

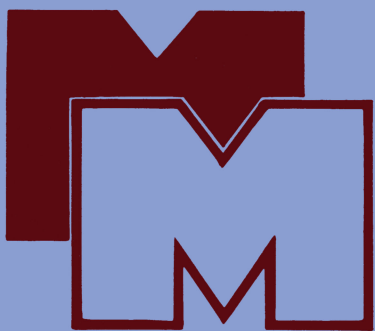
# **Boundary-Value Problems with Free Boundaries for Elliptic Systems of Equations**

by **V. N. MONAKHOV**

**Volume 57**

**TRANSLATIONS OF  
MATHEMATICAL MONOGRAPHS**

American Mathematical Society



**Boundary-Value Problems  
with  
Free Boundaries  
for  
Elliptic Systems of Equations**

*This page intentionally left blank*

10.1090/mmono/057

TRANSLATIONS OF MATHEMATICAL MONOGRAPHS

VOLUME **57**

**Boundary-Value Problems  
with Free Boundaries for  
Elliptic Systems of Equations**

by **V. N. MONAKHOV**

American Mathematical Society · Providence · Rhode Island

# КРАЕВЫЕ ЗАДАЧИ СО СВОБОДНЫМИ ГРАНИЦАМИ ДЛЯ ЭЛЛИПТИЧЕСКИХ СИСТЕМ УРАВНЕНИЙ

В. Н. МОНАХОВ

«НАУКА», СИБИРСКОЕ ОТДЕЛЕНИЕ  
НОВОСИБИРСК 1977

Translated from the Russian by H. H. McFaden  
Translation edited by Lev J. Leifman

1980 *Mathematics Subject Classification*. Primary 30C60, 35J65, 35Q15, 35R35, 76-02, 76B10, 76N15, 76S05; Secondary 30C20, 30E20, 31A25, 35A05, 35M05, 46E35, 47G05, 73B99, 73J06, 73F99, 76C05, 76D15, 76D25, 76F99, 76G05.

**ABSTRACT.** This book is concerned with certain classes of nonlinear problems for elliptic systems of partial differential equations: boundary-value problems with free boundaries. The first part has to do with the general theory of boundary-value problems for analytic functions and applications of it to hydrodynamics. The second presents the theory of quasi-conformal mappings, along with the theory of boundary-value problems for elliptic systems of equations and applications of it to problems in the mechanics of continuous media with free boundaries: problems in subsonic gas dynamics, filtration theory, and problems in elasto-plasticity.

## Library of Congress Cataloging in Publication Data

Monakhov, V. N. (Valentin Nikolaevich)

Boundary value problems with free boundaries for elliptic systems of equations.

(Translations of mathematical monographs; v. 57)

Translation of: Kraevye zadachi so svobodnymi granitsami èllipticheskikh sistem uravnenii.

Bibliography: p.

1. Boundary value problems. 2. Differential equations, Elliptic. 3. Fluid dynamics. I. Title. II. Series.

QA377.M5813 1983

515.3'53

83-2754

ISBN 0-8218-4510-1

ISSN 0065-9282

Copyright © 1983 by the American Mathematical Society

## TABLE OF CONTENTS

Preface	xiii
<b>PART I. Boundary-Value Problems in the Theory of Analytic Functions and their Application to Hydrodynamics</b>	
CHAPTER I. The Boundary-Value Problems of Riemann and Hilbert	3
1. Piecewise smooth curves. Classes of functions on curves	3
2. Cauchy type integrals	9
1. Definition and elementary properties of Cauchy type integrals	9
2. Behavior in the proximity of the curve of integration	11
3. Boundary properties of Cauchy type integrals	14
4. Behavior of the derivative of a Cauchy type integral when approaching the boundary	17
5. Behavior at the endpoints	18
6. The theorem of Muskhelishvili	20
7. The Poincaré-Bertrand formulas for interchanging the in- tegrations in iterated singular integrals	25
8. Inversion formulas for Cauchy type integrals	27
9. The Hilbert inversion formulas	27
3. The Riemann boundary-value problem [the conjunction problem]	28
1. Auxiliary facts	28
2. The problem of determining a piecewise analytic function from a jump	29
3. Analytic extension of functions defined on a collection of closed contours	30
4. The index of a continuous function	30
5. The index of a discontinuous function	32
6. The Riemann problem for Hölder coefficients and a single closed contour	34

7. The Riemann problem with Hölder coefficients in a multiply connected domain	37
8. The Riemann problem with piecewise Hölder coefficients	38
9. The Riemann problem for nonclosed contours	41
10. The Riemann problem for the upper half-plane	41
4. The Hilbert boundary-value problem	42
1. The Hilbert problem with Hölder coefficients for the disk	43
2. The Hilbert problem with piecewise Hölder coefficients	48
3. The Hilbert problem for the upper half-plane	53
4. The Schwarz-Christoffel formula for polygons	55
5. The mixed boundary-value problem for the upper half-plane	57
6. The mixed boundary-value problem with parameters	58
5. Stability of the Hilbert problem	59
6. Singular integral equations. Generalizations of the Riemann and Hilbert boundary-value problems	68
CHAPTER II. Singular Operators in Spaces of Summable Functions. Application to Boundary-Value Problems and to the Study of Boundary Properties of Analytic Functions	
	73
1. Banach spaces, fixed-point principles	73
2. Singular integrals in spaces of summable functions	79
1. Cauchy-Lebesgue type integral and Poisson-Lebesgue integral	79
2. Singular operators defined on the circle and on the real axis	83
3. Singular operators defined on Lyapunov curves	89
4. The interpolation theorem of E. M. Stein	94
3. Boundary properties of conformal mappings, and criteria for univalence	99
4. The classes $H_p$ and $E_p$ of analytic functions	105
1. The classes $H_p$	106
2. Blaschke functions	107
3. Representation of $H_p$ -functions	109
4. Boundary properties of $H_p$ -functions	110
5. The classes $E_p$	116
5. The Hilbert boundary-value problem with piecewise continuous coefficients. Stability of the problem	118
6. Boundary-value problems with measurable coefficients	124
CHAPTER III. The Mixed Boundary-Value Problem with Free Boundary	
	133
1. Statement of the problem. Investigation of representations of the solution in the case of a polygonal line	133
1. Statement of the problem	133
2. Construction of a solution	135
3. Some properties of the solution	137

