

Boundary Value Problems for Elliptic Pseudodifferential Equations

G. I. ÈSKIN

Volume 52

TRANSLATIONS OF
MATHEMATICAL MONOGRAPHS

American Mathematical Society



TRANSLATIONS OF MATHEMATICAL MONOGRAPHS
VOLUME 52

**Boundary Value Problems
for Elliptic
Pseudodifferential
Equations**

G. I. ÈSKIN

This page intentionally left blank

**Boundary Value Problems
for Elliptic Pseudodifferential Equations**

КРАЕВЫЕ ЗАДАЧИ
ДЛЯ ЭЛЛИПТИЧЕСКИХ
ПСЕВДОДИФФЕРЕНЦИАЛЬНЫХ
УРАВНЕНИЙ

Г. П. ЭСКИН

ИЗДАТЕЛЬСТВО «НАУКА»
МОСКВА 1973

Translated from the Russian by S. Smith

1980 *Mathematics Subject Classification*. Primary 35S15;
Secondary 35S05, 47G05, 58G15.

ABSTRACT. The present monograph is devoted to the class of equations known as pseudodifferential equations, which comprises differential and multidimensional singular integral equations as well as integral equations of the first kind and integrodifferential equations whose kernels have a weak singularity. Mixed boundary value problems for elliptic equations are investigated with the use of the Wiener-Hopf method, the asymptotic behavior of their solutions is obtained and various examples are analyzed. It is intended for mathematicians and theoreticians in continuum mechanics.

Library of Congress Cataloging in Publication Data

Eskin, Grigorii Il'ich.

Boundary value problems for elliptic pseudodifferential equations.

Translation of Kraevye zadachi dlia èllipticheskikh psevdodifferentsial'nykh uravnenii.

Bibliography: p.

Includes index.

1. Boundary value problems. 2. Pseudodifferential operators. I. Title.

QA379.E8413 515.3'5

80-39789

ISBN 0-8218-4503-9

Copyright © 1981 by the American Mathematical Society

The paper used in this reprint is acid-free and falls within the guidelines established to ensure permanence and durability. Ⓢ

10 9 8 7 6 5 4 3 94 93 92 91 90 89

TABLE OF CONTENTS

| | |
|--|-----|
| PREFACE TO THE ENGLISH EDITION | vii |
| PREFACE | ix |
| INTRODUCTION | 1 |
| CHAPTER I. GENERALIZED FUNCTIONS AND THE FOURIER TRANSFORM | 7 |
| 1. Generalized functions | 7 |
| 2. Fourier transform | 17 |
| 3. Pseudodifferential operators | 35 |
| 4. Sobolev-Slobodeckii spaces | 46 |
| CHAPTER II. BOUNDARY VALUE PROBLEMS FOR AN ELLIPTIC PSEUDODIFFERENTIAL OPERATOR IN A HALFSPACE | 69 |
| 5. Integral of Cauchy type | 69 |
| 6. Factorization of an elliptic symbol | 80 |
| 7. Pseudodifferential equations in a halfspace | 90 |
| 8. Formulation of boundary value problems for pseudo- differential equations | 102 |
| CHAPTER III. SMOOTHNESS OF SOLUTIONS OF PSEUDODIFFERENTIAL EQUATIONS | 113 |
| 9. Asymptotic behavior of a solution of a pseudodifferential equation in a halfspace | 113 |
| 10. Smooth pseudodifferential operators in a halfspace | 121 |
| 11. Boundary value problems for smooth pseudodifferential operators in a halfspace | 132 |
| 12. General boundary value problems for elliptic pseudo- differential equations in a halfspace | 142 |
| 13. Mixed boundary value problems for an elliptic equation of second order in a halfspace | 150 |

| | |
|---|------------|
| CHAPTER IV. SYSTEMS OF ELLIPTIC PSEUDO-DIFFERENTIAL EQUATIONS IN A HALFSPACE | 163 |
| 14. Conditions for the normal solvability of a pseudodifferential operator on a halfline | 163 |
| 15. An algebra of operators on a halfline containing the pseudodifferential operators | 174 |
| 16. Boundary value problems for a system of elliptic pseudodifferential equations in a halfspace | 184 |
| 17. Construction of the solution of a boundary value problem on a halfline and in a halfspace | 200 |
| CHAPTER V. PSEUDODIFFERENTIAL OPERATORS WITH VARIABLE SYMBOLS | 215 |
| 18. Theorems on the boundedness and composition of pseudodifferential operators | 215 |
| 19. Change of variables formula for pseudodifferential operators | 224 |
| 20. Factorization of elliptic pseudodifferential operators | 237 |
| 21. Pseudodifferential operators in a domain and on a manifold | 250 |
| CHAPTER VI. BOUNDARY VALUE PROBLEMS FOR ELLIPTIC PSEUDODIFFERENTIAL OPERATORS IN A BOUNDED DOMAIN WITH SMOOTH BOUNDARY | 259 |
| 22. Normal solvability of a general boundary value problem in a domain with smooth boundary | 259 |
| 23. Smooth pseudodifferential operators in a domain | 272 |
| 24. Mixed boundary value problems in a domain | 282 |
| 25. Spaces of functions of piecewise constant order of smoothness | 296 |
| CHAPTER VII. APPLICATIONS | 307 |
| 26. A finite element method for strongly elliptic pseudodifferential equations | 307 |
| 27. Asymptotic solution of elliptic pseudodifferential equations with a small parameter | 336 |
| BIBLIOGRAPHY | 363 |
| NOTATION INDEX | 369 |
| SUBJECT INDEX | 373 |

PREFACE TO THE ENGLISH EDITION

The English edition differs from the Russian in that an Introduction and three new sections (§§25–27) have been added. Moreover, various corrections, improvements and remarks have been made throughout the book, especially in Chapter VI.

