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Volume 59

Integral Representation and the Computation of Combinatorial Sums

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American Mathematical Society

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Providence, Rhode Island

Г. П. ЕГОРЫЧЕВ

ИНТЕГРАЛЬНОЕ ПРЕДСТАВЛЕНИЕ И ВЫЧИСЛЕНИЕ КОМБИНАТОРНЫХ СУММ

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ABSTRACT. This book contains investigations on the problem of finding integral representations for and computing finite and infinite sums (generating functions) arising in practice in combinatorial analysis, the theory of algorithms and programming on a computer, probability theory, group theory, function theory, and so on, as well as in physics and other areas of knowledge. A general approach is presented for computing sums (expressions) in closed form by reducing them to one-dimensional and multiple integrals, most often contour integrals. The monograph may be useful to specialists in discrete and continuous mathematics, physicists, engineers, and others interested in computing sums and applying complex analysis in discrete mathematics.

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Dedicated to
Nikolaĭ Ivanovich Slobodchikov,
my school teacher

Foreword

The problem of finding an integral representation for and computing finite and infinite sums (generating functions) arises in the most widely disparate areas of mathematics: combinatorial analysis, graph theory, the theory of algorithms and programming on a computer, probability theory, function theory, and group theory, as well as in physics and other areas of knowledge. The present book contains a systematization of known ideas and methods and an exposition of new ones concerning this problem.

Since the appearance of our book "Combinatorial sums and the method of generating functions" (Krasnoyarsk, 1974), we have received many useful comments from investigators in the USSR and abroad, the books of Gould (USA, 1972) and Kaucký (Czechoslovakia, 1975), both with the title "Combinatorial identities", have been published, and many new results have been obtained. All this has enabled us to better understand the essence of the proposed methods and has significantly enriched the factual material on combinatorial identities.

The formation of the author's views has been greatly influenced by his collaboration with the group of mathematicians led by L. A. Aĭzenberg (Kirenskiĭ Institute of Physics, Siberian Branch of the Academy of Sciences of the USSR, Function Theory Laboratory), as well as by the group led for a long time by Yu. M. Gorchakov (the Group Theory Laboratory of the same institute). This collaboration has contributed to the emergence of new applications of methods being developed in function theory and group theory.

To understand the contents of the book it is necessary to study thoroughly only Chapter 1. The material in the remaining chapters may be read selectively according to the interests and degree of preparation of the reader. Those facts from the theory of multidimensional residues needed to read the book are presented in the Supplement, written by A. P. Yuzhakov. §6.1 was written jointly with L. A. Aĭzenberg, part of §4.2 with B. I. Selivanov, and §6.4 with Yu. D. Ushakov.

The sections and formulas are numbered by chapters. For example, formula (5.20) is formula 20 in Chapter 5. Subsections, definitions, theorems, lemmas, and problems are numbered by sections according to the same scheme. Thus, "Theorem 4.5.2" means that this is the second theorem in §4.5.

The author expresses his deep gratitude to the editor-in-chief G. P. Gavrilov, and to A. P. Yuzhakov, B. I. Selivanov, G. I. Zolotukhina, and E. K. Leĭnartas for a useful discussion and critial comments. V. K. Ivanov and K. A. Rybnikov made valuable remarks in the process of familiarizing themselves with the manuscript, and the author sincerely thanks them for this.

The author is especially grateful to A. M. Aronov for help in putting the finishing touches on the text of the manuscript, and is also thankful to his students A. K. Avramënok, A. I. Karapetyan, A. V. Kurikov, N. I. Sereda, and E. M. Smirnova for help in preparing the manuscript.

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SUPPLEMENT

Facts from the Theory of Multidimensional Residues

A. P. YUZHAKOV

Introduction

It is well known what numerous applications the theory of residues of analytic functions of a single complex variables has. At its basis lie the Cauchy integral theorem, permitting us to replace the contour of integration in the integral of a holomorphic function by another simpler contour, and the residue formula, expressing the integral over a small contour about a singular point in terms of the coefficient of the corresponding Laurent series, and—in the case of poles—in terms of the derivatives.

The theory of multidimensional residues, which goes back to Poincaré [173], is based on the general Stokes formula and its corollary: the Cauchy-Poincaré integral theorem. Serious topological difficulties arise for analytic functions of several complex variables in connection with the fact that the role of a singular point is now played by analytic sets (surfaces) in $\mathbf{C}^n$, which have a complicated structure in the general case, and the apparatus of algebraic topology is needed to overcome these difficulties. Unlike in the case of functions of one variable, integrals over "elementary" cycles cannot always be computed completely even for poles. For example, the Leray residue formula [142] (see §2.4 below) only enables us to lower the order of the integration in the general case. Although the theory of multidimensional residues evolved in a profound and comprehensive way in work of Leray [142], who developed a general theory of residues on a complex analytic manifold, and in work of Dolbeault, Martinelli, and others, it is still far from complete. Multidimensional residues have in recent times found important applications in the investigation of Feynman integrals [16], as well as in combinatorial analysis (see the present book).

In §1 below we present auxiliary concepts and facts on which the theory of multidimensional residues is based. In §§2 and 3 we survey the main concepts and results in this theory and give practical devices for computing residues in some special cases useful in applications. In §4 we present the theory of the logarithmic residue in detail, and in §5 its application to a generalization of the Lagrange expansion to arbitrary implicit functions. The reader will find a more complete exposition of the material given here, along with a bibliography, in the book [240].

§1. Integration of differential forms. The Stokes formula

1.1. ***Differentiable and complex analytic manifolds.*** An *n-dimensional manifold* is defined to be a connected Hausdorff topological space X, each point of which has a neighborhood homeomorphic to a ball in the n-dimensional Euclidean space R^n. A pair (U_a, φ), where U_a is a neighborhood of a point $a \in X$ and φ: $U_a \to \{t \in R^n: |t| < 1\}$ is a homeomorphism, is called a *local coordinate system*, and the values $t = (t_1, \dots, t_n) = \varphi(x)$ are called *local coordinates* of a point $x \in U_\alpha$. Suppose that on X there is a family $\mathfrak{F} = \{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ of local coordinate systems satisfying the following conditions: 1) $\bigcup_{\alpha \in A} U_\alpha = X$; 2) if $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta) \in \mathfrak{F}$ and $U_\alpha \cap U_\beta \neq \varnothing$, then the mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is r times continuously differentiable; 3) if a coordinate system (U, φ) is connected with any coordinate system $(U_\alpha, \varphi_\alpha) \in \mathfrak{F}$ by condition 2), then $(U, \varphi) \in \mathfrak{F}$. Then the pair $(X, \mathfrak{F})$ is called a *differentiable manifold of class* $\mathfrak{S}^r$, $0 \leqslant r \leqslant \infty$ (it is assumed that $r = \infty$ in what follows). A differentiable manifold is said to be *orientable* if $\mathfrak{F}$ can be partitioned into two classes in such a way that if two pairs $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ are in a single class and $U_\alpha \cap U_\beta \neq \varnothing$, then the Jacobian of the mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is positive. These classes are called (opposite) *orientations* of X. If X is a $2n$-dimensional manifold, the mapping

$$\varphi_\alpha: U_\alpha \to \left\{ \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n \cdot |\zeta| = \left(|\zeta_1|^2 + \cdots + |\zeta_n|^2 \right)^{1/2} < 1 \right\}$$

in $(U_\alpha, \varphi_\alpha) \in \mathfrak{F}$ is a homeomorphism onto an open ball of the n-dimensional complex space $\mathbf{C}^n$, and $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a holomorphic mapping, then the pair $(X, \mathfrak{F})$ is called an n-dimensional *complex analytic manifold*. Obviously, an n-dimensional complex analytic manifold is at the same time a (real) $2n$-dimensional differential manifold (and orientable). Some examples of complex analytic manifolds are a domain in $\mathbf{C}^n$, an analytic set without singularities in $\mathbf{C}^n$ ($S = \{z \in \mathbf{C}^n: F_1(z) = \cdots = F_k(z) = 0\}$, where $F_1, \dots, F_k$ are holomorphic functions with $\operatorname{rank}\|\partial F_j/\partial z_v\| = k$), Riemannian manifolds of multivalued analytic functions of several complex variables, etc. (see [202], [5] and [234]).

1.2. ***Differential forms.*** Differential forms are objects of integration over multidimensional surfaces on a manifold and are invariant with respect to the choice of a local coordinate system. A *differential form* of degree p ($p = 0, 1, \dots$) on a differentiable manifold X is defined to be a skew-symmetric covariant tensor field of valence p (see [179]). In a neighborhood of an arbitrary point $a \in X$ a differential form ω can be uniquely represented in local coordinates $(x_1, \dots, x_n)$ by an expression

$$\omega(x) = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant n} a_{i_1 \cdots i_p}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \tag{1}$$

where the $a_{i_1 \cdots i_p}(x)$ are functions of $(x_1, \dots, x_n)$ (which are differentiable sufficiently many times), the dx_j ($j = 1, \dots, n$) are the differentials of the local coordinates, and $\wedge$ is the sign for exterior multiplication and satisfies the anticommutativity condition

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \qquad dx_i \wedge dx_i = 0. \tag{2}$$

In passing to other local coordinates $(y_1, \dots, y_n)$ the expression (1) transforms to

$$\begin{aligned} \omega &= \sum_{j_1, \dots, j_p = 1}^{n} \left(\sum_{i_1, \dots, i_p} a_{i_1 \cdots i_p}(x) \frac{\partial x_{i_1}}{\partial y_{j_1}} \cdots \frac{\partial x_{i_p}}{\partial y_{j_p}} \right) dy_{j_1} \wedge \cdots \wedge dy_{j_p} \\ &= \sum_{i \leqslant j_1 < \cdots < j_p \leqslant n} \left(\sum_{i_1, \dots, i_p} a_{i_1 \cdots i_p}(x) \frac{\partial(x_{i_1}, \dots, x_{i_p})}{\partial(y_{j_1}, \dots, y_{j_p})} \right) dy_{j_1} \wedge \cdots \wedge dy_{j_p}. \end{aligned} \tag{3}$$

A differential form of degree zero is understood to be an ordinary function ω: $X \to R^1, \mathbf{C}^1$. A differential form ω is said to be *regular* if its coefficients in any local representation of the form (1) are functions of class $\mathfrak{S}^\infty$ (differentiable any number of times). Differential forms can be added and multiplied by numbers. The set of regular forms of degree p on a manifold X thus forms a vector space, denoted by $\Omega^p(X)$.

The operation

$$\wedge : \Omega^p(X) \times \Omega^q(X) \to \Omega^{p+q}(X)$$

of *exterior multiplication* is defined for forms; it is realized in local coordinates by the rule for multiplying polynomials, with the rule (2) taken into account. The following properties of the exterior product are obvious:

1) $\omega \wedge (\varphi + \psi) = \omega \wedge \varphi + \omega \wedge \psi$.

2) $\omega \wedge \varphi = (-1)^{p \cdot q} \varphi \wedge \omega$, where p and q are the degrees of the forms ω and φ.

The *exterior differentiation* operator

$$d\colon \Omega^p(X) \to \Omega^{p+1}(X)$$

is defined in local coordinates by the formula

$$dw = \sum_{i_1,\dots,i_p} da_{i_1\cdots i_p}(x) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \tag{4}$$

where

$$da_{i_1\cdots i_p} = \sum_{i=1}^{n} \frac{\partial a_{i_1\cdots i_p}}{\partial x_i} dx_i.$$

The exterior differential has the following properties:

1) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.

2) $d(\omega \wedge \varphi) = d\omega \wedge \varphi(-1)^p\omega \wedge d\varphi$, where p is the degree of the form ω.

3) $d\,d\omega = 0$.