2. Proofs of existence and uniqueness theorems by the continuity method	141
1. The system of equations for parameters	141
2. A priori estimates of the solution	142
3. Local uniqueness of solutions	145
4. An existence theorem	149
5. Global uniqueness of the solution	151
6. Univalence of solutions	152
7. Remarks	153
3. Proof of the existence theorem solely on the basis of a priori estimates of the solution	154
4. An existence theorem in the case of a curvilinear boundary	157
5. Examples of hydrodynamics problems that reduce to a mixed boundary-value problem with free boundary	162
1. The filtration problem for a fluid in an earthen dam	162
2. Determination of the underground contour of a hydraulic structure from a given distribution of pressures	163
3. The inverse problem of the theory of impact	163
4. The inverse problem of transonic flow of a gas	164
6. The mixed boundary-value problem with free boundary for a system of polygonal lines	166
7. Assignment of the boundary conditions in terms of functions of the other parameters	171
8. On boundary-value problems with free boundaries for the case of an unknown domain in the plane of a complex potential	177
9. A brief survey and problems	180
CHAPTER IV. Flows of an Incompressible Fluid with Free Boundaries	183
1. The continuity method in the problem of flow with jet separation past obstacles	183
1. Statement of the problem	183
2. Local uniqueness of solutions	186
3. An existence theorem	190
2. An existence theorem in the general hydrodynamics problem with a single free boundary	191
1. Statement of the problem	191
2. Construction of the solution and its properties	193
3. A priori estimate of the solution	196
4. An existence theorem in the case of polygonal boundaries	200
5. A curvilinear boundary	204
6. Time-dependent flows	210
7. The magnitude of the velocity on the free boundary—a function of one of the coordinates	210



3. Application of the general existence theorem in problems of flow with jet separation past obstacles	211
1. The Kirchhoff scheme	212
2. The Thullen scheme (with two spiral vortices)	212
3. The symmetric scheme of Èfros	213
4. The Riabouchinsky scheme (the "mirror" scheme)	214
5. The modified Lavrent'ev scheme	215
6. Flow of a fluid out of a nozzle	216
7. Flow in channels with partially unknown walls	217
4. Application of the method of finite-dimensional approximation in problems on potential flows of a fluid in a gravitational field	217
5. Detached flows of Riabouchinsky type in a gravitational field with capillary forces taken into account	225
6. A brief survey and problems	231

## **PART II. Generalized Solutions of Quasilinear Systems of Equations in the Mechanics of Continuous Media**

CHAPTER V. Quasiconformal Mappings and Generalized Solutions of Elliptic Systems of Equations on the Plane	239
1. Generalized derivatives. Sobolev spaces	239
2. Potential and singular operators with respect to a domain	244
3. Homeomorphisms of the whole plane and representations of generalized solutions of elliptic systems of equations	259
4. Construction of homeomorphisms of the unit disk onto itself	276
5. Homeomorphisms of infinite domains	291
6. An existence theorem for homeomorphisms of multiply connected domains	297
7. Extension of homeomorphisms	301
1. Extension of homeomorphisms of multiply connected domains	301
2. Extension of homeomorphisms from the boundary	305
8. Mappings of multiply connected domains by solutions of nonlinear $L$ -elliptic systems of equations	307
1. Definition and properties of $L$ -elliptic systems	308
2. The generalized Riemann mapping theorem	310
3. A uniqueness theorem	318
4. Systems of equations that are strongly elliptic in the Lavrent'ev sense	320
5. Equivalence of the definitions of $L$ -ellipticity and strong ellipticity in Lavrent'ev's sense	326
9. Problems	328
CHAPTER VI. Boundary-Value Problems	331
1. On a class of functions defined on curves	331

2. The Hilbert boundary-value problem for quasilinear elliptic systems of equations	335
1. Properties of solutions of the Hilbert problem for analytic functions	335
2. Solvability of the Hilbert problem	340
3. Uniqueness of solutions of the Hilbert problem	345
4. The mixed boundary-value problem with parameters	347
5. Stability of solutions of the Hilbert problem under small changes in the coefficients of the equation and the boundary condition	348
6. The Hilbert problem for nonuniformly elliptic equations	351
7. The Hilbert problem in the case of summable coefficients in the boundary condition	353
3. Boundary-value problems of conjunction with a shift in multiply connected domains	357
1. Statement of the problem	357
2. The boundary-value problem of a jump with a shift	359
3. The connection between the Riemann and Haseman problems	360
4. Stability of solutions of the problem of a jump with a shift	361
5. The Haseman problem for analytic functions in the case of a nondifferentiable shift	362
6. Stability of solutions of the Haseman problem for analytic functions	363
7. The general problem of conjunction with a shift for elliptic systems of equations	364
4. The Poincaré problem with discontinuous coefficients in the boundary condition for a quasilinear second-order elliptic equation	367
1. The Poincaré problem	367
2. Maximum principles	370
5. On certain potential and singular operators in spaces of summable functions	373
6. Boundary-value problems for systems of $2m$ ( $m \geq 1$ ) equations	379
7. Problems	381
 CHAPTER VII. Boundary-Value Problems in Hydrodynamics and Subsonic Gas Dynamics	 383
1. Equations of gas dynamics and their transformation to different variables	383
2. General properties of solutions of boundary-value problems in hydrodynamics and gas dynamics. A priori estimates	388
1. Statement of the problems	388
2. Solvability of problems in the class $\mathfrak{M}(N, \delta)$	392