The author wishes to express his thanks to the American Mathematical Society, for its decision to publish an English edition of his book, and to Dr. S. F. Smith, whose competent translation brought to light a number of inaccuracies in the Russian edition.

This page intentionally left blank

PREFACE

The theory of boundary value problems for elliptic pseudodifferential equations has its roots in (i) the theory of multidimensional singular integral equations on a closed manifold (due to Giraud, Mihlin, Calderón and Zygmund, and others; see Mihlin's monograph [65] and the literature cited there), (ii) the theory of boundary value problems for elliptic differential equations with differential and pseudodifferential boundary conditions (due to Agmon, Douglis, Nirenberg, Browder, Lopatinskii, Slobodeckii, Peetre, Hörmander, Šapiro, Schechter, Agranovič, A. S. Dynin and others; see the survey by Agranovič [4]), and (iii) the theory of integral equations of convolution type on a halfline and singular integral equations on an open contour (see the works of Kreĭn, Gohberg and their students [50], [51], [60] and the monographs of Mushelišvili [67], Gahov [45] and Noble [70]).

In 1962 the author, starting from a paper [73] of Peetre and his own paper [35], considered mixed boundary value problems for multidimensional elliptic differential equations of chiefly second order. By reducing such a mixed boundary value problem to a pair of pseudodifferential equations and applying the Wiener-Hopf method, the author proved its solvability in a Sobolev-Slobodeckii space determined by the factorization index of the corresponding symbol; in addition, he found the smoothness and asymptotic behavior of its solution.

These investigations were jointly continued by M. I. Višik and the author. Together they (i) constructed a theory of elliptic pseudodifferential equations and systems on a manifold with boundary that included a theory of multidimensional singular integral equations in domains with a smooth boundary, (ii) found a statement of boundary value problems for elliptic pseudodifferential equations and systems and proved their normal solvability in weighted spaces of functions of piecewise constant and variable order of smoothness,

and (iii) distinguished a subclass of pseudodifferential equations whose solutions for smooth right sides are smooth up to the boundary.

The normal solvability in L_2 of a singular integral operator was independently investigated by Simonenko [85].

The theory of pseudodifferential operators in \mathbf{R}^n and on a manifold without boundary is due to Kohn and Nirenberg [59] and Hörmander [55] (see also the works of Friedrichs, Seeley, Bokobza-Haggiag and Unterberger, M. I. Višik and the author, and others).

Boundary value problems for elliptic pseudodifferential equations in domains with a smooth boundary have also been studied by Shamir [79], Dynin [33], Boutet de Monvel [16], [17], Šubin [95], Dikanskiĭ [28], Šnirel'man, the author and others.

In the course of writing the book most of these investigations underwent an essential revision: sharper and more complete results have been obtained and new, simpler proofs have been found. The book also contains new results that have been obtained by the author and are being published for the first time.

The author wishes to take this opportunity to express his gratitude to his teacher G. E. Šilov for his help and support. The author also sincerely thanks V. B. Lidskiĭ for his constant interest in the work.

M. S. Agranovič, A. S. Dikanskiĭ and B. P. Panejah have read the manuscript and made many useful remarks, for which the author is deeply grateful.

BIBLIOGRAPHY

Items marked by an asterisk were added for the English translation

1. *Shmuel Agmon, *Lectures on elliptic boundary value problems*, Van Nostrand, Princeton, N. J., 1965.
2. S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*. I, *Comm. Pure Appl. Math.* **12** (1959), 623–727.
3. ———, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*. II, *Comm. Pure Appl. Math.* **17** (1964), 35–92.
4. M. S. Agranovič, *Elliptic singular integro-differential operators*, *Uspehi Mat. Nauk* **20** (1965), no. 5 (125), 3–120 = *Russian Math. Surveys* **20** (1965), no. 5, 1–121.
5. M. S. Agranovič and A. S. Dynin, *General boundary-value problems for elliptic systems in an n -dimensional domain*, *Dokl. Akad. Nauk SSSR* **146** (1962), 511–514 = *Soviet Math. Dokl.* **3** (1962), 1323–1327.
6. M. S. Agranovič and M. I. Višik, *Elliptic problems with a parameter and parabolic problems of general type*, *Uspehi Mat. Nauk* **19** (1964), no. 3(117), 53–161 = *Russian Math. Surveys* **19** (1964), no. 3, 53–157.
7. M. F. Atiyah, *K-theory*, *Lecture Notes*, Harvard Univ., Cambridge, Mass., 1964; reprint, Benjamin, New York, 1968.
8. M. F. Atiyah and R. Bott, *The index problem for manifolds with boundary*, *Differential Analysis (Bombay Colloq., 1964)*, Oxford Univ. Press, London, 1964, pp. 175–186.
9. *Jean-Pierre Aubin, *Approximation of elliptic boundary-value problems*, Wiley, New York, 1972.
10. K. I. Babenko, *On conjugate functions*, *Dokl. Akad. Nauk SSSR* **62** (1948), 157–160. (Russian)
11. *V. A. Babeško, *Asymptotic solution of some two-dimensional convolutions, and their applications*, *Dokl. Akad. Nauk SSSR* **204** (1972), 572–574 = *Soviet Phys. Dokl.* **17** (1972/73), 504–506.
12. Ju. M. Berezanskiĭ, *Expansion in eigenfunctions of selfadjoint operators*, “*Naukova Dumka*”, Kiev, 1965; English transl., Amer. Math. Soc., Providence, R. I., 1968.
13. A. V. Bicadze, *Boundary value problems for second order elliptic equations*, “*Nauka*”, Moscow, 1966; English transl., North-Holland, Amsterdam; Interscience, New York, 1968.
14. *A. Bogomolnii [Bogomol’nyi], G. Eskin and S. Zuchowizkii [Zuhovickii], *Numerical solution of the stamp problem*, *Comput. Methods Appl. Mech. Engrg.* **15** (1978), 241–258.
15. Louis Boutet de Monvel, *Comportement d’un opérateur pseudo-différentiel sur une variété à bord*. I, II, *J. Analyse Math.* **17** (1966), 241–253, 255–304.

