

4. *Class* $\mathfrak{S}[a, b]$. A function $F(z)$ is in class $\mathfrak{S}[a, b]$ if

1) *$F(z)$ is in class $\mathfrak{R}$, and*

2) *$F(z)$ is holomorphic and positive in the intervals $(-\infty, a)$ and $(b, +\infty)$.*

THEOREM A.7. *A function $F(z)$ is in class $\mathfrak{S}[a, b]$ if and only if it admits a representation*

$$F(z) = (b-z)\int_a^b \frac{d\sigma(t)}{t-z}, \tag{A.8}$$

where $\sigma(t)$ is a bounded nondecreasing function.

PROOF. Sufficiency is proved directly. In proving necessity, we may assume without loss of generality that $a = 0$. We set $\zeta = bz/(b-z)$ and $f(\zeta) = F(z)$, and observe that $F(z) \in \mathfrak{S}[a, b]$ if and only if $f(\zeta) \in \mathfrak{S}$. Thus,

$$F(z) = \gamma + \int_0^\infty \frac{d\tau(s)}{s-\zeta} = \gamma + (b-z)\int_0^\infty \frac{d\tau(s)}{\left(\frac{bs}{b+s} - z\right)(b+s)}.$$

Now set

$$t = \frac{bc}{b+s}\left(s = \frac{bt}{b-t}\right), \quad \sigma(t) = \int_0^s \frac{d\tau(s)}{b+s} \quad (0 \leqslant t < b),$$

so that $d\sigma(t) = d\tau(s)/(b+s)$. Then

$$F(z) = \gamma + (b-z)\int_0^{b-0} \frac{d\sigma(t)}{t-z}.$$

If we extend the definition of $\tau(t)$ to b by setting

$$\sigma(b) = \gamma + \int_0^\infty \frac{d\tau(s)}{b+s},$$

we see that the jump of $\sigma(t)$ at b is γ, and we may write

$$F(z) = (b-z)\int_a^b \frac{d\sigma(t)}{t-z},$$

as required.

---

(2) It is an interesting fact that the angle $\Phi(z)$ is equal to the angle at the vertex z of the triangle with vertices at a, b and z.

(!)P.A.7. A function $F(z)$ is in class $\mathfrak{S}[a, b]$ if and only if the functions $F(z)$ and

$$\frac{z-a}{b-z} F(z)$$

are in class $\mathfrak{R}$.

Hint. It follows from Theorems A.6 and A.7 that $F(z) \in \mathfrak{S}[a, b]$ if and only if

$$\frac{1}{b-z} F(z) \in \mathcal{R}\,[a, b].$$

(!)P.A.8. The values of all the functions $F(z) \in \mathfrak{S}[a, b]$ at a fixed point z ($\operatorname{Im} z > 0$) fill out the angle $\Psi(z)$ between the rays $w = \tau$ ($0 \leq \tau < +\infty$) and $w = \tau(b-z)/(a-z)$ ($0 \leq \tau < +\infty$). The magnitude of this angle is

$$\psi(z) = \arg \frac{b-z}{a-z} \quad (0 < \psi(z) < \pi).$$

(!)P.A.9. If $F(z) \in \mathfrak{S}[a, b]$, then

$$1/[(a-z)F(z)] \in \mathcal{R}\,[a, b].$$

Conversely, if $F(z) \in \mathfrak{R}[a, b]$, then

$$1/[(a-z)F(z)] \in \mathcal{S}\,[a, b].$$

5. *Class* $\mathfrak{S}(E_m)$. Let (α_j, β_j) ($j = 1, \ldots, m$) be m nonoverlapping intervals on the real axis $(-\infty, \infty)$. We let E_m denote the set remaining from $(-\infty, \infty)$ after removal of these intervals.

A function $F(z)$ is in class $\mathfrak{S}(E_m)$ if

1) *$F(z)$ is in class $\mathfrak{R}$, and*

2) *$F(z)$ is holomorphic and positive in the intervals (α_j, β_j) ($j = 1, \ldots, m$).*

THEOREM A.8. *A function $F(z)$ is in class $\mathfrak{S}(E_m)$ if and only if it admits a multiplicative representation*

$$F(z) = C \exp \int_{E_m} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) f(t)\, dt, \tag{A.9}$$

where $C > 0$ and $0 \leq f(t) \leq 1$ a.e. on E_m.