3. A priori estimate of the flow velocity	396
4. The parameter problem	403
5. Univalence	410
3. Existence and uniqueness theorems in boundary-value problems in hydrodynamics and subsonic gas dynamics	411
1. Problems on constantly subsonic gas flows in given domains bounded by rigid walls	411
2. Problems of constantly subsonic gas flows with partially unknown (free) boundaries of the flow domain	414
3. Problems on the motion of a fluid in a gravitational field	420
4. Problems with completely unknown boundaries of the flow domain	422
5. Rotational flows of a compressible fluid	422
4. Some problems in gas dynamics with free boundaries in doubly connected domains	425
5. Variational formulas in hydrodynamics problems with free boundaries	430
6. Inverse boundary-value problems for elliptic systems of equations	437
7. Problems	441
CHAPTER VIII. Problems in Filtration Theory for a Fluid with Free Boundaries	443
1. On certain transformations of the equations of motion of a fluid in porous media	443
1. Darcy's law	444
2. A nonlinear law of filtration	447
2. Solvability of a certain class of problems in the linear theory of filtration with free boundaries	450
1. Boundary conditions	450
2. The problem on the generalized hodograph plane	452
3. Selection and properties of the domain in the generalized hodograph plane	453
4. Construction of the quasiconformal mapping	456
5. Other boundary conditions	460
3. On solvability of filtration problems with free boundaries in the case of a nonlinear filtration law	460
1. Reduction to a boundary-value problem	460
2. Properties of the boundary of the domain in the generalized hodograph plane	462
3. Another form for the boundary conditions	465
4. Reduction of problems of filtration type in earthen dams to the Dirichlet problem	465
1. The case of Darcy's law	465
2. Nonlinear law of filtration	468

5. Planar and axisymmetric problems involving filtration of a fluid in nonhomogeneous anisotropic media	471
1. Problems in given domains	471
2. Problems with free boundaries	474
6. Problems involving filtration of a nonhomogeneous fluid with free boundaries for a given time-dependent distribution of temperature and component concentration	482
1. Differential equations	482
2. Boundary conditions	483
3. Quasiconformal mappings dependent on parameters	484
7. Problems	488
1. Steady-state problems	488
2. Time-dependent problems	490
CHAPTER IX. Some Planar Problems with an Unknown Boundary in Elasticity Theory	495
1. Determination of the unknown boundary from a stress tensor given on it	496
2. Determination of the unknown boundary from a stress vector given on it	506
3. The semi-inverse problem with unknown boundary	511
4. Problems	514
Bibliography	517

*This page intentionally left blank*

## PREFACE

The applied sciences (hydrodynamics, geophysics, etc) proposed almost at the same time two types of problems for mathematics: by convention they may be called “direct” and “inverse” problems.

Inverse problems, the theory of which began to take shape only in recent decades, differ from direct problems in that not only the solutions of the differential equations but also the equations themselves or the boundaries (or parts of the boundaries) of their domains are unknown.

The latter type of inverse problems, which (following the terminology adopted in hydrodynamics) we shall call problems with free boundaries, include jet problems in hydrodynamics, elasto-plastic problems, problems of heat propagation in media with a changing phase state (problems of Stefan type), and so on.

This monograph deals with those problems in the mechanics of continuous media with free boundaries which, in the final analysis, reduce to boundary-value problems for quasilinear elliptic systems of first-order equations with operator coefficients.

The book arose from a course of lectures given by the author repeatedly and in several variants first to students at Kazan State University and then (beginning in 1966) to students at Novosibirsk State University.

Therefore, certain sections of the book bear the nature of a textbook and can serve as such in the theory of partial differential equations: boundary-value problems and boundary properties of analytic functions (Chapters I and II), quasiconformal mappings (Chapter V), and boundary-value problems for linear and quasilinear elliptic systems of equations (Chapter VI).

At the end of almost every chapter we state unsolved problems of varying difficulty, some of which may serve as subjects of scientific research or to be assigned as course or degree projects to students specializing in the theory of partial differential equations and its applications to mechanics.

The content of this book is reflected well enough in its Table of Contents. It should also be pointed out that some results of the author and his students are published here for the first time.

The author was greatly aided in the preparation of the book for printing by his students, those on the staff at the Institute of Hydrodynamics, and post-graduate students at the Department of Theoretical Mechanics at Novosibirsk State University: G. V. Alekseev, A. V. Kazhikhov, N. A. Kucher, G. V. Lavrent'ev, A. M. Meïrmanov, A. A. Oleïnik, P. I. Plotnikov, and B. G. Putievskiï, who read through the manuscript and made many useful remarks. In addition, A. V. Kazhikhov and N. A. Kucher also supplied material used in the writing of §5 in Chapter IV and §4 in Chapter VI.