16. ———, *Opérateurs pseudo-différentiels analytiques et problèmes aux limites elliptiques*, Ann. Inst. Fourier (Grenoble) **19** (1970), fasc. 2, 169–268.
17. ———, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11–51.
18. Felix E. Browder, *Estimates and existence theorems for elliptic boundary value problems*, Proc. Nat. Acad. Sci. USA **45** (1959), 365–372.
19. A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.
20. ———, *Singular integral operators and differential equations*, Amer. J. Math. **79** (1957), 901–921.
21. Can Zui Ho [Chan Zui Kho] and G. I. Eskin, *Boundary value problems for parabolic systems of pseudodifferential equations*, Dokl. Akad. Nauk SSSR **198** (1971), 50–53 = Soviet Math. Dokl. **12** (1971), 739–743.
22. H. O. Cordes, *An algebra of singular integral operators with two symbol homomorphisms*, Bull. Amer. Math. Soc. **75** (1969), 37–42.
23. ———, *Pseudo-differential operators on a half-line*, J. Math. Mech. **18** (1968/69), 893–908.
24. *T. A. Cruse, *Application of the boundary-integral equation method to three dimensional stress analysis*, Computers and Structures **3** (1973), 509–527.
25. *A. S. Demidov, *Boundary effects and the asymptotic nature of the degeneration of certain elliptic pseudodifferential operators*, Uspehi Mat. Nauk **27** (1972), no. 1(163), 245–246. (Russian)
26. * ———, *Asymptotic behavior of the solution of a boundary value problem for elliptic pseudodifferential equations with a small parameter multiplying the leading operator*, Trudy Moskov. Mat. Obšč. **32** (1975), 119–146 = Trans. Moscow Math. Soc. **32** (1975), 115–142.
27. * ———, *Elliptic pseudodifferential boundary value problems with a small parameter in the coefficient of the leading operator*, Mat. Sb. **91**(133) (1973), 421–444 = Math. USSR Sb. **20** (1973), 439–463.
28. A. S. Dikanskiĭ, *Problems adjoint to elliptic pseudodifferential boundary value problems*, Dokl. Akad. Nauk SSSR **200** (1971), 1020–1023 = Soviet Math. Dokl. **12** (1971), 1520–1525.
29. * ———, *Conjugate problems of elliptic differential and pseudodifferential boundary value problems in a bounded domain*, Mat. Sb. **91**(133) (1973), 62–77 = Math. USSR Sb. **20** (1973), 67–83.
30. *Nelson Dunford and Jacob T. Schwartz, *Linear operators*. Vol. I, Interscience, New York, 1958.
31. A. S. Dynin, *Singular operators of arbitrary order on a manifold*, Dokl. Akad. Nauk SSSR **141** (1961), 21–23 = Soviet Math. Dokl. **2** (1961), 1375–1377.
32. ———, *Multidimensional elliptic boundary value problems with a single unknown function*, Dokl. Akad. Nauk SSSR **141** (1961), 285–287 = Soviet Math. Dokl. **2** (1961), 1431–1433.
33. ———, *On the theory of pseudodifferential operators on a manifold with boundary*, Dokl. Akad. Nauk SSSR **186** (1969), 251–253 = Soviet Math. Dokl. **10** (1969), 575–578.
34. G. I. Eskin, *Boundary value problems and the parametrix for systems of elliptic pseudodifferential equations*, Trudy Moskov. Mat. Obšč. **28** (1973), 75–116 = Trans. Moscow Math. Soc. **28** (1973), 74–115. (See also Preprint no. 11, Institute for Problems of Mechanics, Acad. Sci. USSR, Moscow, 1972. (Russian))
35. ———, *General boundary value problems for equations of principal type in a plane region with corners*, Uspehi Mat. Nauk **18** (1963), no. 3(111), 241–242. (Russian)
36. ———, *Generalized functions and pseudodifferential operators*, Preprint no. 5, Institute for Problems of Mechanics, Acad. Sci. USSR, Moscow, 1971. (Russian)
37. ———, *Boundary value problems in a half space for elliptic pseudodifferential equations*, Preprint no. 7, Institute for Problems of Mechanics, Acad. Sci. USSR, Moscow, 1971. (Russian)