A differential form ω is said to be *closed* if $d\omega = 0$, and *exact* if $\omega = d\varphi$ for some form φ. The set of closed forms of degree p is denoted by $Z^p(X)$, and the set of exact forms of degree p by $B^p(X)$. By properties 1) and 3) of the exterior differential, $Z^p(X)$ is a subgroup of the group $\Omega^p(X)$, and $B^p(X)$ is a subgroup of $Z^p(X)$. The factor group $Z^p(X)/B^p(X) = H^p(X)$ is called the *p-dimensional cohomology group* of X, and its elements are called *cohomology classes*. Two closed forms ω_1 and ω_2 belonging to the same cohomology class are said to be *cohomologous* ($\omega_1 \approx \omega_2$). Their difference $\omega_1 - \omega_2$ is equal to $d\varphi$ for some form $\varphi \in \Omega^{p-1}(X)$. A mapping

$$f\colon X \to Y \tag{5}$$

of manifolds induces homomorphisms

$$\tilde{f}\colon \Omega^p(Y) \to \Omega^p(X), \qquad f^*\colon H^p(Y) \to H^p(X),$$

defined as follows. If $(x_1,\dots,x_n)$ and $(y_1,\dots,y_m)$ are local coordinates in the neighborhoods U_a and V_b of the points $a \in X$, $b = f(a) \in Y$ and the form $\omega \in \Omega^p(Y)$ is represented in V_b by the expression

$$\omega = \sum_{i_1,\dots,i_p} a_{i_1\cdots i_p}(y)\, dy_{i_1} \wedge \cdots \wedge dy_{i_p},$$

then in U_a

$$\tilde{f}(\omega) = \omega(y(x)) = \sum_{i_1,\dots,i_p} a_{i_1\cdots i_p}(y(x))\, dy_{i_1}(x) \wedge \cdots \wedge dy_{i_p}(x).$$

(It is assumed that (5) is a mapping of class $\mathfrak{S}^\infty$, i.e., local coordinates $y_1,\dots,y_m$ are infinitely differentiable functions of the local coordinates $x_1,\dots,x_n$.) It can be shown that $d \circ \tilde{f} = \tilde{f} \circ d$. Consequently, $\tilde{f}$ maps $Z^p(Y)$ into $Z^p(X)$ and $B^p(Y)$ into $B^p(X)$ and, thus, determines the homomorphism f^*.

In the case of a complex analytic manifold we usually consider forms with complex coefficients, and in place of the differentials dx_j and dy_j, $j = 1,\dots,n$, we take the linear combinations $dz_j = dx_j + i\,dy_j$ and $d\bar{z}_j = dx_j - i\,dy_j$ ($z_j = x_j + iy_j$) of them. A form ω is said to be *holomorphic* if it does not contain terms with $d\bar{z}_j$, $j = 1, \ldots, n$, and its coefficients are holomorphic functions of $z_1, \ldots, z_n$.

1.3. *Singular chains. Homology.* Let us consider integration of differential forms over multidimensional oriented surfaces. It is most convenient to represent the surface of integration in parametric form, with the orientation given by the order of the parameters. To be able to apply algebraic topology we assume a partition of the surface into parts: singular simplexes or cubes. A *smooth singular simplex* of dimension p on a differentiable manifold X is defined to be a pair $\sigma_p = (\Delta_p, g)$, where Δ_p is a rectilinear simplex in R^p, say $\Delta_p = \{t \in R^p\colon t_j \geqslant 0; \Sigma_1^p t_j \leqslant 1\}$, and $g\colon \Delta_p \to X$ is a continuously differentiable mapping. We write $(\Delta_p, g) \simeq (\Delta'_p, g')$ if there exists a diffeomorphism (a differentiable homeomorphism) $\varphi\colon \Delta_p \to \Delta'_p$ with positive Jacobian such that $g' \circ \varphi = g$. Thus, σ_p is determined to within a parametrization. The simplex σ_p is assigned an orientation determined by the order of the coordinates $t_1, \ldots, t_p$ in R^p. If other coordinates $\tau_1, \ldots, \tau_p$ are chosen in R^p, then they determine the same orientation if $\partial(t)/\partial(\tau) = \partial(t_1, \ldots, t_p)/\partial(\tau_1, \ldots, \tau_p) > 0$, and the opposite orientation if $\partial(t)/\partial(\tau) < 0$. The simplex σ_p, with the opposite orientation, is denoted by $-\sigma_p$. A finite linear combination

$$c_p = \sum_i m_i \sigma_p^{(i)} \tag{6}$$

of oriented singular simplexes, where the m_i are integers, is called a *singular chain* or simply a *chain*. Chains can be added and multiplied by integers termwise. The set of singular chains thus forms an abelian group $C_p(X)$.

The set $|\sigma_p| = g(\Delta_p)$ is called the *support* of the simplex $\sigma_p = (\Delta_p, g)$, and the set $|c_p| = \bigcup |\sigma_p^{(i)}|$ is the support of the chain (6). If $\Delta_{p-1}^{(i)}$ is a $(p-1)$-dimensional face of the simplex Δ_p, $i = 0, 1, \ldots, p$, then the $(p-1)$-dimensional singular simplex $\sigma_{p-1}^{(i)} = (\Delta_{p-1}^{(i)}, g|\Delta_{p-1}^{(i)})$ is called a *face* of the simplex σ_p. We choose a coordinate system $(t_1, \ldots, t_p) = t$ in R^p that determines the orientation of σ_p so that the face $\Delta_{p-1}^{(i)}$ lies in the plane $t_1 = 0$, and $t_1 \leqslant 0$ at points $t \in \Delta_p$. Then the parameters $t_2, \ldots, t_p$ determine an orientation of $\sigma_{p-1}^{(i)}$ that is coherent with the orientation of σ_p. The *boundary of the simplex* σ_p is defined to be the sum of its coherently oriented faces:

$$\partial\sigma_p = \sum_{i=1}^{p} \sigma_{p-1}^{(i)}.$$

The *boundary of the chain* (6) is defined by the formula

$$\partial c_p = \sum_i m_i \partial\sigma_p^{(i)}.$$

We have the property

$$\partial\partial c_p = 0. \tag{7}$$

A chain $\gamma \in C_p(X)$ is called a *cycle* if $\partial\gamma = 0$. The relation (7) means that the boundary of a chain is a cycle. Let

$$Z_p(X) = \{\gamma \in C_p(X) : \partial\gamma = 0\}, \qquad B_p(X) = \{\gamma = \partial c,\, c \in C_{p+1}(X)\}.$$

The factor group

$$H_p(X) = Z_p(X)/B_p(X)$$

is called the *p-dimensional homology group* of X. Two cycles γ_1 and γ_2 belonging to the same element (homology class) of the group $H_p(X)$ are said to be *homologous* ($\gamma_1 \sim \gamma_2$). If $\gamma \in B_p(X)$, then $\gamma \sim 0$. A cycle γ is said to be *weakly homologous to zero* ($\gamma \approx 0$) if $k\gamma \sim 0$ for some integer k. The factor group of $Z_p(X)$ by the subgroup of cycles weakly homologous to zero is called a *weak homology group*. In the cases of interest to us it coincides with $H_p(X)$, and we denote it also by $H_p(X)$. If $H_p(X)$ is a finitely generated group, then its dimension (the number of independent generators) is called the *p-dimensional Betti number* of the manifold X. A system $\{\gamma_j\} \subset Z_p(X)$ is called a *p-dimensional homology basis* if each cycle $\gamma \in Z_p(X)$ can be uniquely represented in the form $\gamma \approx \sum_j m_j\gamma_j$, where the m_j are integers. The mapping (5) of manifolds induces a homomorphism $\tilde{f}$: $C_p(X) \to C_p(Y)$ defined by specifying it on singular simplexes $\sigma_p = (\Delta_p, g)$ in X as follows: $\tilde{f}(\sigma_p) = (\Delta_p, f \circ g)$. Obviously, $\partial \circ \tilde{f} = \tilde{f} \circ \partial$. Thus, (5) induces a homomorphism f_*: $H_p(X) \to H_p(Y)$ of the homology groups.

Other definitions of homology groups, definitions equivalent to those presented here in the case of manifolds, along with methods for computing them, can be found in the books [216], [230], and [80].

1.4. ***Integration of differential forms over chains.*** The *integral of a form* $\omega \in \Omega^p(X)$ over an oriented singular simplex $\sigma_p = (\Delta_p, g)$ is defined by

$$\int_{\sigma_p} \omega = \int_{\Delta_p} \tilde{g}(\omega) = \int_{|\Delta_p|} A(t)\, dt_1 \cdots dt_p, \tag{8}$$

where $\tilde{g}(\omega) = A(t)\, dt_1 \wedge \cdots \wedge dt_p$ is a form on $\Delta_p \subset R^p$; if the form ω has the expression (1), then

$$A(t) = \sum_{i_1,\ldots,i_p} a_{i_1\cdots i_p}(x(t)) \cdot \frac{\partial(x_{i_1},\ldots,x_{i_p})}{\partial(t_1,\ldots,t_p)}.$$

The last integral in (8) is understood as an ordinary p-fold integral in R^p. The *integral of a form over the chain* (6) is defined by

$$\int_{c_p} \omega = \sum_i m_i \int_{\sigma_p^{(i)}} \omega. \tag{9}$$

REMARK 1. We often parametrize a chain (surface of integration) in the large, without partitioning it into simplexes. In this case the integral is computed by a formula like (8), and then (9) becomes a consequence of the additive property of a multiple integral.

THEOREM 1 (change of variables formula). *If $f\colon X \to Y$ is a mapping of manifolds, and $\gamma \in C_p(X)$ and $\omega \in \Omega^p(Y)$, then*

$$\int_\gamma \tilde{f}(\omega) = \int_{\underset{\sim}{f}(\gamma)} \omega. \tag{10}$$

In view of (9), it suffices to prove this for the case when $\gamma = \sigma_p = (\Delta_p, g)$ is a simplex. Then

$$\int_{\sigma_p} \tilde{f}(\omega) = \int_{\sigma_p} \omega(f(x)) = \int_{\Delta_p} \tilde{g}(\omega(f(x))) = \int_{\Delta_p} \omega(f(g(t)))$$

and

$$\int_{\underset{\sim}{f}(\sigma_p)} \omega = \int_{(\Delta_p, f\circ g)} \omega = \int_{\Delta_p} (\widetilde{f\circ\omega})(\omega) = \int_{\Delta_p} \omega(f(g(t))).$$

COROLLARY 2. *If $f\colon X \to Y$ is a homeomorphism, then*

$$\int_\gamma \omega = \int_{\underset{\sim}{f}(\gamma)} \widetilde{f^{-1}}(\omega).$$

REMARK 2. Theorem 1 enables us to make a change of variables in $\int\omega$ when f is not a homeomorphism, but either $\gamma = \underset{\sim}{f}(\gamma')$ or $\omega = \tilde{f}(\omega')$.

1.5. *The general Stokes formula. Integration by parts.*

THEOREM 3. *If $\omega \in \Omega^{p-1}(X)$ and $\gamma \in C_p(X)$, then Stokes' formula holds:*

$$\int_{\partial\gamma} \omega = \int_\gamma d\omega. \tag{11}$$

COROLLARY 4. *If $\gamma \in Z_p(X)$ and $\omega \in B^p(X)$, then*

$$\int_\gamma \omega = 0,$$

i.e., the integral of an exact form over a cycle is equal to zero.

Indeed, $\omega = d\varphi$ for some $\varphi \in \Omega^{p-1}(X)$ and $d\gamma = 0$, so, by Stokes' formula,

$$\int_\gamma \omega = \int_\gamma d\varphi = \int_{\partial\gamma} \varphi = 0.$$

COROLLARY 5. *The integral of a closed form over a cycle weakly homologous to zero is equal to zero.*

Indeed, if $d\omega = 0$ and $k\gamma = \partial c$, where $c \in C_{p+1}(X)$, then

$$\int_\gamma \omega = \frac{1}{k}\int_{k,\gamma} \omega = \frac{1}{k}\int_{\partial c} \omega = \frac{1}{k}\int_c d\omega = 0.$$

COROLLARY 6. *If* $\gamma \approx \gamma_1$ *(cycles) and* $\omega = \omega_1 + d\varphi$ *(forms), then*

$$\int_\gamma \omega = \int_{\gamma_1} \omega_1.$$

Thus, the integral of a closed form ω over a cycle γ depends only on the weak homology class $\{\gamma\} \in H_p(X)$ to which γ belongs and the cohomology class $\{\omega\} \in H^p(X)$ containing ω. It is therefore possible to define the integral of a cohomology class $\{\omega\} \in H^p(X)$ over a homology class $\{\gamma\} \in H^p(X)$ by the equality $\int_{\{\gamma\}}\{\omega\} = \int_\gamma \omega$.