It would be impossible to overestimate the constant help given the author by the Editor, S. N. Antontsev. He suggested a number of improvements in the presentation, wrote §4 in Chapter VII at the author's request, and wrote §3 in Chapter VI jointly with the author.

The author expresses his sincere gratitude to all these persons.

V. N. Monakhov

## BIBLIOGRAPHY

1. Lars V. Ahlfors, *Lectures in quasiconformal mappings*, Van Nostrand, Princeton, N.J., 1966.
2. B. D. Annin, *Two-dimensional problems of elasto-plasticity*, Izdat. Novosibirsk. Gos. Univ., Novosibirsk, 1968. (Russian)
3. S. N. Antontsev, *Subsonic gas flows in multiply connected regions*, Fluid Dynamics Trans. Vol. 5, Part II (Ninth Sympos. Adv. Problems and Methods in Fluid Mech., Kazimierz, 1969), PWN, Warsaw, 1971, pp. 7–18.
4. S. N. Antontsev and N. A. Kucher, *The Poincaré problem with discontinuous coefficients of the boundary condition for a quasilinear elliptic equation*, Dinamika Sploshnoi Sredy Vyp. 5 (1970), 118–124. (Russian)
5. S. N. Antontsev and V. D. Lelyukh, *Some conjunction problems of rotational and potential subsonic flows*, Dinamika Sploshnoi Sredy Vyp. 1 (1969), 134–153. (Russian)
6. S. N. Antontsev and V. N. Monakhov, *On the solvability of a class of conjunction problems with shift*, Dokl. Akad. Nauk SSSR **205** (1972), 263–266; English transl. in Soviet Math. Dokl. **13** (1972).
7. ———, *On some problems in the filtration of two-phased incompressible fluids*, Dinamika Sploshnoi Sredy Vyp. 2 (1969), 156–167. (Russian)
8. ———, *On a general quasilinear model of the filtration of immiscible fluids*, Dinamika Sploshnoi Sredy Vyp. 3 (1969), 5–17. (Russian)
9. ———, *Some nonstationary problems in the filtration of nonhomogeneous fluids with free (unknown) boundaries*, Dinamika Sploshnoi Sredy Vyp. 3 (1969), 18–32. (Russian)
10. ———, *On some nonstationary problems with unknown boundaries*, Some Problems of Math. and Mech. (M. A. Lavrent'ev Seventieth Birthday Vol.), “Nauka”, Leningrad, 1970, pp. 75–87; English transl. in Amer. Math. Soc. Transl. (2) **104** (1976).
11. ———, *The Riemann-Hilbert boundary value problem with discontinuous boundary conditions for quasilinear elliptic systems of equations*, Dokl. Akad. Nauk SSSR **175** (1967), 511–513; English transl. in Soviet Math. Dokl. **8** (1967).
12. ———, *Boundary value problems with discontinuous boundary conditions for quasilinear systems of  $2m$  ( $m \geq 1$ ) first order equations*, Izv. Sibirsk. Otdel. Akad. Nauk SSSR **1967**, no. 8, 65–73. (Russian)
13. V. M. Babich, *On the extension of functions*, Uspekhi Mat. Nauk **8** (1953), no. 2(54), 111–113. (Russian)
14. Herbert Beckert, *Existenzbeweis für permanente Kapillarwellen einer schweren Flüssigkeit entlang eines Karals*, Arch. Rational Mech. Anal. **13** (1963), 15–45.
15. Paul W. Berg, *The existence of subsonic Helmholtz flows of a compressible fluid*, Comm. Pure Appl. Math. **15** (1962), 289–347.