38. ———, *The conjugacy problem for equations of principal type with two independent variables*, Trudy Moskov. Mat. Obšč. **21** (1970), 245–292 = Trans. Moscow Math. Soc. **21** (1970), 263–316.
39. * ———, *A variational-difference method for solving elliptic pseudodifferential equations*, Uspehi Mat. Nauk **28** (1973), no. 5(173), 255–256. (Russian)
40. * ———, *The numerical solution of some pseudo-differential equations of elasticity by finite element methods*, Sémin. Inst. Recherche Informat. Automat. (Paris, 1977) (to appear).
41. * ———, *Asymptotics of solutions of elliptic pseudodifferential equations with a small parameter*, Dokl. Akad. Nauk SSSR **211** (1973), 547–550 = Soviet Math. Dokl. **14** (1973), 1080–1084.
42. * ———, *Asymptotics near the boundary of spectral functions of elliptic self-adjoint boundary problems*, Israel J. Math. **22** (1975), 214–246.
43. * ———, *General mixed problems for elliptic differential equations*, Centro Internaz. Mat. Estivo, II Ciclo (Bressanone, 1977), Napoli, 1977.
44. K. O. Friedrichs, *Pseudo-differential operators*, Lecture Notes, Courant Inst. Math. Sci., New York Univ., New York, 1968; rev. ed., 1970.
45. F. D. Gahov, *Boundary value problems*, Fizmatgiz, Moscow, 1958; English transl., Pergamon Press, Oxford; Addison-Wesley, Reading, Mass., 1966.
46. *L. A. Galin, *Contact problems in the theory of elasticity*, GITTL, Moscow, 1953; English transl., North Carolina State College, Raleigh, N. C., 1961.
47. F. R. Gantmaher, *The theory of matrices*, 2nd ed., “Nauka”, Moscow, 1968; English transl. of 1st ed., vols. I, II, Chelsea, New York, 1959.
48. Lars Gårding, *Transformation de Fourier des distributions homogènes*, Bull. Soc. Math. France **89** (1961), 381–428.
49. I. M. Gel'fand and G. E. Šilov, *Generalized functions*. Vols. 1, 2, 3, Fizmatgiz, Moscow, 1958; English transl., Academic Press, New York, 1964, 1967, 1968.
50. I. C. Gohberg and I. A. Fel'dman, *Convolution equations and projection methods for their solution*, “Nauka”, Moscow, 1971; English transl., Amer. Math. Soc., Providence, R. I., 1974.
51. I. C. Gohberg and M. G. Kreĭn, *Systems of integral equations on a half line with kernels depending on the difference of the arguments*, Uspehi Mat. Nauk **13** (1958), no. 2(80), 3–72; English transl., Amer. Math. Soc. Transl. (2) **14** (1960), 217–287.
52. I. C. Gohberg and N. Ja. Krupnik, *The algebra generated by the one-dimensional singular integral operators with piecewise continuous coefficients*, Funkcional. Anal. i Priložen. **4** (1970), no. 3, 26–36 = Functional Anal. Appl. **4** (1970), 193–201.
53. *R. V. Gol'dšteĭn, I. S. Kleĭn and G. I. Èskin, *A variational-difference method for solving certain integral and integro-differential equations of three-dimensional problems of elasticity theory*, Preprint no. 33, Institute for Problems of Mechanics, Acad. Sci. USSR, Moscow, 1973. (Russian) RZ Mat. **1974** 561073.
54. Lars Hörmander, *Linear partial differential operators*, Springer-Verlag, Berlin; Academic Press, New York, 1963.
55. ———, *Pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 501–517.
56. ———, *Pseudo-differential operators and hypoelliptic equations*, Proc. Sympos. Pure Math., vol. 10, Amer. Math. Soc., Providence, R. I., 1967, pp. 138–183.
57. ———, *Fourier integral operators*. I, Acta Math. **127** (1971), 79–183.
58. *D. Ivanenko and A. Sokolov, *The classical theory of fields (new problems)*, 2nd ed., GITTL, Moscow, 1951; German transl. of 1st ed., Akademie-Verlag, Berlin, 1953.
59. J. J. Kohn and L. Nirenberg, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 269–305.

60. M. G. Kreĭn, *Integral equations on a half line with kernels depending on the difference of the arguments*, Uspehi Mat. Nauk 13 (1958), no. 5 (83), 3–120; English transl., Amer. Math. Soc. Transl. (2)22 (1962), 163–288.
61. *Jacques-Louis Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod; Gauthier-Villars, Paris, 1969.
62. J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*. Vols. 1, 2, 3, Dunod, Paris, 1968, 1970; English transl., Springer-Verlag, Berlin and New York, 1972, 1973.
63. Ja. B. Lopatinskiĭ, *A method of reducing boundary value problems for a system of differential equations of elliptic type to regular integral equations*, Ukrain. Mat. Z. 5 (1953), no. 2, 123–151; English transl., Amer. Math. Soc. Transl. (2)89 (1970), 149–183.
64. *A. I. Lur'e, *Elasticity theory*, "Nauka", Moscow, 1970. (Russian)
65. S. G. Mihlin, *Multidimensional singular integrals and integral equations*, Fizmatgiz, Moscow, 1962; English transl., Pergamon Press, New York, 1965.
66. ———, *Variational methods in mathematical physics*, 2nd rev. aug. ed., "Nauka", Moscow, 1970; English transl. of 1st ed., Macmillan, New York, 1964.
67. N. I. Mushelišvili, *Singular integral equations*, 3rd ed., "Nauka", Moscow, 1968; English transl. of 1st ed., Noordhoff, Groningen, 1953; reprinted, 1972.
68. *J. C. Nédélec, *Curved finite element methods for the solution of singular integral equations on surfaces in R^3* , Comput. Methods Appl. Mech. Engrg. 8 (1976), 61–80.
69. *L. Nirenberg, *Pseudo-differential operators*, Proc. Sympos. Pure Math., vol. 16, Amer. Math. Soc., Providence, R. I., 1970, pp. 149–167.
70. B. Noble, *Methods based on the Wiener-Hopf technique for the solution of partial differential equations*, Pergamon Press, New York, 1958.
71. O. A. Oleĭnik and E. V. Radkevič, *Equations of second order with nonnegative characteristic form*, Itogi Nauki: Mat. Anal., 1969, VINITI, Moscow, 1971; English transl., Plenum Press, New York, 1973.
72. R. E. A. C. Paley and Norbert Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc., Providence, R. I., 1934.
73. Jaak Peetre, *Mixed problems for higher order elliptic equations in two variables*. I, Ann. Scuola Norm. Sup. Pisa (3) 15 (1961), 337–353.
74. ———, *Another approach to elliptic boundary problems*, Comm. Pure Appl. Math. 14 (1961), 711–731.
75. Z. Ja. Šapiro, *On general boundary value problems for equations of elliptic type*, Izv. Akad. Nauk SSSR Ser. Mat. 17 (1953), 539–562. (Russian)
76. Martin Schechter, *General boundary value problems for elliptic partial differential equations*, Comm. Pure Appl. Math. 12 (1959), 457–486.
77. L. Schwartz, *Théorie des distributions*. Tomes 1, 2, Actualités Sci. Indust., nos. 1089, 1122, Hermann, Paris, 1950, 1951.
78. R. T. Seeley, *Integro-differential operators on vector bundles*, Trans. Amer. Math. Soc. 117 (1965), 167–204.
79. Eliahu Shamir, *Elliptic systems of singular integral operators*. I, Trans. Amer. Math. Soc. 127 (1967), 107–124.
80. ———, *Mixed boundary value problems for elliptic equations in the plane. The L_p theory*, Ann. Scuola Norm. Sup. Pisa (3) 17 (1963), 117–139.
81. * ———, *Regularization of mixed second-order elliptic problems*, Israel J. Math. 6 (1968), 150–168.
82. * ———, *Boundary value problems for elliptic convolution systems*, Centro Internaz. Mat. Estivo, II Ciclo (Stresa, 1968), Edizione Cremonese, Rome, 1969, pp. 308–331.
83. G. E. Šilov, *Mathematical analysis. Special course*, 2nd ed., Fizmatgiz, Moscow, 1961; English transl., Pergamon Press, New York, 1965.