As a special case of Corollary 4 we have

COROLLARY 7. *If* ω *is a holomorphic form of degree* n *on an* n*-dimensional complex analytic manifold* X, *and* σ *is an* $(n + 1)$*-dimensional chain in* X, *then* $\int_{\partial\sigma} \omega = 0$.

It suffices to show that ω is a closed form. In local coordinates $z = (z_1, \ldots, z_n)$ the form ω can be written as

$$f(z)\,dz = f(z_1, \ldots, z_n)\,dz_1 \wedge \cdots \wedge dz_n,$$

where f is a holomorphic function ($\partial f/\partial \bar{z}_j = 0$ for $j = 1, \ldots, n$; see [202] and [5]). Then

$$d\omega = df \wedge dz_1 \wedge \cdots \wedge dz_n = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j \wedge dz_1 \wedge \cdots \wedge dz_n = 0,$$

because $dz_j \wedge dz_j = 0$.

For $X = D \subset \mathbf{C}^n$ we get

THEOREM 8 (Cauchy-Poincaré). *Suppose that the function* $f(z) = f(z_1, \ldots, z_n)$ *is holomorphic in a domain* $D \subset \mathbf{C}^n$. *Then for any* $(n + 1)$*-dimensional chain* σ *in* D

$$\int_{\partial\sigma} f(z)\,dz = \int_{\partial\sigma} f(z_1, \ldots, z_n)\,dz_1 \wedge \cdots \wedge dz_n = 0.$$

The following formulas for integration by parts are a consequence of Stokes' formula and the property 2 of the exterior product.

PROPOSITION 9. *If* $\varphi \in \Omega^p(X)$, $\psi \in \Omega^q(X)$, *and* $\gamma \in C_{p+q+1}(X)$, *then*

$$\int_\gamma d\varphi \wedge \psi = \int_{\partial\gamma} \varphi \wedge \psi - (-1)^p \int_\gamma \varphi \wedge d\psi.$$

If $\gamma \in Z_{p+q+1}(X)$, *then*

$$\int_\gamma d\varphi \wedge \psi = (-1)^{p-1} \int_\gamma \varphi \wedge d\psi.$$

PROPOSITION 10. *If* $f(z)$ *and* $\varphi(z)$ *are holomorphic functions in a domain* $D \subset \mathbf{C}^n$ *and* $\gamma \in Z_n(D)$, *then*

$$\int_\gamma \frac{\partial f}{\partial z_j} \cdot \varphi \cdot dz = -\int_\gamma f \cdot \frac{\partial \varphi}{\partial z_j} dz, \tag{12}$$

$$\int_\gamma \frac{\partial^{m_1+\cdots+m_n} f}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} \cdot \varphi \cdot dz = (-1)^{m_1+\cdots+m_n} \int_\gamma f \cdot \frac{\partial^{m_1+\cdots+m_n} \varphi}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} dz. \tag{13}$$

Equality (12) follows from Stokes' formula and the equality

$$d(f \cdot \varphi \cdot dz_1 \wedge \cdots [j] \cdots \wedge dz_n) = \left(\frac{\partial f}{\partial z_j} \varphi + f \cdot \frac{\partial \varphi}{\partial z_j} \right) dz,$$

where $[j]$ means that dz_j is omitted in $dz_1 \wedge \cdots \wedge dz_n$. The equality (13) follows from (12).

§2. Elements of the general theory of multidimensional residues

2.1. ***The subject of the theory.*** In a narrow sense, the main problem of the theory of multidimensional residues is the study and computation of integrals of holomorphic functions of n complex variables (1) over n-dimensional cycles (closed n-dimensional surfaces) in the domain of regularity of these functions. In a broad sense, it is the computation of integrals of closed differential forms over cycles on a complex analytic manifold. It is common to consider integrals of forms having singularities on an analytic set of codimension 1 (such a set can be given by an equation $g(z) = 0$, with g holomorphic, in a neighborhood of each of its points).

The corollaries of Stokes' theorem permit us to replace an integral of a closed form over a cycle by an integral of a simpler form cohomologous to the

(1) More precisely, of holomorphic forms of degree n.

given one over a simpler cycle homologous to the initial one. From Corollaries 5 and 6 of §1 we have

PROPOSITION 1. *Let* $\{\gamma_j\}$ *be a basis for the p-dimensional homology of a manifold X, and let* ω *be a closed form of degree p on X. Then for any cycle* $\gamma \in Z_p(X)$

$$\int_\gamma \omega = \sum_j k_j \int_{\gamma_j} \omega, \tag{14}$$

where the k_j *are the expansion coefficients of* γ *in the basis* $\{\gamma_j\}$: $\gamma \approx \sum_j k_j \gamma_j$.

The problem of computing the integral of a closed form over a cycle thus reduces to: 1) a study of the homology group of the manifold X (determination of its dimension and construction of a homology basis); 2) computation of the integrals over the basis cycles; 3) determination of the expansion coefficients of the cycle of integration in the basis. Unlike in the one-dimensional case, the solution of these problems for the multidimensional case involves considerable difficulties. Below we present some methods and results enabling us in many cases to solve these problems completely or in part.

2.2. ***Application of Alexander-Pontryagin duality.***(2) We give some facts from algebraic topology (see [152] and [6]). Let X be an n-dimensional orientable manifold with fixed orientation determined by the order of the local coordinates $(x_1, \dots, x_n)$, and let $v_r = (\Delta_r, \varphi)$ and $u_q = (\Delta_q, \psi)$, $r + q = n$, be singular simplexes in X having a common point $x^{(0)}$ "interior" to both simplexes and being "in general position" at this point. Then the *intersection index* of the simplexes v_r and u_q is defined by

$$\chi(v_r, u_q) = \operatorname{sgn} \left. \frac{\partial(x_1, \dots, x_n)}{\partial(t_1, \dots, t_r, \tau_1, \dots, \tau_q)} \right|_{x^{(0)}}, \tag{15}$$

where $(x_1, \dots, x_n)$ are local coordinates in a neighborhood of the point $x^{(0)} \in X$, and $(t_1, \dots, t_r)$ and $(\tau_1, \dots, \tau_q)$ are coordinates in $R^r \supset \Delta_r$ and $R^q \supset \Delta_q$ determining the orientation of the simplexes v_r and u_q (to say that v_r and u_q are "in general position" at $x^{(0)}$ means that the Jacobian in (15) is nonzero). If $|v_r| \cap |u_q| = \varnothing$, then we set $\chi(v_r, u_q) = 0$. The *intersection index* of two chains $c_r = \sum_i n_i v_r^{(i)}$ and $l_q = \sum_j m_j u_q^{(j)}$ is defined by

$$\chi(c_r, l_q) = \sum_{i,j} n_i m_j \chi\left(v_r^{(i)}, u_q^{(j)}\right).$$

(2) The idea of applying Alexander-Pontryagin duality is due to Martinelli, who used it in generalizing the Cauchy integral formula to holomorphic functions of n complex variables and to $(n + l)$-fold integrals, where $0 \leqslant l \leqslant n - 1$ [152]. On the initiative of V. K. Ivanov this idea was developed by the author for multiple residues in [237], [238], and elsewhere.

The *linking coefficient* of two cycles $\sigma_{r-1} \in B_{r-1}(X)$ and $\gamma_q \in B_q(X)$ such that $r + q = n$ and $|\sigma_{r-1}| \cap |\gamma_q| = \varnothing$ is defined by

$$\mathfrak{V}(\sigma_{r-1}, \gamma_q) = \chi(c_r, \gamma_q) = (-1)^r \chi(\sigma_{r-1}, l_{q+1}),$$

where $\partial c_r = \sigma_{r-1}$ and $\partial l_{q+1} = \gamma_q$.

The linking coefficient does not depend on the choice of the chains c_r and l_{q+1}. We note the following properties of the linking coefficient:

1) $\mathfrak{V}(\sigma_{r-1}, \gamma_q) = (-1)^{q(r-1)-1}\mathfrak{V}(\gamma_q, \sigma_{r-1})$.

2) $\mathfrak{V}(m_1\sigma_1 + m_2\sigma_2, \gamma) = m_1\mathfrak{V}(\sigma_1, \gamma) + m_2\mathfrak{V}(\sigma_2, \gamma)$.

3) If $\sigma \approx 0$ in $X \setminus |\gamma|$, then $\mathfrak{V}(\sigma, \gamma) = 0$.

THEOREM 2 (Alexander-Pontryagin duality). *Let S^n be an n-dimensional spherical manifold (a space homeomorphic to a sphere) and let T be a compact polyhedron* (3) *in S^n. Then the weak homology groups $H_{r-1}(T)$ and $H_q(S^n \setminus T)$ are isomorphic for $r, q \geqslant 0$, $r + q = n$. Moreover, for every basis $\sigma_1, \ldots, \sigma_p$ of the $(r-1)$-dimensional homology of the polyhedron T there exists a corresponding dual basis $\gamma_1, \ldots, \gamma_p$ of the q-dimensional homology of the set $S^n \setminus T$ such that*

$$\mathfrak{V}(\sigma_j, \gamma_i) = \delta_{ji} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Let us apply Theorem 2 to the problem of computing the integrals (14). In Proposition 1 we set $X = D = \mathbf{C}^n \setminus T$, where T is the singular set of the form ω. Since the space $\mathbf{C}^n \simeq R^{2n}$ is homeomorphic to the $2n$-dimensional sphere with a "deleted" point, $\mathbf{C}^n$ can be regarded as imbedded in the spherical space $S^{2n} = \mathbf{C}^n \cup \{(\infty)\}$. The exterior of any ball in $\mathbf{C}^n$ is a neighborhood of the point (∞) in it. If $\overline{T} = T \cup \{(\infty)\}$ is a polyhedron in S^{2n}, then Theorem 2 can be applied to T and $S^{2n} \setminus \overline{T} = \mathbf{C}^n \setminus T$ (it is sometimes expedient to replace $\mathbf{C}^n$ by a domain $B \subset \mathbf{C}^n$ homeomorphic to a ball; then we can supplement B to form the spherical space S^{2n} by identifying the boundary ∂B with a single point). To find the q-dimensional Betti number of the domain D (the dimension of the group $H_q(D)$), it now suffices to find the dimension of the group $H_{r-1}(\overline{T})$, where $\overline{T} = T \cup \{(\infty)\}$, $r + q = 2n$. Instead of a q-dimensional basis $\{\gamma_j\}_1^p$ for the homology of D we can construct an $(r-1)$-dimensional basis $\{\sigma_j\}_1^p$ for the homology of the singular set $\overline{T}$. The expansion coefficients $\{k_j\}_1^p$ for the integration cycle $\gamma \in Z_q(D)$ in the basis $\{\gamma_j\}$ dual to $\{\sigma_j\}$ are found as the linking coefficients:

$$\mathfrak{V}(\sigma_j, \gamma) = \mathfrak{V}\left(\sigma_j, \sum_{i=1}^p k_i\gamma_i\right) = \sum_{i=1}^p k_i\mathfrak{V}(\sigma_j, \gamma_i) = \sum_{i=1}^p k_i\delta_{ji} = k_j.$$

(3) S^n admits a triangulation in which T is a subcomplex, i.e., consists of finitely many simplexes.

To find the integrals of the form ω over the basis cycles γ_j it is also not necessary to determine directly the basis $\{\gamma_j\}$ dual to $\{\sigma_j\}$. For this it suffices to choose p homologically independent q-dimensional cycles $\Gamma_1, \ldots, \Gamma_p$ in D over which the integrals of ω are as simple as possible. Then the integrals over the basis cycles can be found from the system of linear equations

$$\int_{\Gamma_j} \omega = \sum_{i=1}^{p} k_{ji} \int_{\gamma_i} \omega, \qquad j = 1, \ldots, p,$$

where $k_{ji} = \mathfrak{V}(\sigma_i, \Gamma_j)$, and $\det \|k_{ji}\| \neq 0$ is the condition for the cycles Γ_j ($j = 1, \ldots, p$) to be homologically independent.