16. Lipman Bers, *Mathematical aspects of subsonic and transonic gas dynamics*, Wiley, New York, and Chapman & Hall, London, 1958.
17. ———, *Quasiconformal mappings and Teichmüller's theorem*, Analytic Functions, Princeton Univ. Press, Princeton, N.J., 1960, pp. 89–119.
18. A. V. Bitsadze, *On the problem of equations of mixed type*, Trudy Mat. Inst. Steklov. **41** (1953); German transl., VEB Deutscher Verlag Wiss., Berlin, 1957.
19. Garrett Birkhoff, *Hydrodynamics. A study in logic, fact and similitude*, Princeton Univ. Press, Princeton, N.J., 1950.
20. Garrett Birkhoff and E. H. Zarantonello, *Jets, wakes, and cavities*, Academic Press, 1957.
21. B. V. Boyarskiĭ [Bogdan Bojarski], *Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients*, Mat. Sb. **43(85)** (1957), 451–503. (Russian)
22. A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.
23. I. A. Charnyi, *Underground hydrodynamics and gas dynamics*, Gostoplekhizdat, Moscow, 1963. (Russian)
24. G. P. Cherepanov, *On a method of solving a problem of elastico-plasticity*, Prikl. Mat. Mekh. **27** (1963), 428–435; English transl. in J. Appl. Math. Mech. **27** (1963).
25. Royal Eugene Collins, *Flow of fluids through porous materials*, Reinhold, New York, 1961.
26. R. Courant, *Dirichlet's principle, conformal mapping, and minimal surfaces*, Interscience, 1950.
27. I. I. Danilyuk, *On Hilbert's problem with measurable coefficients*, Sibirsk. Mat. Zh. **1** (1960), 171–197. (Russian)
28. ———, *A problem with oblique derivative*, Sibirsk. Mat. Zh. **3** (1962), 17–55. (Russian)
29. ———, *A generalized Cauchy formula for axisymmetric fields*, Sibirsk. Mat. Zh. **4** (1963), 48–85. (Russian)
30. Basile Demtchenko, *Problèmes mixtes harmoniques en hydrodynamique des fluides parfaits*, Gauthier-Villars, Paris, 1933.
31. Nelson Dunford and Jacob T. Schwartz, *Linear operators*. Vol. I, Interscience, 1958.
32. F. D. Gakhov, *Boundary-value problems*, Fizmatgiz, Moscow, 1958; English transl. of 2nd (1963) ed., Pergamon Press, Oxford, and Addison-Wesley, Reading, Mass., 1966.
33. ———, *Riemann's boundary value problem for a system of  $n$  pairs of functions*, Uspekhi Mat. Nauk **7** (1952), no. 4(50), 3–54. (Russian)
34. F. D. Gakhov and I. M. Mel'nik, *Singular contour points in the inverse boundary value problem of the theory of analytic functions*, Ukrain. Mat. Zh. **11** (1959), 25–37. (Russian)
35. L. A. Galin, *Plane elastico-plastic problem. Plastic zones in the vicinity of circular apertures in plates and girders*, Prikl. Mat. Mekh. **10** (1946), 367–386. (Russian; English summary)
36. P. R. Garabedian, H. Lewy and M. Schiffer, *Axially symmetric cavitation flow*, Ann. of Math. (2) **56** (1952), 560–602.
37. T. G. Gegeliya, *Boundedness of singular operators*, Soobshch. Akad. Nauk Gruzin. SSR **20** (1958), 517–523. (Russian)
38. Robert Gerber, *Sur les solutions exactes des équations du mouvement avec surface libre d'un liquide pesant*, J. Math. Pures Appl. (9) **34** (1955), 185–299.
39. David Gilbarg, *Unsteady flows with free boundaries*, Z. Angew. Math. Phys. **3** (1952), 34–42.
40. M. A. Gol'dshtik, *A mathematical model of separated flows in an incompressible liquid*, Dokl. Akad. Nauk SSSR **147** (1962), 1310–1313; English transl. in Soviet Phys. Dokl. **7** (1962/63).
41. G. M. Goluzin, *Geometric theory of functions of a complex variable*, GITTL, Moscow, 1952; English transl. of 2nd (1966) ed., Transl. Math. Mono., vol. 26, Amer. Math. Soc., Providence, R.I., 1969.
42. M. I. Gurevich, *Theory of jets in ideal fluids*, Fizmatgiz, Moscow, 1961; English transl., Academic Press, 1965.

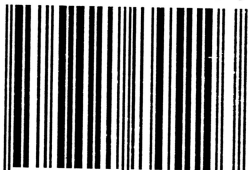
43. L. G. Guzevskii and G. V. Lavrent'ev, *Application of the method of finite-dimensional approximation in problems of jet flows*, *Dinamika Sploshnoi Sredy Vyp.* 1 (1969), 75–91. (Russian)
44. M. I. Kaikin, *Existence theorems for a class of inverse mixed boundary-value problems of the theory of analytic functions*, *Trudy Kazan. Aviats. Inst. Vyp.* 64 (1961), 3–24. (Russian)
45. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, 1934.
46. S. A. Kristianovich, *Motion of ground water which does not conform to Darcy's law*, *Prikl. Mat. Mekh.* 4 (1940), no. 1, 33–52. (Russian; English summary)
47. R. Huron, *Contribution à l'étude de l'unicité des solutions du problème de représentation conforme de Helmholtz*, *Ann. Fac. Sci. Univ. Toulouse* (4) 15 (1951), 5–78.
48. N. V. Kusnutdinova, *On the behavior of solutions of the Stefan problem as time increases without bound*, *Dinamika Sploshnoi Sredy Vyp.* 2 (1969), 168–177. (Russian)
49. B. V. Kvedelidze, *Linear discontinuous boundary value problems in the theory of functions, singular integral equations and some of their applications*, *Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze* 23 (1956), 3–158. (Russian)
50. N. E. Zhukovskii [Joukowsky], *A modification of Kirchoff's method of determining a two-dimensional motion of a fluid, given a constant velocity along an unknown streamline*, *Mat. Sb.* 15 (1890/91), 121–276; reprinted as *Trudy Tsentral. Aéro-gidrodinam. Inst. Vyp.* 41 (1930); also reprinted in at least three collections of his *Works*. (Russian)
51. Wilfred Kaplan, *Close-to-convex schlicht functions*, *Michigan Math. J.* 1 (1952), 169–185.
52. A. V. Kazhikhov, *Some self-similar problems of time-dependent filtration, and their numerical solution*, *Dinamika Sploshnoi Sredy Vyp.* 3 (1969), 33–44. (Russian)
53. ———, *On the existence of a separation flow of Ryabushinskii type in a gravitational field with capillary forces taken into account*, *Dinamika Sploshnoi Sredy Vyp.* 1 (1969), 92–99. (Russian)
54. M. Keldysh and F. Frankl, *Die äussere Neumann'sche Aufgabe für nichtlineare elliptische Differentialgleichungen mit Anwendung auf die Theorie des Flügels im kompressiblen Gas*, *Izv. Akad. Nauk SSSR* (7) Otd. Mat. Estestv. Nauk 1934, 561–601. (Russian; German summary)
55. M. V. Keldysh and L. I. Sedov, *Applications of the theory of functions of a complex variable to hydrodynamics and aerodynamics (review of some works of the Moscow school)*, *Appl. Theory of Functions in Continuum Mech. (Proc. Internat. Sympos., Tbilisi, 1963), Part II: Fluid and Gas Mech., Math. Methods*, "Nauka", Moscow, 1965, pp. 13–42; English transl., *ibid.*, pp. 43–64.
56. N. E. Kochin, I. A. Kibel' and N. V. Roze, *Theoretical hydromechanics*. I, 5th ed., GITTL, Moscow, 1955; English transl., Wiley, 1964.
57. M. A. Krasnosel'skii, *Topological methods in the theory of nonlinear integral equations*, GITTL, Moscow, 1956; English transl., Macmillan, 1964.
58. Julien Kravtchenko, *Sur le problème de représentation conforme de Helmholtz; théorie des sillages et des poutes*, *J. Math. Pures Appl.* (9) 20 (1941), 35–234, 235–303.
59. O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and quasilinear elliptic equations*, "Nauka", Moscow, 1964; English transl., Academic Press, 1968.
60. O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and quasilinear equations of parabolic type*, "Nauka", Moscow, 1967; English transl., *Transl. Math. Mono.*, vol. 23, Amer. Math. Soc., Providence, R.I., 1968.
61. M. A. Lavrent'ev, *Variational methods for boundary value problems for systems of elliptic equations*, *Izdat. Akad. Nauk SSSR*, Moscow, 1962; English transl., Noordhoff, 1963.
62. ———, *Boundary value problems in the theory of univalent functions*, *Mat. Sb.* 1(43) (1936), 815–844; English transl. in *Amer. Math. Soc. Transl.* (2) 32 (1963).
63. ———, *Sur certaines propriétés des fonctions univalentes et leurs applications à la théorie des sillages*, *Mat. Sb.* 4 (1938), 391–458. (Russian; French summary)
64. ———, *Certain boundary-value problems for systems of elliptic type*, *Sibirsk. Mat. Zh.* 3 (1962), 715–728. (Russian)
65. ———, *The general problem of the theory of quasiconformal mappings of plane regions*, *Mat. Sb.* 21(63) (1947), 285–320; English transl. in *Amer. Math. Soc. Transl.* (1) 2 (1962).