84. ———, *Mathematical analysis. Second special course*, "Nauka", Moscow, 1965; English transl., *Generalized functions and partial differential equations*, Gordon & Breach, New York, 1968.
85. I. B. Simonenko, *A new general method for investigating linear operator equations of singular integral equation type*. I, II, *Izv. Akad. Nauk SSSR Ser. Mat.* **29** (1965), 567–586, 757–782. (Russian)
86. L. N. Slobodeckii, *Generalized Sobolev spaces and their application to boundary value problems for partial differential equations*, Leningrad. Gos. Ped. Inst. Učen. Zap. **197**(1958), 54–112; English transl., *Amer. Math. Soc. Transl.* (2)**57** (1966), 207–276.
87. A. I. Snirel'man, *Convolution equations in a half space*, *Mat. Sb.* **82**(124) (1970), 476–493 = *Math. USSR Sb.* **11** (1970), 441–458.
88. S. L. Sobolev, *Applications of functional analysis in mathematical physics*, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; English transl., *Amer. Math. Soc.*, Providence, R. I., 1963.
89. Norman Steenrod, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, N. J., 1951.
90. Elias M. Stein, *Note on singular integrals*, *Proc. Amer. Math. Soc.* **8** (1957), 250–254.
91. * ———, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.
92. * Gilbert Strang and George J. Fix, *An analysis of the finite element method*, Prentice-Hall, Englewood Cliffs, N. J., 1973.
93. * ———, *A Fourier analysis of the finite element variational method*, *Centro Internaz. Mat. Estivo II Ciclo* (Erice, 1971), Edizione Cremonese, Rome, 1973, pp. 793–840.
94. M. A. Subin, *On the index of families of Wiener-Hopf operators*, *Mat. Sb.* **84**(126) (1971), 537–558 = *Math. USSR Sb.* **13** (1971), 529–551.
95. ———, *Factorization of matrices depending on a parameter, and elliptic equations in a half space*, *Mat. Sb.* **85**(127) (1971), 65–84 = *Math. USSR Sb.* **14** (1971), 65–84.
96. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Clarendon Press, Oxford, 1937.
97. B. R. Vainberg and V. V. Grušin, *Uniformly nonelliptic problems*. I, II, *Mat. Sb.* **72**(114) (1967), 602–636; **73**(115) (1967), 126–154 = *Math. USSR Sb.* **1** (1967), 543–568; **2** (1967), 111–134.
98. I. N. Vekua, *New methods for solving elliptic equations*, OGIZ, Moscow, 1948; English transl., North-Holland, Amsterdam; Interscience, New York, 1967.
99. N. P. Vekua, *Systems of singular integral equations and some boundary value problems*, 2nd rev. ed., "Nauka", Moscow, 1970; English transl. of 1st ed., Noordhoff, Groningen, 1967.
100. M. I. Višik and G. I. Eskin, *Boundary value problems for general singular integral equations in a bounded domain*, *Dokl. Akad. Nauk SSSR* **155** (1964), 24–27 = *Soviet Math. Dokl.* **5** (1964), 325–329.
101. ———, *Convolution equations in a bounded domain*, *Uspehi Mat. Nauk* **20** (1965), no. 3(123), 89–152 = *Russian Math. Surveys* **20** (1965), no. 3, 85–151.
102. ———, *Elliptic convolution equations in a bounded domain and their applications*, *Uspehi Mat. Nauk* **22** (1967), no. 1(133), 15–76 = *Russian Math. Surveys* **22** (1967), no. 1, 13–75.
103. ———, *Convolution equations of variable order*, *Trudy Moskov. Mat. Obsč.* **16** (1967), 25–50 = *Trans. Moscow Math. Soc.* **1967**, 27–52.
104. ———, *Mixed boundary value problems for elliptic systems of differential equations*, *Tbilis. Saĥelmc Univ. Gamogeneb. Math. Inst. Srom.* **2** (1969), 31–48. (Russian)
105. ———, *Variable order Sobolev-Slobodeckii spaces with weighted norms and their application to mixed boundary value problems*, *Sibirsk. Mat. Ž.* **9** (1968), 973–997 = *Siberian Math. J.* **9** (1968), 723–740.
106. ———, *Normally solvable problems for elliptic systems of convolution equations*, *Mat. Sb.* **74**(116) (1967), 326–356 = *Math. USSR Sb.* **3** (1967), 303–330.

