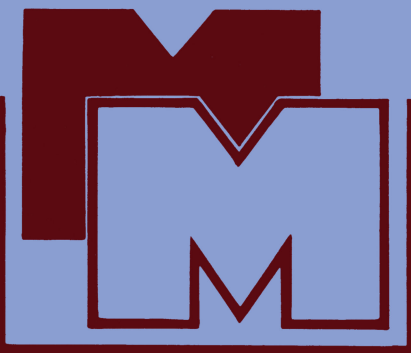
Boundary-Value Problems with Free Boundaries for Elliptic Systems of Equations

by V. N. MONAKHOV



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КРАЕВЫЕ ЗАДАЧИ СО СВОБОДНЫМИ ГРАНИЦАМИ ДЛЯ ЭЛЛИПТИЧЕСКИХ СИСТЕМ УРАВНЕНИЙ

В. Н. МОНАХОВ

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ABSTRACT. This book is concerned with certain classes of nonlinear problems for elliptic systems of partial differential equations: boundary-value problems with free boundaries. The first part has to do with the general theory of boundary-value problems for analytic functions and applications of it to hydrodynamics. The second presents the theory of quasi-conformal mappings, along with the theory of boundary-value problems for elliptic systems of equations and applications of it to problems in the mechanics of continuous media with free boundaries: problems in subsonic gas dynamics, filtration theory, and problems in elastico-plasticity.

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PREFACE

The applied sciences (hydrodynamics, geophysics, etc) proposed almost at the same time two types of problems for mathematics: by convention they may be called "direct" and "inverse" problems.

Inverse problems, the theory of which began to take shape only in recent decades, differ from direct problems in that not only the solutions of the differential equations but also the equations themselves or the boundaries (or parts of the boundaries) of their domains are unknown.

The latter type of inverse problems, which (following the terminology adopted in hydrodynamics) we shall call problems with free boundaries, include jet problems in hydrodynamics, elastico-plastic problems, problems of heat propagation in media with a changing phase state (problems of Stefan type), and so on.

This monograph deals with those problems in the mechanics of continuous media with free boundaries which, in the final analysis, reduce to boundary-value problems for quasilinear elliptic systems of first-order equations with operator coefficients.

The book arose from a course of lectures given by the author repeatedly and in several variants first to students at Kazen State University and then (beginning in 1966) to students at Novosibirsk State University.

Therefore, certain sections of the book bear the nature of a textbook and can serve as such in the theory of partial differential equations: boundary-value problems and boundary properties of analytic functions (Chapters I and II), quasiconformal mappings (Chapter V), and boundary-value problems for linear and quasilinear elliptic systems of equations (Chapter VI).

At the end of almost every chapter we state unsolved problems of varying difficulty, some of which may serve as subjects of scientific research or to be assigned as course or degree projects to students specializing in the theory of partial differential equations and its applications to mechanics.

The content of this book is reflected well enough in its Table of Contents. It should also be pointed out that some results of the author and his students are published here for the first time.

The author was greatly aided in the preparation of the book for printing by his students, those on the staff at the Institute of Hydrodynamics, and post-graduate students at the Department of Theoretical Mechanics at Novosibirsk State University: G. V. Alekseev, A. V. Kazhikhov, N. A. Kucher, G. V. Lavrent'ev, A. M. Meĭrmanov, A. A. Oleĭnik, P. I. Plotnikov, and B. G. Putievskiĭ, who read through the manuscript and made many useful remarks. In addition, A. V. Kazhikhov and N. A. Kucher also supplied material used in the writing of §5 in Chapter IV and §4 in Chapter VI.

It would be impossible to overestimate the constant help given the author by the Editor, S. N. Antontsev. He suggested a number of improvements in the presentation, wrote §4 in Chapter VII at the author's request, and wrote §3 in Chapter VI jointly with the author.

The author expresses his sincere gratitude to all these persons.

V. N. Monakhov

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