66. ———, *A fundamental theorem of the theory of quasiconformal mappings of two-dimensional regions*, Izv. Akad. Nauk SSSR Ser. Mat. **12** (1948), 513–554; English transl. in Amer. Math. Soc. Transl. (1) **2** (1962).
67. M. A. Lavrent'ev and A. V. Bitsadze, *On the problem of equations of mixed type*, Dokl. Akad. Nauk SSSR **70** (1950), 373–376. (Russian)
68. M. A. Lavrent'ev and B. V. Shabat, *Geometrical properties of solutions of nonlinear systems of partial differential equations*, Dokl. Akad. Nauk SSSR **112** (1957), 810–811. (Russian)
69. ———, *Methods of the theory of functions of a complex variable*, 2nd ed., GITTL, Moscow, 1958; German transl. of 3rd (1965) ed., VEB Deutscher Verlag Wiss., Berlin, 1967.
70. Jean Leray, *Les problèmes de représentation conforme de Helmholtz; théories des sillages et des proues*, Comment. Math. Helv. **8** (1935/36), 149–185, 250–263.
71. Jean Leray and Jules Schauder, *Topologie et équations fonctionnelles*, Ann. Sci. École Norm. Sup. (3) **51** (1934), 45–78.
72. Jean Leray and Alexandre Weinstein, *Sur un problème de représentation conforme posé par la théorie de Helmholtz*, C. R. Acad. Sci. Paris **198** (1934), 430–432.
73. Li Chung [Li Zhong], *On the existence of homeomorphic solutions of a system of quasilinear partial differential equations of elliptic type*, Acta Math. Sinica **13** (1963), 454–461; English transl. in Chinese Math. Acta **4** (1963).
74. M. J. Lighthill, *A new method of two-dimensional aerodynamic design*, Ministry of Supply [London], Aeronaut. Res. Council, Reports and Memoranda, No. 2112 (1945); reprinted in *Aeronaut. Res. Council Tech. Rep. for the Year 1945*, Vol. I (65), HMSO, London, 1955, pp. 105–157.
75. E. B. McLeod, Jr., *The explicit solution of a free boundary problem involving surface tension*, J. Rational Mech. Anal. **4** (1955), 557–567.
76. I. M. Mel'nik, *An exceptional case in the Riemann boundary value problem*, Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze **24** (1957), 149–162. (Russian)
77. L. G. Mikhaïlov, *A new class of singular integral equations and its application to differential equations with singular coefficients*, Akad. Nauk Todzhik. SSR, Dushanbe, 1963; English transl., Noordhoff, 1970.
78. N. N. Moiseev and A. M. Ter-Krikorov, *Study of the motion of a heavy fluid at speeds close to critical*, Mosk. Fiz.-Tekhn. Inst. Trudy Vyp. 3 (1959), 25–59. (Russian)
79. V. N. Monakhov, *A boundary value problem in function theory*, Izv. Vyssh. Uchebn. Zaved. Matematika **1960**, no. 1(14), 154–165; erratum, *ibid.*, no. 4(17), 218. (Russian)
80. ———, *On a boundary value problem in function theory*, Trudy Kazan. Aviats. Inst. Vyp. 61 (1960), 13–21. (Russian)
81. ———, *On the inverse mixed boundary value problem*, Studies in Contemporary Problems of the Theory of Functions of a Complex Variable, Fizmatgiz, Moscow, 1961, pp. 375–380. (Russian)
82. ———, *On special cases in the inverse mixed boundary value problem*, Trudy Kazan. Aviats. Inst. Vyp. 64 (1961), 25–45. (Russian)
83. ———, *Inverse mixed boundary value problem for several unknown arcs*, Dokl. Akad. Nauk SSSR **141** (1961), 800–802; English transl. in Soviet Math. Dokl. **2** (1961).
84. ———, *Solvability questions of inverse mixed boundary value problems for analytic functions*, Functional Anal. and Theory of Functions, No. 1, Izdat. Kazan. Univ., Kazan, 1963, pp. 56–71. (Russian)
85. ———, *On uniqueness theorems in hydrodynamics problems with free boundaries*, Trudy Sem. Obratn. Kraev. Zadacham **1** (1964), 81–87. (Russian)
86. ———, *On theorems on existence of solutions in hydrodynamic problems with free boundaries*, Trudy Sem. Obratn. Kraev. Zadacham **2** (1964), 142–152. (Russian)
87. ———, *Unique solvability of two-dimensional problems of gas dynamics with free boundaries*, Dokl. Akad. Nauk SSSR **164** (1965), 982–984; English transl. in Soviet Math. Dokl. **6** (1965).