107. *_____, *Convolution equations in a bounded domain in spaces with weighted norms*, Mat. Sb. 69(111) (1966), 65–110; English transl., Amer. Math. Soc. Transl. (2) 67 (1968), 33–82.
108. *_____, *General boundary value problems with discontinuous boundary conditions*, Dokl. Akad. Nauk SSSR 198 (1964), 25–28 = Soviet. Math. Dokl. 5 (1964), 1154–1157.
109. *_____, *Parabolic convolution equations in a bounded domain*, Mat. Sb. 71(113) (1966), 162–190; English transl. in Amer. Math. Soc. Transl. (2) 95 (1970), 131–162.
110. *M. I. Višik and L. A. Ljusternik, *Regular degeneration and boundary layer for linear differential equations with a small parameter*, Uspehi Mat. Nauk 12 (1957), no. 5(77), 3–122; English transl., Amer. Math. Soc. Transl. (2) 20 (1962), 239–364.
111. V. S. Vladimirov, *Equations of mathematical physics*, 2nd rev. ed., “Nauka”, Moscow, 1971; English transl. of 1st ed., Dekker, New York, 1971.
112. L. R. Volevič and B. P. Panejah, *Some spaces of generalized functions and imbedding theorems*, Uspehi Mat. Nauk 20 (1965), no. 1(121), 3–74 = Russian Math. Surveys 20 (1965), no. 1, 1–73.
113. I. I. Vorovič, V. M. Aleksandrov and V. A. Babeško, *Nonclassical mixed problems of elasticity theory*, “Nauka”, Moscow, 1974. (Russian)

NOTATION INDEX

- α_{\pm} , 165
 \mathfrak{A} , 147, 186, 261
 $\mathfrak{A}(\omega)$, 186
 $C_0^\infty(\mathbf{R}^n)$, 7
 $C_0^\infty(U)$, 14
 $C_0'(\mathbf{R}^n)$, 52
 $C^\infty(M)$, 254
deg, 109
 D^k , 10
 $D_{\alpha+i\beta}^{(0)}$, $D_{\alpha+i\beta}$, 121
 $\hat{D}_{\alpha+i\beta}'$, 275
 $\hat{D}_{\alpha+i\beta}^{(0)}(G)$, 278
 $\hat{D}_{\alpha,\gamma}(G)$, 315
 F , 17, 22
 $H_s(\mathbf{R}^n)$, $\tilde{H}_s(\mathbf{R}^n)$, 46
 H_s' , 50
 H_s^\pm , 53, 163
 \tilde{H}_s^\pm , 56
 $H_s(\mathbf{R}_+^n)$, 60, 163
 H_s^* , 62
 $H_{s,r}(\mathbf{R}^n)$, $H_{s,r}(\mathbf{R}_+^n)$, 151
 H , 244
 $\dot{H}_s(G)$, 251
 $H_s(G)$, 251
 $H_s(M)$, 254
 $H_{s,N}(\mathbf{R}^n)$, $H_{s,N}$, $H_{s,N}(\mathbf{R}_+^n)$, 272
 $\dot{H}_{s,N}(G)$, $H_{s,N}(G)$, 274
 H_s'' , 283
 $H_{s,N}'$, $H_{s,N}^{\pm}$, $H_{s,N}(\mathbf{R}_\pm^n)$, 283
 $H_{s,N}(\Gamma_1)$, $H_{s,N}(\Gamma_2)$, 293
 $\dot{H}_{(s)}(G)$, $H_{(s)}(G)$, 296
 $H_{(s)}(\Gamma)$, 298
 $H_{(s),N}(\Gamma_1)$, $H_{(s),N}(\Gamma_2)$, 299
 $\dot{H}_{s,N,\sigma(x)}(G)$, $H_{s,N,\sigma(x)}(G)$, $H_{s,\sigma(x)}(\Gamma)$, 309
 $H^{s,r}(\mathbf{R}^n)$, 345
 $H^{s(x),r}(\mathbf{R}^n)$, 345
 $\dot{H}^{s(x),r}(G)$, $H^{s(x),r}(G)$, 345
 $\mathfrak{I}_s^{(1)}(\mathbf{R}^n)$, $\mathfrak{I}_s^{(2)}(\mathbf{R}^n)$, 185
 $\mathfrak{I}_s^{(3)}(\mathbf{R}^n)$, $\mathfrak{I}_s^{(4)}(\mathbf{R}^n)$, 187, 188
 $\mathfrak{I}_s^{(1)}(G)$, $\mathfrak{I}_s^{(2)}(G)$, 260
 $\mathfrak{I}_{s-1/2,N}^{(1)}$, $\mathfrak{I}_{s-1/2,N}^{(2)}$, 291
 $\mathfrak{I}_{(s),N}^{(1)}(G)$, $\mathfrak{I}_{(s),N}^{(2)}(G)$, 300
 $\mathfrak{I}_{s,N,\sigma(x)}^{(1)}(G)$, $\mathfrak{I}_{s,N,\sigma(x)}^{(2)}(G)$, 309
 $K(M)$, 193
 l , 15
 L_N , 61
 m_1, m_2 , 142
 m_\pm , 143, 185
 \underline{m} , 192
 M , 253
 \mathfrak{M} , 174