Now let $q = r = n$ and $\omega = f(z)\, dz = f(z_1, \ldots, z_n)\, dz_1 \wedge \cdots \wedge dz_n$, where f is a holomorphic function in D. The foregoing arguments and Proposition 2 then gives us

THEOREM 3 (on residues). *Suppose that $f(z)$ is a holomorphic function in the domain $D = \mathbf{C}^n \setminus T$, and the set $\overline{T} = T \cup \{(\infty)\}$ is a polyhedron in the space $\mathbf{C}^n$, completed to form the sphere $S^{2n} = \mathbf{C}^n \cup \{(\infty)\}$ by a single point (∞) at infinity. If $\{\sigma_j\}_1^p$ is a basis for the $(n-1)$-dimensional homology of the singular set $\overline{T}$, and $\{\gamma_j\}_1^p$ is the corresponding dual basis for the n-dimensional homology of D, then for any cycle $\gamma \in Z_n(D)$*

$$\int_{\gamma} f(z)\, dz = (2\pi i)^n \sum_{j=1}^{p} k_j R_j,$$

where

$$k_j = \mathfrak{V}(\sigma_j, \gamma),$$

$$R_j = \frac{1}{(2\pi i)^n} \int_{\gamma_j} f(z)\, dz, \qquad j = 1, \ldots, p.$$

By analogy with the one-variable case, R_j can be called the *residue* of the function $f(z)$ with respect to the basis cycle γ_j.

Application of Alexander-Pontryagin duality is especially efficient when $n = 2$, because in this case the study of the two-dimensional homology group of a domain in R^4 reduces to the study of the one-dimensional homology group of a two-dimensional surface.

EXAMPLE 1. Let $f(w, z)$ be an entire function in $\mathbf{C}^2$, and m and n relatively prime integers. Then for any cycle $\gamma \in Z_2(\mathbf{C}^2 \setminus T)$, where $T = \{(w, z) \in \mathbf{C}^2: aw^m - bz^n = 0\}$, we have

$$\int_{\gamma} f(w, z)/(aw^m - bz^n)\, dw \wedge dz = 0.$$

Indeed, the set $\bar{T} = T \cup \{(\infty)\}$ is homeomorphic to the two-dimensional sphere S^2, and $H_1(S^2) \simeq 0$. Consequently, $H_2(\mathbf{C}^2 \setminus T) \simeq 0$. It remains to use Corollary 5 in §1.

EXAMPLE 2. Let $T = \{(w, z) \in \mathbf{C}^2: wz - 1 = 0\}$. Then the set $\bar{T} = T \cup \{(\infty)\}$ is homeomorphic to the two-dimensional sphere with two points identified (the points at $z = 0$ and $z = \infty$ are identified with the point (∞)). Consequently, the dimension of the groups $H_1(\bar{T}) \simeq H_2(\mathbf{C}^2 \setminus T)$ is equal to 1. As dual basis cycles we can take

$$\sigma = \{(w, z) \in \mathbf{C}^2: \operatorname{Im} z = \operatorname{Im} w = 0, \operatorname{Re} z > 0, wz = 1\} \in Z_1(\bar{T})$$

and

$$\gamma = \{|w| = |z| = 2\} \in Z_2(\mathbf{C}^2 \setminus T).$$

Obviously, $\mathfrak{B}(\sigma, \gamma) = \chi(c, \gamma) = 1$, where $c = \{(w, z): |w| \leqslant 2, |z| = 2\}$. The residue of the function $f(w, z)/(wz - 1)$, where f is an entire function, with respect to the cycle γ is

$$R = \frac{1}{(2\pi i)^2} \int_\gamma \frac{f(w, z)\, dw \wedge dz}{wz - 1} = \sum_{k=0}^{\infty} \frac{1}{(2\pi i)^2} \int_\gamma \frac{f\, dw \wedge dz}{(wz)^{k+1}}$$
$$= \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \frac{\partial^{2k} f(0,0)}{\partial w^k \partial z^k}.$$

See §3 below for further examples. Residues of arbitrary rational functions of two variables are considered in [237] and [239] with the use of Alexander-Pontryagin duality.

2.3. *Application of de Rham duality.* Instead of Alexander-Pontryagin duality it is sometimes convenient to use de Rham duality according to an analogous scheme.

THEOREM 4 (de Rham duality) [184], [142]. *For any homomorphism γ: $H_p(X) \to \mathbf{C}^1$, where X is a differentiable manifold and $\mathbf{C}^1$ is the group of complex numbers, there exist a unique element $h^* = \{w\} \in H^p(X)$ (here $H^p(X)$ is the cohomology group of complex-valued differential forms on X) such that $\int_h h^* = \int_\gamma \omega = \lambda(h)$ for any $h = \{\gamma\} \in H_p(X)$.*(4)

The de Rham theorem and Stokes' formula (11) give us the following important corollaries.

COROLLARY 5. *A cycle $\gamma \in Z_p(X)$ is weakly homologous to zero ($\gamma \approx 0$) if and only if $\int_\gamma \omega = 0$ for any form $\omega \in Z^p(X)$.*

(4) We observe that any closed form $\omega \in Z^p(X)$ (its cohomology class $\{\omega\} \in H^p(X)$) determines a homomorphism λ: $H_p(X) \to \mathbf{C}^1$ by the equality $\lambda(h) = \int_h \{\omega\}$.

COROLLARY 6. *A closed form $\omega \in Z_p(X)$ is exact if and only if $\int_\gamma \omega = 0$ for any cycle $\gamma \in Z_p(X)$.*

COROLLARY 7. *If the condition $\det\|a_{jk}\| \neq 0$ holds for the cycles $\gamma_j \in Z_p(X)$, $j = 1,\ldots, q$, and the closed forms $\omega_j \in Z^p(X)$, $j = 1,\ldots, q$, where $a_{jk} = \int_{\gamma_k} \omega_j$, and if any form $\omega \in Z^p(X)$ has a representation $\omega \approx \sum_1^q c_j\omega_j$ for some complex numbers $c_1,\ldots, c_q$, then the cycles $\gamma_1,\ldots, \gamma_q$ make up a basis for the p-dimensional homology, and the forms $\omega_1,\ldots, \omega_q$ make up a basis for the p-dimensional cohomology of the manifold X.*

The bases $\{\gamma_j\}_1^q$ and $\{\omega_j\}_1^q$ are said to be *dual in the de Rham sense* if $\int_{\gamma_j} \omega_k = \delta_{jk}$.

THEOREM 8 (on residues). *If $\gamma_1,\ldots, \gamma_q$ is a basis for the p-dimensional homology, and $\omega_1,\ldots, \omega_q$ is a basis for the p-dimensional cohomology of X that is dual to the former basis in the de Rham sense, then for any cycle $\gamma \in Z_p(X)$ and any closed form $\omega \in Z^p(X)$*

$$\int_\gamma \omega = \sum_{j=1}^{q} N_j R_j,$$

where $N_j = \int_\gamma \omega_j$ are the expansion coefficients of the cycle γ in the basis $\{\gamma_j\}$, i.e. $\gamma \approx \sum_j N_j\gamma_j$, and $R_j = \int_{\gamma_j} \omega$ is the residue (period) of the form ω with respect to the basis cycle γ_j.

In function theory, as a rule, we deal with domains of holomorphy in $\mathbf{C}^n$ and their generalizations—Stein manifolds (see [79], [202] and [5])—so it is useful to bear in mind the following theorem of Serre.

THEOREM 9 [79]. *If X is a Stein manifold (in particular, a domain of holomorphy in $\mathbf{C}^n$), then for any closed regular form $\omega \in Z^p(X)$ there exists a form cohomologous to it that is holomorphic in X.*

Thus, in the de Rham theorem and its corollaries the form $\omega \in h^* \in H^p(X)$ can be assumed and be holomorphic in the case of a Stein manifold.

COROLLARY 10. *If X is a Stein manifold of complex dimension n, then $H^p(X) \simeq H_p(X) \simeq 0$ for $p > n$.*

EXAMPLE 3. The form

$$\omega_1 = \frac{1}{(2\pi i)^n}\frac{dz}{z}, \qquad \text{where } \frac{dz}{z} = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n},$$

and the cycle $\gamma_1 = \{|z_1| = \cdots = |z_n| = 1\}$ make up dual bases for the n-dimensional cohomology and homology in the domain $D = \mathbf{C}^n \setminus \{z_1 \cdots z_n = 0\}$.

Indeed, since D is a domain of holomorphy, any form $\omega \in Z^n(D)$ is by Theorem 9 cohomologous to a holomorphic form, which can be expressed by $f(z)\,dz$, with f a holomorphic function in D. The function f can be expanded into a multiple Laurent series in D:

$$f(z) = \sum_{\alpha} c_\alpha z^\alpha = \sum_{\alpha_1,\ldots,\alpha_n=-\infty}^{\infty} c_{\alpha_1\cdots\alpha_n} z_1^{\alpha_1}\cdots z_n^{\alpha_n}.$$

If $\alpha = (\alpha_1,\ldots,\alpha_n) \neq -I = (-1,\ldots,-1)$, for example, $\alpha_j \neq -1$, then

$$c_\alpha Z^\alpha\,dz = d\left(\frac{c_\alpha}{\alpha_j+1} z_1^{\alpha_1}\cdots z_j^{\alpha_j+1}\cdots z_n\,dz_1\wedge\cdots[j]\cdots\wedge dz_n\right) \approx 0.$$

Thus, $\omega \approx f(z)\,dz \approx c_{-I}(dz/z)$. Obviously, it remains to apply Corollary 7. See [240], §13, for other examples of the use of de Rham duality.

2.4. ***The Leray theory of residues*** **([142], [171]).** ***Froissart's decomposition theorem*** **[127].** Let X be a complex analytic manifold, and S a complex analytic submanifold of it with codimension 1 (its dimension is smaller by 1). We shall define the coboundary homomorphism

$$\delta\colon C_p(S) \to C_{p+1}(X\setminus S). \qquad (16)$$

To do this, we assign to each point $z \in S$ an oriented curve $\delta z \in X\setminus S$ homeomorphic to a circle, in such a way that the following conditions hold:

a) $\delta S = \bigcup_{z\in S}\delta z$ is a smooth surface.

b) If $z^{(1)} \neq z^{(2)}$, then $\delta z^{(1)} \cap \delta z^{(2)} = \varnothing$.

c) In some neighborhoods U_a of a point $a \in S$ there exists a local system of coordinates $z_1,\ldots,z_n$ in which $S \cap U_a = \{z_1 = 0\}$ and $\delta a = \{|z_1| = \varepsilon, z_2 = \cdots = z_n = 0\}$, where ε is a sufficiently small positive number.

It suffices to define the homomorphism (16) for a singular simplex σ_p. It can be assumed that $|\sigma_p| \subset U_a$; otherwise, we break up σ_p into finer simplexes. We set $\delta\sigma_p = \bigcup_{z\in\sigma_p}\delta_z$, with the corresponding orientation ($\delta\sigma_p$ is homeomorphic to $\{|z_1| = \varepsilon\}\times\sigma_p$). Obviously, $\partial\delta\sigma_p = -\delta\partial\sigma_p$. Consequently, (16) induces a homomorphism $\delta\colon H_p(S)\to H_{p+1}(X\setminus S)$.

Further, suppose that the form $\omega \in Z^p(X\setminus S)$ has a first-order polar singularity on S. This means that in a neighborhood U_a of an arbitrary point $a \in S$ the form $\omega(z)\cdot s(z)$, where $s(z)$ is a holomorphic function defining S in U_a ($S\cap U_a = \{z\colon s(z) = 0\}$), can be extended to a regular (of class $\mathfrak{S}^\infty$) function in U_a. The form ω has a representation

$$\omega(z) = \frac{\varphi(z)\wedge ds(z)}{s(z)} + \theta(z)$$

in U_a, where φ and θ are regular forms in U_a. The form $\varphi|_{S \cap U_a}$ is closed and does not depend on the choice of the local coordinate system and the function $s(z)$ defining S in U_a. The form $\operatorname{res}[\omega] \in Z^{p-1}(S)$ defined locally by the equality $\operatorname{res}[\omega]_{U_a} = \varphi|_{S \cap U_a}$ is called the *residue form* of the form ω.