PROOF. Sufficiency is verified directly. Necessity will follow if we can show that whenever $F(z) \in \mathfrak{S}(E_m)$ the function $f(t)$ in (A.3) vanishes in the intervals (α_j, β_j) ($j = 1, \ldots, m$). This is easily seen, noting that since the function $\ln F(z)$ is holomorphic the inversion formula for $\ln F(z)$ in these intervals is

$$f(t) = \frac{1}{\pi} \arg F(t) \quad (\alpha_j < t < \beta_j)$$

and since $F(t) > 0$ it follows that $f(t) = 0$ for $t \in (\alpha_j, \beta_j)$, Q.E.D.

We now introduce the notation

$$\omega_j(z) = \begin{cases} \dfrac{\beta_j - z}{\alpha_j - z} & \text{if} \quad -\infty < \alpha_j < \beta_j < \infty, \\ z - \beta_j & \text{if} \quad \alpha_j = -\infty, \\ \dfrac{1}{\alpha_j - z} & \text{if} \quad \beta_j = +\infty. \end{cases}$$

It is readily checked that

$$\omega_j(z) = C_j \exp \int_{\alpha_j}^{\beta_j} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) dt, \tag{A.10}$$

where $C_j > 0$.

P.A.10. A function $F(z)$ is in class $\mathcal{S}(E_m)$ if and only if both functions $F(z)$ and $\omega_1(z) \cdots \omega_m(z) F(z)$ are in class $\mathfrak{R}$.

Hint. Use the representations (A.9) and (A.10).

P.A.11. A function $F(z)$ of class $\mathcal{C}$, holomorphic on the arc $z = e^{i\theta}$ $(\tau < \theta < 2\pi - \tau)$, assumes pure imaginary values on this arc such that $(1/i)F(e^{i\theta}) = \operatorname{Im} F(e^{i\theta}) > 0$ $(\tau < \theta < 2\pi - \tau)$ *if and only if*

$$F(z) = i \,|\, F(0) \,|\, \exp \left\{ -\frac{1}{2} i \int_{-\tau}^{\tau} \frac{e^{it} + z}{e^{it} - z} f(t) \, dt \right\}, \tag{A.11}$$

where $-1 \leq f(t) \leq 1$ for $-\tau \leq t \leq \tau$.

P.A.12. The representation (A.11) is valid if and only if both functions

$$F(z) \quad \text{and} \quad \frac{z - e^{i\tau}}{1 - z e^{i\tau}} F(z)$$

are in class $\mathcal{C}$.

P.A.13 (Löwner). Let $\mathfrak{B}(K_1, K_2)$ denote the class of functions holomorphic in an open disk K_1, mapping it into a disk K_2. Let $\mathfrak{B}(\gamma_1, \gamma_2)$ denote the class of all $f(z) \in \mathfrak{B}(K_1, K_2)$ which are holomorphic on an arc γ_1 of the circumference of K_1 and map it into an arc γ_2 of the circumference of K_2. Let $\gamma_1(\zeta)$ be an arc through an endpoint of γ_1 and a fixed point $\zeta \in K_1$, and $\gamma_2(\zeta)$ an arc through an endpoint of γ_2, within K_2, forming with γ_2 an angle equal to the angle between $\gamma_1(\zeta)$ and γ_1. *The set of values of all functions $f(z) \in \mathfrak{B}(\gamma_1, \gamma_2)$ at ζ fills out the lune bounded by γ_2 and $\gamma_2(\zeta)$.*

Hint. Use P.A.7 or P.A.12.

Notes on the Appendix

Theorems A.1 and A.2 and the Stieltjes inversion formula are classical and need no comment.

Ahiezer and Kreĭn [2] gave the first characterization of the classes of functions indicated in P.A.11, using a factor of the type $\omega_j(z)$ (see P.A.12); they used a multiplicative representation of the functions in this class. This approach was suggested by a paper of Markov [8] and provided a new approach to Löwner's results (see Ahiezer and Kreĭn [8], and also P.A.13).

The class $\mathfrak{S}$ and its various characterizations were introduced and used systematically by Kreĭn [6], [12], [13] in connection with the theory of generalized resolvents and spectral functions of a string.

The classes $\mathfrak{R}[a, b]$ and $\mathfrak{S}[a, b]$ also arose naturally in investigations of the power moment problem on a finite interval (see Chapter IV, §7). The class $\mathfrak{S}[a, b]$ is also important in describing the extensions of a hermitian operator, under the assumption that its norm cannot exceed some prescribed upper bound (see Kreĭn [8]).

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