- $n_{\pm}(\omega)$, 184
 $O_{\alpha+i\beta}^{\infty}$, 30, 216
 $O'_{\alpha+i\beta}$, 81
 $O_{\mathbb{Q}}$, 165
 $\hat{O}_{\alpha+i\beta}^{\infty}$, 216, 255
 p , 15, 61, 252
 p' , 52, 257
 p'_{+} , 130
 p_{\pm} , 153
 p'' , 283
 p'_1, p'_2 , 284
 P_n , 39
 P_{\pm} , 169
 P_k , 319
 \mathbb{R}_{+} , 18
 $S(\mathbb{R}^n)$, 7
 $S'(\mathbb{R}^n)$, 9
 $S(U)$, 14
 $S'(U)$, 15
 S_{α}^0 , 35
 $S_{\alpha}^0(\mathbb{R}^n)$, 215
 $S'_{\alpha}(\mathbb{R}^n)$, 215
 $S_{\alpha}^0(\mathbb{R}^n \times \mathbb{R}^n)$, 224
 $S'_{\alpha}(\mathbb{R}^n \times \mathbb{R}^n)$, 225
 $S'(M)$, 254
 t_{+} , 11
 t_{-} , 12
 $T^{\alpha}(M)$, 255
 $T_0^{\alpha}(M)$, 255
 $\text{Vect}(M)$, 192
 $W_{s,N}(\mathbb{R}^n), W_{s,N}(\mathbb{R}^n_{+})$, 282, 283
 $W_{s,N}^{(1)}, W_{s,N}^{(2)}$, 287
 $W_{s,N}(G)$, 293
 $W_{s,N}^{(1)}(G), W_{s,N}^{(2)}(G)$, 294
 $W_{(s),N}(G)$, 299
 x' , 15
 Y_k , 44
 Z , 93
 $Z(\kappa)$, 102
 Δ , 39
 $\hat{\Delta}$, 148
 $\theta(t)$, 10
 θ^{+} , 71
 θ^{-} , 72
 κ , 88, 253
 $\Lambda^{\alpha+i\beta}$, 38
 Λ_{\pm} , 57
 $\hat{\Lambda}_{\pm}$, 57
 Λ_s , 254
 $\hat{\Lambda}'$, 275
 Π' , 52
 Π^0 , 71
 Π^{+} , 71
 Π^{-} , 71
 Π_{δ} , 72
 $\Pi_{\delta}^{(1)}$, 73
 ω , 91
 \forall , 9
 $\ddot{\cdot}$, 17
 $\dot{\cdot}$, 19, 91
 $\ddot{\cdot}$, 150
 $*$, 7
 \oplus , 192
 \approx , 193
 (\cdot, \cdot) , 9, 17
 $\langle \cdot, \cdot \rangle_s$, 46
 $[\cdot, \cdot]$, 257
 $|\cdot|$, 7
 $\|[\cdot]\|_m$, 7

- $\| \cdot \|_s$, 46, 255
 $\| \cdot \|_\lambda$, 48
 $[\cdot]_s$, 50
 $\| \cdot \|_s^+$, 60
 $\| \cdot \|_m^+$, 61
 $\| \cdot \|_{s,r}$, $\| \cdot \|_{s,r}^+$, 151
 $[\cdot]_s^+$, $[\cdot]_s^-$, 152
 $\| \cdot \|_1$, $\| \cdot \|_2$, 188
 $\| \cdot \|$, 244
 $\| \cdot \|_{s,N}$, 272, 274
 $\| \cdot \|_{s,N}^+$, 274
 $[[\cdot]]_{s,N}$, 283
 $[\cdot]_{s,N}$, 283
 $[\cdot]_s''$, 283
 $\| \cdot \|_{(s)}$, 296
 $\| \cdot \|_{(s)}^+$, 296
 $[\cdot]_{(s)}$, 298
 $\| \cdot \|_{s,N,\sigma(x)}$, 309
 $\langle \cdot \rangle_{s,r}$, 345
 $\langle \cdot \rangle_{s(x),r}$, 345

This page intentionally left blank

SUBJECT INDEX

- adjoint of a pseudodifferential operator, 227
- averaging kernel, 48

- base space, 192
- boundary value problem, normally solvable, 266
- bundle(s)
 - cotangent, 255
 - isomorphic vector, 192, 193
 - trivial vector, 192
 - vector, 192

- C^∞ -manifold without boundary, compact
 - n -dimensional, 253
- Cauchy principal value, 40
- coboundary operator, 96, 258
- coboundary value problem, 140
- compact n -dimensional C^∞ -manifold without boundary, 253
- condition
 - local, 120
 - nonlocal, 120
 - Šapiro-Lopatinskiĭ, 104, 109, 261, 290
 - transmission, 4
- continuous function, support of, 14
- continuous functional on S , 9
- convergent sequence of generalized functions, 11
- convolution
 - of fundamental functions, 7
 - of a generalized function and a fundamental function, 26
 - of a generalized function and a generalized function with compact support, 28
- convolution type, integral operator of, 36
- coordinate neighborhood, 254
- coordinate system
 - local, 250, 254
 - special local, 251

- cotangent bundle, 255
- cotangent vector, 255

- delta function, 9
- derivative of a generalized function, 10
- differences of vector bundles, equivalent formal, 193
- direct product
 - of generalized functions, 15
 - of regular functionals, 15
- direct sum of vector bundles, 192
- double layer potential, 97
- Douglis-Nirenberg type, elliptic system of, 282
- dual space, 62

- elliptic homogeneous polynomial, 90
- elliptic matching problem, 295
- elliptic pseudodifferential operator, 223, 256
- elliptic symbol, 80, 163
 - factorization index of, 88, 253
 - homogeneous factorization of, 80
 - strongly, 89, 207, 307
- elliptic system of Douglis-Nirenberg type, 282
- equation(s)
 - paired, 153
 - pseudodifferential, 91, 92
- equivalent atlases, 254
- equivalent formal differences of vector bundles, 193
- expansion formula for an integral of Cauchy type, 76

- factorization of an elliptic symbol, homogeneous, 80
- factorization index of an elliptic symbol, 88, 253
- family of Fredholm operators, index of, 194
- fiber, 192