88. \_\_\_\_\_, *On boundary value problems with free boundaries for elliptic systems of equations*, Trudy Sem. Obratn. Kraev. Zadacham **1** (1964), 88–92. (Russian)
89. \_\_\_\_\_, *Some properties of solutions of nonlinear systems of equations, which are elliptic in the sense of Lavrent'ev*, Dinamika Sploshnoi Sredy Vyp. 15 (1974), 89–103. (Russian)
90. \_\_\_\_\_, *On a class of boundary value problems with free boundaries for elliptic systems*, Trudy Sem. Obratn. Kraev. Zadacham **1** (1964), 93–95. (Russian)
91. \_\_\_\_\_, *Solvability of filtration problems with free boundaries*, Dokl. Akad. Nauk SSSR **156** (1964), 1320–1322; English transl. in Soviet Phys. Dokl. **9** (1964/65).
92. \_\_\_\_\_, *On some plane problems of elasticity theory with unknown boundaries*, Dinamika Sploshnoi Sredy Vyp. 1 (1969), 242–257. (Russian)
93. \_\_\_\_\_, *Boundary value problems for subsonic gas dynamics with free boundaries*, Mechanics of Continuous Media (Materials Internat. Conf., 1966), Bolgar. Akad. Nauk, Sofia, 1968, pp. 47–61. (Russian; English summary)
94. \_\_\_\_\_, *Solvability and the principle of topological similarity for free boundary problems in gas dynamics and filtration theory*, Fluid Dynamics Trans., Vol. 4 (Proc. Eighth Sympos., Tarda, 1967), PWN, Warsaw, 1969, pp. 91–104.
95. \_\_\_\_\_, *Boundary value problems with free boundaries for elliptic systems of equations*. Parts I, II, Novosibirsk. Gos. Univ., Novosibirsk, 1968, 1969. (Russian)
96. V. N. Monakhov and S. N. Antontsev, *On some problems with discontinuous boundaries in the theory of motion of a fluid or gas with free boundaries*, Mechanics of Continuous Media (Materials Internat. Conf., 1966), Bolgar. Akad. Nauk, Sofia, 1968, pp. 63–73. (Russian; English summary)
97. N. I. Muskhelishvili, *Singular integral equations*, 2nd ed., Fizmatgiz, Moscow, 1962; English transl. of 1st ed., Noordhoff, 1953; reprinted, 1972.
98. Mitio Nagumo, *A theory of degree of mapping based on infinitesimal analysis*, Amer. J. Math. **73** (1951), 485–496.
99. I. P. Natanson, *Theory of functions of a real variable*, 2nd rev. ed., GITTL, Moscow, 1957; English transl., Vols. 1, 2, Ungar, New York, 1955, 1961.
100. Nguen Din Ci [Nguyen Dinh Thi], *On a problem with free boundary for a parabolic equation*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1966**, no. 2, 40–54. (Russian)
101. Kiyoshi Noshiro, *Cluster sets*, Springer-Verlag, 1960.
102. M. T. Nuzhin and N. B. Il'inskiĭ, *A method of constructing the underground contour of hydraulic structures*, Kazan. Gos. Univ., Kazan, 1963. (Russian)
103. O. A. Oleĭnik, *Mathematical problems of boundary layer theory*, Uspekhi Mat. Nauk **23** (1968), no. 3 (141), 3–65; English transl. in Russian Math. Surveys **23** (1968).
104. Seymour V. Parter, *On mappings of multiply connected domains by solutions of partial differential equations*, Comm. Pure Appl. Math. **13** (1960), 167–182.
105. P. I. Plotnikov, *Some problems on the flow of a turbulent fluid*, Dinamika Sploshnoi Sredy Vyp. 1 (1969), 124–133. (Russian)
106. P. Ja. Polubarinova-Kochina, *Theory of ground water movement*, GITTL, Moscow, 1952; English transl., Princeton Univ. Press, Princeton, N. J., 1962.
107. \_\_\_\_\_ (editor), *The development of research on filtration theory in the USSR (1917–1967)*, "Nauka", Moscow, 1969. (Russian)
108. G. Pólya and G. Szegő, *Isoperimetric inequalities in mathematical physics*, Princeton Univ. Press, Princeton, N. J., 1951.
109. I. I. Privalov, *Boundary properties of analytic functions*, 2nd ed., GITTL, Moscow, 1950; German transl., VEB Deutscher Verlag Wiss., Berlin, 1956.
110. G. N. Pykhteev, *Solution of the inverse problem for a plane cavitation flow past a curvilinear arc*, Prikl. Mat. Mekh. **20** (1956), 373–381. (Russian)
111. D. Riabouchinsky [Ryabushinskiĭ], *Sur la détermination d'une surface d'après les données qu'elle porte*, C. R. Acad. Sci. Paris **189** (1929), 629–632.