THEOREM 11 (Leray residue formula). *If $\gamma \in Z_{p-1}(S)$ and the form $\omega \in Z^p(X \setminus S)$ has a first-order polar singularity on S, then*

$$\int_{\delta\gamma} \omega = 2\pi i \int_\gamma \operatorname{res}[\omega]. \tag{17}$$

THEOREM 12 (Leray [142]). *For every form $\omega \in Z^p(X \setminus S)$ there exists a form $\tilde{\omega} \in \{\omega\} \in H^p(X \setminus S)$ having a first-order polar singularity on S.*

The cohomology class $\{\operatorname{res}[\omega]\} \in H^{p-1}(S)$ is called the *residue class* of the cohomology class $\{\omega\} \in H^p(X \setminus S)$ and denoted by $\operatorname{Res}[\omega]$. The mapping $\operatorname{Res}\colon H^p(X \setminus S) \to H^{p-1}(S)$ is a homomorphism. From (17) we get the formula

$$\int_{\delta\gamma} \omega = 2\pi i \int_\gamma \operatorname{Res}[\omega].$$

Let $S_1, \ldots, S_k$ be complex analytic submanifolds of codimension 1 which are "in general position". The latter means that at any point $a \in S_{j_1} \cap \cdots \cap S_{j_r}$, $\{j_1, \ldots, j_r\} \subset \{1, \ldots, k\}$, the vectors $\operatorname{grad} s_j(z)|_a$, $j = j_1, \ldots, j_r$, where $s_j(z)$, where $s(z)$ is a holomorphic function defining S_j in a neighborhood U_a of a (i.e., $S_j \cap U_a = \{s_j(z) = 0\}$), are linearly independent. We define the multiple coboundary homomorphism

$$\delta^k\colon H_p(S_1 \cap \cdots \cap S_k) \to H_{p+k}(X \setminus S_1 \cup \cdots \cup S_k)$$

as the composition of the homomorphisms δ:

$$H_p(S_1 \cap \cdots \cap S_k) \xrightarrow{\delta} H_{p+1}(S_1 \cap \cdots \cap S_{k-1} \setminus S_k)$$
$$\xrightarrow{\delta} \cdots H_{p+k-1}(S_1 \setminus S_2 \cup \cdots \cup S_k) \xrightarrow{\delta} H_{p+k}(X \setminus S_1 \cup \cdots \cup S_k).$$

The *multiple residue class* Res^k is defined as the composition

$$H^{p+k}(X \setminus S_1 \cup \cdots \cup S_k) \xrightarrow{\operatorname{Res}} H^{p+k-1}(S_1 \setminus S_2 \cup \cdots \cup S_k)$$
$$\to \cdots \to H^{p+1}(S_1 \cap \cdots \cap S_{k-1} \setminus S_k) \xrightarrow{\operatorname{Res}} H^p(S_1 \cap \cdots \cap S_k).$$

For a form $\omega \in Z^{p+k}(X \setminus S_1 \cup \cdots \cup S_k)$ and a cycle $\gamma \in Z_p(S_1 \cap \cdots \cap S_k)$ we have the Leray multiple residue formula

$$\int_{\delta^k \gamma} \omega = (2\pi i)^k \int_\gamma \operatorname{Res}^k[\omega]. \tag{18}$$

An algorithm is given in [142] (Chapter VI) and in [171] (Chapter III, §4.3) for determining the residue class for semimeromorphic forms (i.e., forms having polar singularities on the submanifolds $S_1, \ldots, S_k$).

THEOREM 13 (Froissart decomposition theorem [127]). *Let $\Sigma_1, \ldots, \Sigma_m$ and $S_0, S_1, \ldots, S_k$ be two families of complex $(n-1)$-dimensional submanifolds of the complex projective space $\mathbf{C}P^n$ which are in general position, with $S_0 = \mathbf{C}P^{n-1}$ a complex hyperplane at infinity, $\mathbf{C}P^n \setminus S_0 = \mathbf{C}^n$, and $X = \mathbf{C}^n \cap (\cap_1^m \Sigma_j)$. Then*

$$H_q(X \setminus (S_1 \cup \cdots \cup S_k)) = \bigoplus_{h \in \{1, \ldots, k\}} \delta^{|h|} H_{q-|h|}\left(X \cap \left(\bigcap_{j \in h} S_j\right)\right)$$
$$= H_q(X) \oplus \bigoplus_{j=1}^{k} \delta H_{q-1}(S_j) \oplus \bigoplus_{1 \leqslant i < j \leqslant k} \delta^2 H_{q-2}(X \cap S_i \cap S_j) \oplus \cdots,$$

where $|h|$ is the number of elements in the set h, and $\oplus$ is the direct sum symbol.

Froissart's decomposition theorem shows that in sufficiently nice cases every cycle $\gamma \in H_q(X \setminus S_1 \cup \cdots \cup S_k)$ can be represented as a sum of coboundary cycles such that the integrals over them reduce (by the Leray residue formulas) to integrals over cycles of lower dimension on submanifolds.

§3. Local residues of certain meromorphic functions

We apply the results of the preceding section to compute the residues of certain meromorphic functions with respect to cycles lying in a sufficiently small neighborhood of the singular set.

3.1. ***The case $n = 2$ [239] (application of Alexander-Pontryagin duality).*** Consider the integral

$$\int_\Gamma \frac{f(w, z)}{g(w, z)} dw \wedge dz, \tag{19}$$

where f and g are holomorphic functions in a domain $D \subset \mathbf{C}^2$, $\Gamma \in Z_2(U \setminus T)$, $T = \{(w, z) \in \mathbf{C}^2 \colon g(w, z) = 0\}$, and U is a sufficiently small neighborhood of a point $(a, b) \in T$. We assume that $(a, b) = (0, 0)$. According to the Weierstrass preparation theorem [202], g can be represented in the form $g = \mathcal{P} \cdot \Omega$ in some neighborhood U of $(0, 0)$, where $\mathcal{P}(w, z) = w^m + a_1(z) w^{m-1} + \cdots + a_m(z)$ is a Weierstrass polynomial, with $a_j(z)$ and $\Omega(w, z)$ holomorphic functions such that $a_j(0) = 0$ and $\Omega(0,0) \neq 0$. The Weierstrass polynomial $\mathcal{P}$ can be decomposed into a product of irreducible Weierstrass polynomials: $\mathcal{P} = \mathcal{P}_1^{r_1} \cdots \mathcal{P}_k^{r_k}$, where

$$\mathcal{P}_j(w, z) = \prod_{\nu=1}^{m_j} \left[w - w_j(\varepsilon_{m_j,\nu} \cdot z) \right], \qquad \varepsilon_{m_j,\nu} = e^{2\pi\nu \cdot i/m_j},$$

$$w_j(z) = \sum_{n=n_0}^{\infty} b_{jn} z^{n/m_j}. \tag{20}$$

The series (20) can be found with the help of a Newton diagram (see [45]). Note that only finitely many terms of this series are required.

The irreducible polynomial $\mathcal{P}_j$ defines a surface $T_j = \{z \in U: \mathcal{P}_j(z) = 0\}$ in U that is homeomorphic to a disk (m_j-sheeted over the z-plane). The set $T \cap U = \cup_1^k T_j$ is thus homeomorphic to k disks with centers identified. We complete U to form a four-dimensional sphere by identifying its boundary ∂U with a single point Q. According to Theorem 2 in §2, we then have $H_2(U \setminus T) \simeq H_1(\overline{T})$, where $\overline{T} = \cup_1^k \overline{T}_j$, $\overline{T}_j = T_j \cup \{Q\}$. The set T is homeomorphic to k two-dimensional spheres with two common points ((0,0) and Q). It is easy to compute (for example, with the help of the Euler-Poincaré formula; see [80] and [216]) that the dimension of the group $H_1(\overline{T})$ is $k - 1$. In particular, $H_2(U \setminus T) \simeq H_1(\overline{T}) \simeq 0$ for $k = 1$.

We construct dual bases for the groups $H_1(\overline{T})$ and $H_2(U \setminus T)$. Let t_j be an oriented curve on T_j joining the point (0,0) with $Q = \partial U \cap T$, for example, $t_j = \{(w, z): z = \tau,\ w = w_j(\tau),\ 0 \leqslant \tau \leqslant \tau_0,\ j = 1, \ldots, k\}$. We take $\sigma_j = t_j - t_k \in Z_1(\overline{T})$, $j = 1, \ldots, k - 1$, and, respectively, $\gamma_j = \{(w, z): z = \varepsilon \cdot e^{i\varphi};\ w = w_j(\varepsilon e^{i\varphi} + \delta e^{i\theta}),\ 0 \leqslant \varphi \leqslant 2\pi m_j,\ 0 \leqslant \theta \leqslant 2\pi\}$, $j = 1, \ldots, k - 1$. For ε sufficiently small and $0 < \delta < \varepsilon$ we have $\gamma_j \in Z_2(U \setminus T)$. It is easy to verify that $\mathfrak{B}(\sigma_j, \gamma_s) = \delta_{js}$, $j, s = 1, \ldots, k - 1$. Thus, $\{\sigma_j\}_1^{k-1}$ and $\{\gamma_j\}_1^{k-1}$ form dual bases in $\overline{T}$ and $U \setminus T$ (see §3.2).

A residue with respect to a basis cycle can be found by repeated application of the residue formula for functions of a single complex variable with respect to a pole:

$$R_j = \frac{1}{(2\pi i)^2} \int_{\gamma_j} \frac{f\,dw \wedge dz}{g} = \frac{1}{(2\pi i)^2} \int_{z=\varepsilon e^{i\varphi}} dz \int_{|w-w_j(z)|=\delta} \frac{f\,dw \wedge dz}{\mathcal{P}_1^{r_1} \cdots \mathcal{P}_k^{r_k} \Omega}$$

$$= m_j \operatorname*{res}_{z=0} \left[\frac{1}{(r_j - 1)!} \frac{\partial^{r_j - 1}}{\partial w^{r_j - 1}} \left(\frac{f \cdot (w - w_j(z))^{r_j}}{\Omega \cdot \mathcal{P}_1^{r_1} \cdots \mathcal{P}_k^{r_k}} \right) \right] \Bigg|_{w = w_j(z)}.$$

3.2. ***The case*** $n \geqslant 2$ ***(under certain additional assumptions).*** Consider the integral

$$\int_\Gamma \frac{f(z)}{g(z)}dz = \int_\Gamma \frac{f(z_1,\dots,z_n)}{g(z_1,\dots,z_n)}dz_1 \wedge \cdots \wedge dz_n, \tag{21}$$

where f and g are holomorphic functions in a domain $D \subset \mathbf{C}^n$, $\Gamma \in Z_n(U \setminus T)$, $T = \{z \in D\colon g(z) = 0\}$, and U is a sufficiently small neighborhood of a point $a \in T$ (it can be assumed that $a = 0 = (0,\dots,0)$). We examine the case where $g = g_1^{r_1} \cdots g_k^{r_k}$, $g_j(0) = 0$, and

$$\left.\frac{\partial(g_{j_1},\dots,g_{j_n})}{\partial(z_1,\dots,z_n)}\right|_0 \neq 0$$

for any collection $\{j_1,\dots,j_n\} \subset \{1,\dots,k\}$.