- formal differences of vector bundles, equivalent, 193
- Fourier transform
 of a fundamental function, 17
 of a generalized function, 22
- Fredholm operator(s), 164
 index of a family of, 194
- function
 delta, 9
 fundamental, 9
 generalized, 9, 254
 homogeneous, 30
 trial, 316
- functional
 continuous (on S), 9
 regular, 9
 semilinear, 9
- fundamental function(s), 9
 convolution of, 7
 Fourier transform of, 17
- generalized function(s), 9, 254
 convergent sequence of, 11
 derivative of, 10
 direct product of, 15
 in a domain, 22
 Fourier transform of, 14
 restriction of, 22
 support of, 14
 translation of, 11
- generalized pseudodifferential operator, 225
 kernel of, 231
- homogeneous factorization of an elliptic symbol, 80
- homogeneous function, 30
- homogeneous polynomial, elliptic, 90
- homogeneous symbol, principal, 91, 252, 256
- imbedding theorem, Sobolev, 53
- index
 factorization, 88, 253
 of a family of Fredholm operators, 193
 of an operator, 164
- integral of Cauchy type, expansion formula for, 76
- integral operator
 of convolution type, 36
 singular, 40
- integrodifferential operator, Zygmund-Calderón, 44
- isomorphic vector bundles, 192, 193
- kernel
 averaging, 48
 of a generalized pseudodifferential operator, 231
- left regularizer, 167
- Liouville's theorem, 78
- local condition, 120
- local coordinate system, 250, 254
 special, 251
- localization of a pseudodifferential operator, 252
- manifold without boundary, compact
 n -dimensional C^∞ -, 253
- matching problem, elliptic, 295
- matrices, transition, 192
- Mellin transform, 19
- "minus" operator, 348
- "minus" symbol, 134, 240, 248
- neighborhood, coordinate, 254
- Newtonian potential, 39
- nonlocal condition, 120
- normally solvable boundary value problem, 266
- normally solvable operator, 164
- operator
 coboundary, 96, 258
 elliptic pseudodifferential, 223, 256
 Fredholm, 164
 generalized pseudodifferential, 225
 index of, 164
 "minus", 348
 normally solvable, 164
 order of, 218
 "plus", 347
 pseudodifferential, 35, 36, 216
 Riesz, 44
 singular integral, 40
 Zygmund-Calderón integrodifferential, 44
- order of an operator, 218
- paired equations, 153
- Paley-Wiener theorem, 58
- "plus" operator, 347
- "plus" symbol, 105, 240, 248
- polynomial, homogeneous elliptic, 90
- potential
 double layer, 97
 Newtonian, 39
 single layer, 258
- potential type, pseudodifferential operator of, 96, 258
- principal homogeneous symbol, 92, 252, 256

- principal value, Cauchy, 40
- problem
- coboundary value, 140
 - elliptic matching, 295
- product of a generalized function and a smooth function, 16
- pseudodifferential equation, 91
- in \mathbb{R}_+^n , 92
- pseudodifferential operator, 35, 36, 216
- adjoint of, 227
 - in a domain, 252
 - elliptic, 223, 256
 - generalized, 225
 - localization of, 252
 - of potential type, 96, 258
 - smooth in a domain, 4
 - smooth in \mathbb{R}_+^n , 121
 - symbol of, 35, 174, 179, 216
- pseudolocal property, 233
- regular functional(s), 9
- direct product of, 15
- regular summand, 358
- regularizer, 165
- left, 167
 - right, 167
- restriction of a generalized function, 15
- Riesz operator, 44
- Šapiro-Lopatinskii condition, 104, 109, 261, 290
- semilinear functional, 9
- single layer potential, 258
- singular integral, Cauchy principal value of, 40
- singular integral operator, 40
- singular summand, 358
- smooth pseudodifferential operator
- in a domain, 4
 - in \mathbb{R}_+^n , 121
- Sobolev imbedding theorem, 53
- Sobolev-Slobodeckii space, 46, 254
- space
- base, 192
 - dual, 62
 - Sobolev-Slobodeckii, 46, 254
- special local coordinate system, 251
- strongly elliptic symbol, 89, 207
- summand
- regular, 358
 - singular, 358
- support
- of a continuous function, 14
 - of a generalized function, 14
- symbol
- elliptic, 80, 163
 - "minus", 134, 240, 248
 - "plus", 105, 240, 248
 - principal homogeneous, 92, 252, 256
 - of a pseudodifferential operator, 35, 174, 179, 216
 - strongly elliptic, 89, 207, 307
- system
- local coordinate, 250, 254
 - special local coordinate, 251
- theorem
- Liouville's, 78
 - Paley-Wiener, 58
 - Sobolev imbedding, 53
- transform
- Fourier, 17, 22
 - Mellin, 19
- transition matrices, 192
- translation of a generalized function, 11
- transmission condition, 4
- trial function, 316
- trivial vector bundle, 192
- vector, cotangent, 255
- vector bundle(s), 192
- base space of, 192
 - direct sum of, 192
 - equivalent formal differences of, 193
 - fiber of, 192
 - isomorphic, 192, 193
 - transition matrices of, 192
 - trivial, 192
- Zygmund-Calderon integrodifferential operator, 44

This page intentionally left blank

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy an article for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication (including abstracts) is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Executive Director, American Mathematical Society, P.O. Box 6248, Providence, Rhode Island 02940.

The owner consents to copying beyond that permitted by Sections 107 or 108 of the U.S. Copyright Law, provided that a fee of \$1.00 plus \$.25 per page for each copy be paid directly to the Copyright Clearance Center, Inc., 21 Congress Street, Salem, Massachusetts 01970. When paying this fee please use the code 0065-9282/89 to refer to this publication. This consent does not extend to other kinds of copying, such as copying for general distribution, for advertising or promotion purposes, for creating new collective works, or for resale.