112. V. S. Rogozhin, *On the uniqueness of the solution of the exterior inverse boundary value problem*, Uchen. Zap. Kazan. Gos. Univ. **117** (1957), no. 2, 38–41. (Russian)
113. A. B. Shabat, *On a scheme for the plane motion of a fluid when there is a trench on the bottom*, Ž. Prikl. Mekh. Tekhn. Fiz. **1962**, no. 4, 68–80. (Russian)
114. ———, *Two problems in splicing solutions of Dirichlet's problem*, Dokl. Akad. Nauk SSSR **150** (1963), 1242–1245; English transl. in Soviet Phys. Dokl. **8** (1963/64).
115. B. V. Shabat, *Mappings effected by solutions of nonlinear systems of partial differential equations*, Studies in Contemporary Problems of the Theory of Functions of a Complex Variable, Fizmatgiz, Moscow, 1960, pp. 451–461. (Russian)
116. Z. Schapiro [Z. Va. Shapiro], *Sur l'existence des représentations quasi-conformes*, C. R. (Dokl.) Acad. Sci. URSS **30** (1941), 690–692.
117. L. I. Sedov, *Two-dimensional problems in hydrodynamics and aerodynamics*, GITTL, Moscow, 1950; English transl., Interscience, 1965.
118. Ya. I. Sekerzh-Zen'kovich, *Sur la théorie des sillages*, Trudy Tsentral. Aëro-gidrodinam. Inst. Vyp. 299 (1937). (Russian; French summary)
119. James Serrin, *On plane and axially symmetric free boundary problems*, J. Rational Mech. Anal. **2** (1953), 563–575.
120. Max Shiffman, *On the existence of subsonic flows of a compressible fluid*, J. Rational Mech. Anal. **1** (1952), 605–652.
121. I. B. Simonenko, *The Riemann boundary value problem for  $n$  pairs of functions with measurable coefficients and its application to the study of singular integrals in  $L_p$  spaces with weights*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 277–306. (Russian)
122. S. L. Sobolev, *Applications of functional analysis in mathematical physics*, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; English transl., Transl. Math. Mono., vol. 7, Amer. Math. Soc., Providence, R. I., 1963.
123. V. V. Sokolovskii, *On nonlinear filtration of ground water*, Prikl. Mat. Mekh. **13** (1949), 525–536.
124. ———, *Some shapes of uniformly strong masses*, Inzh. Zh. Mekh. Tverd. Tela **1968**, no. 2, 44–51; English transl. in Mech. Solids **3** (1968) (1972).
125. S. Stoilow, *Leçons sur les principes topologiques de la théorie des fonctions analytiques*, 2nd ed., Gauthier-Villars, Paris, 1956.
126. G. G. Tumashev and M. T. Nuzhin, *Inverse boundary value problems and their applications*, 2nd ed., Izdat. Kazan. Univ., Kazan, 1965. (Russian)
127. O. M. Turovskii, *A case of the inverse mixed boundary value problem*, Dokl. Akad. Nauk SSSR **168** (1966), 292–295; English transl. in Soviet Math. Dokl. **7** (1966).
128. I. N. Vekua, *Generalized analytic functions*, Fizmatgiz, Moscow, 1959; English transl., Pergamon Press, Oxford, and Addison-Wesley, Reading, Mass., 1962.
129. N. P. Vekua, *Systems of singular integral equations and some boundary value problems*, GITTL, Moscow, 1950; English transl., Noordhoff, 1967.
130. V. S. Vinogradov, *On a boundary value problem for linear elliptic systems of first order in the plane*, Dokl. Akad. Nauk SSSR **118** (1958), 1059–1062. (Russian)
131. ———, *On some boundary value problems for quasilinear elliptic systems of first order on the plane*, Dokl. Akad. Nauk SSSR **121** (1958), 579–581. (Russian)
132. L. I. Volkovskii, *Quasiconformal mappings*, Izdat. L'vov. Gos. Univ., L'vov, 1954. (Russian)
133. Stefan Warschawski, *Über das Randverhalten der Ableitung der Abbildungsfunktion bei konformer Abbildung*, Math. Z. **35** (1932), 322–456.
134. Alexander Weinstein, *Non-linear problems in the theory of fluid motion with free boundaries*, Proc. Sympos. Appl. Math., Vol. 1, Amer. Math. Soc., Providence, R. I., 1949, pp. 1–18.
135. L. C. Woods, *Compressible subsonic flow in two-dimensional channels with mixed boundary conditions*, Quart J. Mech. Appl. Math. **7** (1954), 263–282.
136. A. Zygmund, *Trigonometric series*, 2nd rev. ed., Vols. I, II, Cambridge Univ. Press, 1959.

ISBN 978-0-8218-4510-3



9 780821 845103

MMONO/57