By our assumptions, for any n functions $g_{j_1},\dots,g_{j_n}$ among $g_1,\dots,g_{k-1}$ there are functions $\varphi_1,\dots,\varphi_n$ that are holomorphic in a sufficiently small neighborhood U of 0 and such that $\varphi_j(z) \neq 0$ in U and

$$g_k(z) \equiv g_{j_1}(z)\varphi_1(z) + \cdots + g_{j_n}(z)\varphi_n(z).$$

For this, it suffices to make the substitution $\zeta_\nu = g_{j_\nu}(z)$, $\nu = 1,\dots,n$, expand g_k in a series in powers of $\zeta_1,\dots,\zeta_n$, and regroup. Then in U

$$\frac{1}{g_{j_1}\cdots g_{j_n}} = \sum_{\nu=1}^n \frac{\varphi_\nu}{g_{j_1}\cdots[\nu]\cdots g_{j_n}g_k}. \tag{22}$$

Using representations of this form sufficiently many times, we represent the fraction $1/g$ in the form

$$\frac{1}{g_1^{r_1}\cdots g_k^{r_k}} = \sum_{\alpha,\beta}\frac{\psi_{\alpha,\beta}}{g_{\alpha_1}^{\beta_1}\cdots g_{\alpha_n}^{\beta_n}},$$

where $\alpha = \{\alpha_1,\dots,\alpha_n\} \in \{1,\dots,k\}$, $\alpha_n = k$, $\beta = (\beta_1,\dots,\beta_m)$, and $\beta_1 + \cdots + \beta_n = r_1 + \cdots + r_k$. Thus, computation of the integral (21) reduces to computation of integrals of the form

$$I_{\alpha,\beta} = \int_\Gamma \frac{\psi_{\alpha,\beta}(z)\,dz}{g_{\alpha_1}^{\beta_1}(z)\cdots g_{\alpha_n}^{\beta_n}(z)}. \tag{23}$$

In precisely the same way as in Example 3 of §2 it can be shown that the cycle

$$\gamma_\alpha = \left\{z \in U\colon |g_{\alpha_1}(z)| = \cdots = |g_{\alpha_n}(z)| = \varepsilon\right\},$$

where ε is a sufficiently small positive number, and the form

$$\omega_\alpha = \frac{1}{(2\pi i)^n} \frac{dg_{\alpha_1} \wedge \cdots \wedge dg_{\alpha_n}}{g_{\alpha_1} \cdots g_{\alpha_n}}$$

make up dual (in the de Rham sense) bases for the n-dimensional homology and cohomology of the domain $U_\alpha = U \setminus T_{\alpha_1} \cup \cdots \cup T_{a_n}$, where $T_j = \{z \in U: g_j(z) = 0\}$. Consequently, in U_a we have $\Gamma \sim k_\alpha \gamma_a$, where $k_\alpha = \int_\Gamma \omega_\alpha$. Then the integral (23) is

$$I_{\alpha,\beta} = \int_\Gamma \frac{\psi_{\alpha,\beta}\, dg_{\alpha_1} \wedge \cdots \wedge dg_{\alpha_n}}{(\partial(g_\alpha)/\partial(z)) g_{\alpha_1}^{\beta_1} \cdots g_{\alpha_n}^{\beta_n}}$$
$$= \frac{k_\alpha}{(\beta_1 - 1)! \cdots (\beta_n - 1)!} \frac{\partial^{\beta_1 + \cdots + \beta_n - n}}{\partial w_1^{\beta_1 - 1} \cdots \partial w_n^{\beta_n - 1}} \left[\frac{\psi_{\alpha,\beta}}{\partial(g_\alpha)/\partial(z)} \right] \Bigg|_{(0)},$$

where $w_j = g_{\alpha j}(z), j = 1, \ldots, n$ $(w = g_\alpha(z))$.

3.3. *Residues of meromorphic functions with linear singularities (the case of general position).* [5] We consider an integral of the form (21), where f is an entire function, $g = g_1^{r_1} \cdots g_k^{r_k}$, $g_j(z) = \sum_1^n a_{j\nu} z_\nu + b_j$ (linear functions such that the analytic planes $S_j = \{z \in C^n: g_j(z) = 0\}, j = 1, \ldots, k$, are in general position), and $\Gamma \in Z_n(\mathbf{C}^n \setminus S_1 \cup \cdots \cup S_k)$. Theorem 13 of §2 is applicable under our assumptions. Since $H_n(\mathbf{C}^n) \simeq 0$ and $H_{n-p}(S_{j_1} \cap \cdots \cap S_{j_p}) \simeq 0$ for $p < n$, it follows that

$$H_n(\mathbf{C}^n \setminus S_1 \cup \cdots \cup S_k) \simeq \bigoplus_{\alpha \in A} \delta^n H_0(S_{\alpha_1} \cap \cdots \cap S_{\alpha_n}),$$

where A is the set of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{1, \ldots, k\}$ for which $S_{\alpha_1} \cap \cdots \cap S_{\alpha_n} \neq \varnothing$. Since $S_{\alpha_1} \cap \cdots \cap S_{\alpha_n} = a_\alpha$ is a point for $\alpha \in A$, the cycles $\gamma_\alpha = \delta^n a_\alpha = \{|g_{\alpha_1}(z)| = \cdots = |g_{\alpha_n}(z)| = \varepsilon\}$, $\alpha \in A$, where ε is sufficiently small, form a basis for the n-dimensional homology of the domain $\mathbf{C}^n \setminus S_1 \cup \cdots \cup S_k$. Obviously, the forms

$$\omega_\alpha = \frac{1}{(2\pi i)^n} \frac{dg_{\alpha_1} \wedge \cdots \wedge dg_{\alpha_n}}{g_{\alpha_1} \cdots g_{\alpha_n}}, \qquad \alpha \in A,$$

([5]) See [238] for the arbitrary case.

make up the dual n-dimensional cohomology basis. The residue of the form $\omega = (f/g)\,dz$ with respect to the basis cycle γ_α is

$$R_\alpha = \frac{1}{(2\pi i)^n}\int_{\gamma_\alpha}\omega = \frac{1}{(r_{\alpha_1}-1)!\cdots(r_{\alpha_n}-1)!\Delta_\alpha}\frac{\partial^{r_{\alpha_1}+\cdots+r_{\alpha_n}-n}}{\partial g_{\alpha_1}^{r_{\alpha_1}-1}\cdots\partial g_{\alpha_n}^{r_{\alpha_n}-1}} \times\left[\frac{f}{g}g_{\alpha_1}^{r_{\alpha_1}}\cdots g_{\alpha_n}^{r_{\alpha_n}}\right]\Bigg|_{a_\alpha},$$

where

$$\Delta_\alpha = \frac{\partial(g_{\alpha_1},\dots,g_{\alpha_n})}{\partial(z_1,\dots,z_n)} = \det\|a_{j\nu}\|_{j=\alpha_1,\dots,\alpha_n;\ \nu=1,\dots,n}.$$

§4. Multidimensional analogues of the logarithmic residue

Just as the logarithmic residue is connected with the number of zeros of a holomorphic function in the case of analytic functions of a single complex variable, the multidimensional analogues of the logarithmic residue are connected with the number of common zeros of a system of holomorphic functions.

4.1. ***The multiplicity of a common zero of a system of holomorphic functions. The theorem on the logarithmic residue in $\mathbf{C}^n$, connected with integration over the whole boundary*** **([33], [157], [202], [243]).** Suppose that a system of n holomorphic functions

$$w_j = f_j(z) = f_j(z_1,\dots,z_n), \qquad j = 1,\dots,n, \tag{24}$$

or, in other words, a holomorphic mapping $w = f(z)$, $f = (f_1,\dots,f_n)$, is defined in a domain $G \subset \mathbf{C}^n$. A point $a = (a_1,\dots,a_n) \in G$ is called a *zero of the system* (24) if $f_j(a) = 0$, $j = 1,\dots,n$ ($f(a) = 0$). Let $\mathfrak{E}_f = \{z \in G: f_1(z) = \cdots = f_n(z) = 0\}$ be the set of zeros of the system (24) in G. It is assumed that $\mathfrak{E}_f$ is discrete and the Jacobian

$$\frac{\partial(f)}{\partial(z)} = \frac{\partial(f_1,\dots,f_n)}{\partial(z_1,\dots,z_n)}$$

is not identically zero in G. A zero a of the system (24) is said to be *simple* if $(\partial(f)/\partial(z))|_\alpha \neq 0$.

PROPOSITION 1. *Let U_a be a neighborhood of a zero a of the system* (24) *such that $U_a \cap (\mathfrak{E}_f \setminus \{a\}) = \varnothing$. Then for any number $\varepsilon > 0$ there exists a point $\zeta = (\zeta_1,\dots,\zeta_n)$, $|\zeta| < \varepsilon$ ($|\zeta| = (|\zeta_1|^2 + \cdots + |\zeta_n|^2)^{1/2}$), such that the system*

$$w = f(z) - \zeta = (f_1(z) - \zeta_1,\dots,f_n(z) - \zeta_n) \tag{25}$$

has only simple zeros in U_a. *Moreover, for* ε *sufficiently small their number does not depend on the choice of* ζ.

Under the hypotheses of this proposition the number of simple zeros of the system (25) is called the *multiplicity of the zero a* of the system (24).

THEOREM 2 (on the logarithmic residue). *Suppose that the system* (24) *of holomorphic functions has a discrete set* $\mathfrak{S}_f$ *of zeros in the domain* $G \subset \mathbf{C}^n$, *and that* $\partial(f)/\partial(z) \not\equiv 0$ *in* G. *Then the number of zeros (counting their multiplicities)*(6) *of* (24) *in a domain* $D \Subset G$ *with piecewise smooth boundary* ∂D, $\partial D \cap \mathfrak{S}_f = \varnothing$, *is determined by the formula*

$$N(f, D) = \int_{\partial D} \omega(f(z)), \tag{26}$$

where

$$\omega(f) = \frac{(n-1)!}{(2\pi i)^n} \frac{1}{|f|^{2n}} \sum_{j=1}^{n} \bar{f}_j df_j \wedge d\bar{f}_1 \wedge df_1 \wedge \cdots [j] \cdots \wedge d\bar{f}_n \wedge df_n$$

($|f| = (\sum_1^n |f_j|^2)^{1/2}$) *is the Martinelli-Bochner kernel (see* [5] *and* [202]).

THEOREM 3 (the Rouché principle). *Suppose that the system* (24) *satisfies the conditions of Theorem* 2, *while the system* $g = (g_1, \ldots, g_n)$ *of holomorphic functions in G satisfies the inequality*

$$|g(z)| < |f(z)| \qquad \left(\text{or } \|g(z)\| < \|f(z)\|, \text{ where } \|f\| = \max\{f_1, \ldots, f_n\}\right) \tag{27}$$

on ∂D. *Then the system*

$$f + g = (f_1 + g_1, \ldots, f_n + g_n) \tag{28}$$

of functions and the system (24) *have the same number of zeros in D, counting multiplicities.*

LEMMA 1. *Suppose that the conditions of Theorem 2 hold and neighborhoods* U_a *of the points* $a \in \mathfrak{S}_f \cap D$ *are chosen in such a way that* $\overline{U}_a \subset D$ *and* $\overline{U}_a \cap \overline{U}_b = \varnothing$ *for* $a \neq b$, $a, b \in \mathfrak{S}_f \cap D$. *Then*

$$\int_{\partial D} \omega(f) = \sum_{a \in \mathfrak{S}_f \cap D} \int_{\partial U_a} \omega(f). \tag{29}$$

Indeed, it can be verified directly that the form $\omega(f(z))$ is closed and regular in $G \setminus \mathfrak{S}_f$. We have

$$\partial D \sim \sum_{a \in \mathfrak{S}_f \cap D} \partial U_a,$$

(6) More precisely, the sum of the multiplicities of these zeros.

because

$$\partial D - \sum_{\alpha \in \mathfrak{E}_f \cap D} \partial U_a = \partial\Big(D \setminus \bigcup_{a \in \mathfrak{E}_f \cap D} U_a\Big).$$

Thus, (29) follows by Corollary 5 in §1.

LEMMA 2 (special case of Theorem 2). *Suppose that the conditions of Theorem 2 hold and all the zeros of the system* (24) *in the domain D are simple. Then the number of these zeros is determined by* (26).

PROOF. Take U_a to be the connected component of the set $\{z \in G: |f(z)| < \varepsilon\}$ containing the point $a \in \mathfrak{E}_f \cap D$, where ε is sufficiently small. By assumption, $(\partial(f)/\partial(z))|_a \neq 0$, so the mapping (24) is biholomorphic in U_a for ε sufficiently small. Making the change of variables (24) and applying Stokes' formula, we get

$$\int_{\partial U_a} \omega(f(z)) = \int_{|w|=\varepsilon} \omega(w)$$

$$= \frac{(n-1)!}{(2\pi i)^n} \int_{|w|=\varepsilon} \frac{1}{\varepsilon^{2n}} \sum_{j=1}^{n} \bar{w}_j\, dw_j \wedge d\bar{w}_1 \wedge dw_1 \wedge \cdots [j] \cdots \wedge d\bar{w}_n \wedge dw_n$$

$$= \frac{(n-1)!}{(2\pi i)^n} \frac{1}{\varepsilon^{2n}} \int_{|w|\leqslant\varepsilon} n \cdot d\bar{w}_1 \wedge dw_1 \wedge \cdots \wedge d\bar{w}_n \wedge dw_n$$

$$= \frac{n!}{\pi^n \varepsilon^{2n}} \int_{|w|\leqslant\varepsilon} du_1 \wedge av_1 \wedge \cdots \wedge du_n \wedge dv_n = \frac{n!}{\pi^n \varepsilon^{2n}} V_{2n}(\varepsilon) = 1, \tag{30}$$

where $V_{2n}(\varepsilon) = \pi^n \varepsilon^{2n}/n!$ is the volume of the $2n$-dimensional ball of radius ε, and $w_j = u_j + iv_j$ $(j = 1, \ldots, n)$.([7])

Lemma 2 follows from (29) and (30).

LEMMA 3. *Under the conditions of Theorem* 3,

$$\int_{\partial D} \omega(f) = \int_{\partial D} \omega(f+g). \tag{31}$$

PROOF. According to (27), $|f(z) + tg(z)| \geqslant |f(z)| - t|g(z)| > 0$([8]) for $z \in \partial D$ and $0 \leqslant t \leqslant 1$, so the form $\omega(f(z) + tg(z))$ is continuous on the compact set $\partial D \times [0, 1]$. Consequently, the integral $I(t) = \int_{\partial D} \omega(f + tg)$ is a continuous function of t on $[0, 1]$. But $I(t)$ is an integer, because $\int_{\partial D} \omega(f + tg) = \int_{(f+tg)\partial D} \omega(w)$, $(f + tg)\partial D \sim n(t) \cdot \{w \in \mathbf{C}^n: |w| = 1\}$ in $\mathbf{C}^n \setminus \{0\}$ (the domain

([7]) It is assumed that the orientation of the space $\mathbf{C}^n_w = R^{2n}_{(u,v)}$ is determined by the order of the coordinates $u_1, v_1, \ldots, u_n, v_n$.

([8]) The inequality $\|g(z)\| < \|f(z)\|$ also implies that $|f(z) + tg(z)| > 0$.

of regularity of the form $\omega(w)$), where $n(t)$ is an integer, and $\int_{|w|=1}\omega(w) = 1$ (see (30)). Consequently, $I(t) = \text{const}$. Then $I(0) = I(1)$, i.e., (31) is valid.

PROOF OF PROPOSITION 1. By Sard's theorem (see, for example, [184]),

$$\operatorname{meas} f\left(\left\{z \in G\colon \frac{\partial(f)}{\partial(z)} = 0\right\}\right) = 0.$$

Hence, for any $\varepsilon > 0$ there is a point $\zeta = (\zeta_1, \dots, \zeta_n)$, $|\zeta| < \varepsilon$, such that $\partial(f)/\partial(z) \neq 0$ for any $z \in G \cap f^{-1}(\zeta)$. The system (25) will have only simple zeros in G for such a point ζ. If $\varepsilon < \min_{z\in\partial U_a}|f(z)|$, then $\mathfrak{E}_{f-\zeta} \cap \partial U_a = \varnothing$, and, according to Lemma 3,

$$\int_{\partial U_a}\omega(f) = \int_{\partial U_a}\omega(f-\zeta). \tag{32}$$

By Lemma 2, the integral $\int_{\partial U_a}\omega(f-\zeta)$ is equal to the number of zeros of (25) in U_a. The equality (32) implies that this number does not depend on the choice of ζ if $|\zeta| < \min_{z\in\partial U_a}|f(z)|$.

PROOF OF THEOREM 2. From the definition of the multiplicity of a zero and from (32) it follows that the multiplicity of a zero of (24) is determined by

$$m_a = \int_{\partial U_a}\omega(f(z)). \tag{33}$$

This and Lemma 1 give us Theorem 2.

Theorem 3 is deduced from Theorem 2 and Lemma 3. Here the discreteness of the set $\mathfrak{E}_{f+g} \cap D$ follows from the fact that $f + g \neq 0$ on ∂D (the compact analytic set $\mathfrak{E}_{f+g}$ is discrete in the domain $D \subset \mathbf{C}^n$).

REMARK 1. Obviously, the multiplicity of a simple zero is equal to 1. It can be shown that if $(\partial(f)/\partial(z))|_a = 0$, $a \in \mathfrak{E}_f$, then the multiplicity m_a of a zero of (24) is greater than 1.

THEOREM 4 (generalization of the logarithmic residue theorem). *Suppose that the system* (24) *and the domain $D \Subset G$ satisfy the conditions of Theorem* 2, *and φ is a holomorphic function in G. Then*

$$\int_{\partial D}\varphi(z)\omega(f(z)) = \sum_{a_\nu\in\mathfrak{E}_f\cap D} m_\nu\varphi(a_\nu), \tag{34}$$

where m_ν is the multiplicity of the zero a_ν of (24).

PROOF. Since φ is a holomorphic function and $\omega(f)$ is a closed form containing the factor $dz = dz_1 \wedge \cdots \wedge dz_n$, it follows that $\varphi\cdot\omega(f)$ is also a closed form. Similarly, (29) gives us

$$\int_{\partial D}\varphi\cdot\omega(f) = \sum_{a_\nu\in\mathfrak{E}_f\cap D}\int_{\partial U_\nu}\varphi\cdot\omega(f), \tag{35}$$

where U_ν is the connected component of the set $\{z \in G\colon |f(z)| < \varepsilon\}$ containing a_ν, with ε sufficiently small. By the continuity of $\varphi(z)$, (33) implies that

$$\lim_{\varepsilon\to 0}\int_{\partial U_\nu(\varepsilon)}\varphi(z)\omega(f(z)) = \lim_{\varepsilon\to 0}\int_{\partial U_\nu(\varepsilon)}\varphi(a_\nu)\omega(f(z)) = \varphi(a_\nu)\cdot m_\nu.$$

This and (35) give us (34).

4.2. *Theorems on the logarithmic residue in* $\mathbf{C}^n$**, *connected with n-fold integrals* ([153], [202], [215], [243]).**

THEOREM 5 (on the logarithmic residue). *Under the conditions of Theorem* 2 *the number of zeros of the system* (24) *in the domain D is expressed by*

$$N(f, D) = \frac{1}{(2\pi i)^n}\int_\Gamma \frac{df}{f} = \frac{1}{(2\pi i)^n}\int_\Gamma \frac{df_1(z)}{f_1(z)}\wedge\cdots\wedge\frac{df_n(z)}{f_n(z)}, \tag{36}$$

where $\Gamma = \{z \in \partial D\colon |f_2(z)| = \cdots = |f_n(z)| = \varepsilon\}$, *with* $0 < \varepsilon < \min_{z\in\partial D}\|f(z)\|$ *and* $\|f\| = \max(|f_1|,\ldots,|f_n|)$. *The orientation of the cycle* Γ *is chosen to be coherent with that of the surface* $\{z \in D\colon |f_2(z)| = \cdots = |f_n(z)| = \varepsilon\}$, *as determined by the order of the parameters* u_1, v_1 *and* $\theta_2,\ldots,\theta_n$, *where* $f_1(z) = u_1 + iv_1$ *and* $f_j(z) = \varepsilon\cdot e^{i\theta_j}$, $j = 2,\ldots, n$, *is the parametrization of this surface at the points with* $\partial(f)/\partial(z) \neq 0$.

We also have the following modification of Theorem 5.

THEOREM 6. *Suppose that the system* (24) *satisfies the conditions of Theorem* 2, *and D is a connected component* (*a union of finitely many connected components*) *of the set* $\{z \in G\colon |f_j(z)| < \rho_j, j = 1,\ldots, n\}$, $D \Subset G$. *Then the number of zeros of the system* (24) *in D is determined by* (36), *where* $\Gamma = \{z \in D\colon f_j(z)| = \rho_j,\ j = 1,\ldots, n\}$.

THEOREM 7 (generalized theorem on the logarithmic residue). *If* φ *is a holomorphic function in the domain G, then under the conditions of Theorems* 5 *and* 6

$$\frac{1}{(2\pi i)^n}\int_\Gamma \varphi(z)\frac{df(z)}{f(z)} = \sum_{a_\nu\in\mathfrak{E}_f\cap D} m_\nu\varphi(a_\nu), \tag{37}$$

where m_ν *is the multiplicity of the zero* a_ν *of the system* (24), *and*

$$\frac{df}{f} = \frac{df_1}{f_1}\wedge\cdots\wedge\frac{df_n}{f_n}.$$

The next result turns out to be useful in applications.

THEOREM 8. *Suppose that the system of functions* (24), *the open set D, and the cycle* Γ *satisfy the conditions of Theorem* 6, *and the system*

$$g = (g_1,\ldots, g_n) \tag{38}$$

of holomorphic functions in the domain G satisfies on Γ *the inequalities*

$$|f_j(z) - g_j(z)| < |f_j(z)| = \rho_j, \qquad j = 1, \ldots, n. \tag{39}$$

Then the systems (24) *and* (38) *have the same number of zeros in D, counting multiplicities. Moreover, for an arbitrary holomorphic function* $\varphi(z)$ *in G*

$$\frac{1}{(2\pi i)^n} \int_\Gamma \varphi(z) \frac{dg_1(z)}{g_1(z)} \wedge \cdots \wedge \frac{dg_n(z)}{g_n(z)} = \sum_{b_\nu \in \mathfrak{G}_g \cap D} m_\nu \varphi(b_\nu), \tag{40}$$

where $\mathfrak{G}_g$ *is the set of zeros of the system* (38), *and* m_ν *is the multiplicity of the zero* $b_\nu \in \mathfrak{G}_g$.

LEMMA 4. *Under the conditions of Theorem* 8 *there is a number* $\delta_0 > 0$ *such that if* $0 < \delta < \delta_0$, *then*

$$\Gamma \sim \gamma(\delta) = \{z \in D: |g_1(z)| = \cdots = |g_n(z)| = \delta\}$$

in the domain

$$G_g = G \setminus \{z \in G: g_1(z) \cdots g_n(z) = 0\}.$$

PROOF. Let $r_j = \max_{z \in \bar{D}} |f_j(z) - g_j(z)|$, $j = 1, \ldots, n$. The inequality $r_j < \rho_j$ follows from (39) and the fact that the maximum modulus of a holomorphic function in the Weil polyhedron $\bar{D}$ is attained on its distinguished boundary Γ (see [79], §15). Take $\delta_0 = \min\{\rho_1 - r_1, \ldots, \rho_n - r_n\}$, $\delta < \delta_0$, and consider the family of n-dimensional cycles $\Gamma_p = \{z \in \bar{D}: |g_j(z)| = \delta,\ j = 1, \ldots, p; |f_\nu(z)| = \rho_\nu,\ \nu = p+1, \ldots, n\}$, $p = 0, 1, \ldots, n$. Obviously, $|\Gamma_p| \subset G_g$, because $g_j(z) \neq 0$ on Γ_p by the choice of δ. We have $\Gamma = \Gamma_0 \sim \Gamma_1 \sim \cdots \sim \Gamma_n = \gamma(\delta)$ in G_g. Indeed, $\Gamma_p - \Gamma_{p+1} = \partial B_p$, where $B_p = \{z \in D: |g_j(z)| = \delta,\ j = 1, \ldots, p-1; |g_p(z)| \geqslant \delta, |f_p(z)| \leqslant \rho_p; |f_\nu(z)| = \rho_\nu,\ \nu = p+1, \ldots, n\} \subset G_g$. The lemma is proved.

PROOF OF THEOREM 6. We enclose each point $a_\nu \in \mathfrak{G}_f \cap D$ in a neighborhood $\bar{U}_\nu \Subset D$, with $\bar{U}_\nu \cap \bar{U}_\mu = \varnothing$ for $\nu \neq \mu$. Take

$$0 < \varepsilon < \min_{\bar{D} \setminus \cup_\nu U_\nu} \|f(z)\|$$

(note that $f(z) \neq 0$ in $D / \cup_\nu U_\nu$). Then each connected component of the cycle $\gamma(\varepsilon) = \{z \in D: |f_1(z)| = \cdots = |f_n(z)| = \varepsilon\}$ falls in one of the neighborhoods U_ν. Let

$$\gamma_\nu = \gamma(\varepsilon) \cap U_\nu \cdot \gamma(\varepsilon) = \sum_{a_\nu \in \mathfrak{G}_f \cap D} \gamma_\gamma.$$

We have $\Gamma \sim \gamma(\varepsilon)$ in $G_f = \{z \in G: f_1(z), \dots, f_n(z) = 0\}$, since $\Gamma - \gamma(\varepsilon) = \partial B$, where $B = \{z \in D: |f_j(z)| = \varepsilon\tau + \rho_j(1 - \tau),\ 0 \leqslant \tau \leqslant 1, j = 1, \dots, n\}$. By the Cauchy-Poincaré theorem,

$$\frac{1}{(2\pi i)^n} \int_\Gamma \frac{df}{f} = \sum_{a_\nu \in \mathfrak{S}_f \cap D} \frac{1}{(2\pi i)^n} \int_{\gamma_\nu} \frac{df}{f}. \tag{41}$$

If a_ν is a simple zero of (24), then the neighborhood U_ν can be chosen small enough that the mapping (24) is biholomorphic in it. Then, making the change of variables $w = f(z)$, we obtain

$$\frac{1}{(2\pi i)^n} \int_{\gamma_\nu} \frac{df(z)}{f(z)} = \frac{1}{(2\pi i)^n} \int_{\substack{|w_1| = \varepsilon \\ \cdots \\ |w_n| = \varepsilon}} \frac{dw}{w} = 1. \tag{42}$$

Let a_ν be a zero of multiplicity m_ν. By Proposition 1, there is a point $\zeta \in C_n$, $\|\zeta\| < \varepsilon$, such that (25) has m_ν simple zeros in U_ν, namely, $a_{\nu j}$, $j = 1, \dots, m_\nu$. According to Lemma 4 (with $g(z) = f(z) - \zeta$),

$$\begin{aligned} \gamma_\nu &= \{z \in U_\nu: |f_1(z)| = \cdots = |f_n(z)| = \varepsilon\} \\ &\sim \gamma_\nu(\delta) = \{z \in U_\nu: |f_j(z) - \zeta_j| = \delta, j = 1, \dots, n\} \end{aligned}$$

in $G_{f-\zeta}$ when $0 < \delta < \varepsilon - \|\zeta\|$. For δ sufficiently small the cycle $\gamma_\nu(\delta)$ splits into connected components $\gamma_{\nu j}$, $j = 1, \dots, m_\nu$, each of which lies in a neighborhood $U_{\nu j}$ of some zero $a_{\nu j}$ of (25) in which the system (25) is biholomorphic. Using (42), we then get that

$$\begin{aligned} \frac{1}{(2\pi i)^n} \int_{\gamma_\nu} \frac{d(f(z) - \zeta)}{f(z) - \zeta} &= \frac{1}{(2\pi i)^n} \int_{\gamma_\nu(\delta)} \frac{d(f - \zeta)}{f - \zeta} \\ &= \sum_{j=1}^{m_\nu} \frac{1}{(2\pi i)^n} \int_{\gamma_{\nu j}} \frac{d(f - \zeta)}{f - \zeta} = m_\nu. \end{aligned}$$

Passing to the limit as $\zeta \to 0$, we have

$$\frac{1}{(2\pi i)^n} \int_{\gamma_\nu} \frac{df}{f} = \lim_{\zeta \to 0} \frac{1}{(2\pi i)^n} \int_{\gamma_\nu} \frac{d(f - \zeta)}{f - \zeta} = m_\nu. \tag{43}$$

Theorem 6 follows from (41) and (43).

Proof of Theorem 5. First of all, note that $\Gamma \subset G_f$ under the conditions of Theorem 5. Indeed, $|f_2(z)| = \cdots = |f_n(z)| = \varepsilon > 0$ and $|f_1(z)| > \varepsilon > 0$ at the points of Γ, since $\|f\| > \varepsilon$ on ∂D. Take $D_1 = \{z \in D: |f_j(z)| < \varepsilon, j = 1, \dots, n\}$ and $\Gamma_1 = \{z \in D: |f_1(z)| = \cdots = |f_n(z)| = \varepsilon\}$. Obviously, $D_1 \cap \mathfrak{S}_f = D \cap \mathfrak{S}_f$.

We have $\Gamma \sim \Gamma_1$ in G_f, because $\Gamma - \Gamma_1 = \partial B$, where $B = \{z \in \bar{D}: |f_1(z)| \geqslant \varepsilon, |f_2(z)| = \cdots = |f_n(z)| = \varepsilon\}$. By the Cauchy-Poincaré theorem and Theorem 6, applied to D_1 and Γ_1, this gives us Theorem 5.

Theorem 7 can be proved in the same way as Theorems 5 and 6. Instead of (42) we then get

$$\frac{1}{(2\pi i)^n}\int_{\gamma_\nu}\varphi(z)\frac{df(z)}{f(z)} = \frac{1}{(2\pi i)^n}\int_{\substack{|w_j|=\varepsilon \\ j=1,\dots,n}}\varphi(f^{-1}(w))\frac{dw}{w} = \varphi(a_\nu),$$

and instead of (43)

$$\frac{1}{(2\pi i)^n}\int_{\gamma_\nu}\varphi\frac{df}{f} = \lim_{\zeta\to 0}\frac{1}{(2\pi i)^n}\int_{\gamma_\nu}\varphi\frac{d(f-\zeta)}{f-\zeta}$$
$$= \lim_{\zeta\to 0}\sum_{j=1}^{m_\nu}\varphi(a_{\nu j}) = m_\nu\cdot\varphi(a_\nu).$$

PROOF OF THEOREM 8. As we remarked in the proof of Lemma 4, $|f_j(z) - g_j(z)| < \rho_j$ in $\bar{D}$. Consequently,

$$\|f(z) - g(z)\| < \|f(z)\| \quad \text{on } \partial D = \bigcup_{j=1}^{n}\{z \in \partial D: f_j(z) = \rho_j\},$$

and, by Theorem 3 (see the remark in parentheses), the systems (24) and (38) thus have the same number of zeros in D. Since (see Lemma 4) $\mathfrak{E}_g \cap D = \mathfrak{E}_g \cap D_1$ for $\delta < \delta_0$, where $D_1 = \{z \in D: |g_j(z)| < \delta, j = 1,\dots,n\}$, Lemma 4 and Theorem 6 give us that

$$\frac{1}{(2\pi i)^n}\int_\Gamma\frac{dg}{g} = \frac{1}{(2\pi i)^n}\int_{\gamma(\delta)}\varphi\frac{dg}{g} = \sum_{b\in\mathfrak{E}_g\cap D} m_b\varphi(b),$$

i.e., (40).

§5. Application of the multiple logarithmic residue for determining implicit functions and inverting holomorphic mappings

With the help of the logarithmic residue for holomorphic functions of one variable it is easy to derive the Lagrange expansion [236] (the Bürmann-Lagrange expansion [202]) of a holomorphic function $f(z)$ in a series of powers of a variable ζ connected with z by the equation $z = a + \zeta\psi(z)$. Similarly, the multiple logarithmic residue can be used to generalize the Lagrange expansion to the multidimensional case (see [88], [173] and [241]; some generalizations are obtained in [169], [196], and elsewhere without using residues). Such a generalization is given below for arbitrary implicit functions.

5.1. ***Generalization of the Lagrange expansion to systems of implicit functions*** **([241], [242]).** We solve the following problem. Suppose that

$$F_j(w, z), \qquad j = 1, \ldots, n, \tag{44}$$

and $\Phi(w, z)$ are holomorphic functions of the variables $w = (w_1, \ldots, w_m)$ and $z = (z_1, \ldots, z_n)$ in a neighborhood of the point $(0, 0) \in C^{m+n}_{(w,z)}$, with

$$F_j(0,0) = 0, \quad j = 1, \ldots, n, \qquad \left.\frac{\partial(F)}{\partial(z)}\right|_{(0,0)} = \left.\frac{\partial(F_1, \ldots, F_n)}{\partial(z_1, \ldots, z_n)}\right|_{(0,0)} \neq 0.$$

It is required to find an explicit representation of the function $\Phi(w, \varphi(w))$, where $z = \varphi(w)$, $\varphi = (\varphi_1, \ldots, \varphi_n)$, is an implicit vector-valued function determined by the system of equations

$$F_j(w, z) = 0, \qquad j = 1, \ldots, n, \tag{45}$$

in a neighborhood of the point $(0, 0) \in C^{m+n}_{(w,z)}$. In particular (for $\Phi(w, z) \equiv z_j$), it is required to find implicit functions $z_j = \varphi_j(w)$, $j = 1, \ldots, n$, determined by (45). We shall assume that

$$\frac{\partial F_j(0,0)}{\partial z_k} = \delta_{jk}, \qquad j, k = 1, \ldots, n, \tag{46}$$

where δ_{jk} is the Kronecker symbol—otherwise, we replace (45) by the equivalent system

$$\left\| \frac{\partial F_j(0,0)}{\partial z_k} \right\|^{-1} F(w, z) = 0.$$

The following theorems hold under these assumptions.

THEOREM 1. *There exist numbers* $\delta > 0$ *and* $\varepsilon > 0$ *such that in the closed polydisk* $\overline{V}_\delta = \{w \ni C^m\colon |w_j| \leqslant \delta,\ j = 1, \ldots, m\}$ *the function* $\Phi(w, \varphi(w))$ *has the integral representation*

$$\Phi(w, \varphi(w)) = \frac{1}{(2\pi i)^n} \int_{\Gamma_\varepsilon} \frac{\Phi(w, \zeta)(\partial(F)/\partial(\zeta))\, d\zeta}{F(w, \zeta)}, \tag{47}$$

where $F = F_1 \cdots F_n$, $\Gamma_\varepsilon = \{\zeta \in C^n\colon |\zeta_1| = \cdots = |\zeta_n| = \varepsilon\}$, *and* $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n$. *If* $\Phi(w, z) \equiv z_j$, $j = 1, \ldots, n$, *then* (47) *gives an integral representation of the implicit functions* $z_j = \varphi_j(w) = \varphi_j(w_1, \ldots, w_n)$ *determined by the system* (45).

THEOREM 2. *Let $h(w) = (h_1(w), \dots, h_n(w))$ be a system of functions holomorphic in a neighborhood of the point $0 \in \mathbf{C}^m$, $h(0) = 0$. Then in some neighborhood of 0 the function $\Phi(w, \varphi(w))$ can be expressed by the following uniformly and absolutely convergent function series*:

$$\Phi(w, \varphi(w)) = \sum_{\beta \geqslant 0} \frac{(-1)^{|\beta|}}{\beta!} \frac{\partial^{|\beta|}}{\partial z^\beta} \left[\Phi(w, z) \cdot \theta^\beta(w, z) \frac{\partial(F)}{\partial(z)} \right] \bigg|_{z = h(w)}, \quad (48)$$

where $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index, $\beta_j \geqslant 0$, $|\beta| = \beta_1 + \cdots + \beta_n$, $\beta! = \beta_1! \cdots \beta_n!$, $\partial^{|\beta|}/\partial z^\beta = \partial^{|\beta|}/(\partial z_1^{\beta_1} \cdots \partial z_n^{\beta_n})$, and $\theta^\beta = \theta_1^{\beta_1} \cdots \theta_n^{\beta_n}$, with $\theta_j(w, z) = F_j(w, z) - z_j + h_j(w)$, $j = 1, \dots, n$.

REMARK 1. It is most often convenient to take either $h_j(w) = 0$ or $h_j(w) = -F_j(w, 0)$.

THEOREM 3. *In a neighborhood of the point $0 = (0, \dots, 0) \in \mathbf{C}^m$ the function $\Phi(w, \varphi(w))$ can be represented by a multiple power series*

$$\Phi(w, \varphi(w)) = \sum_{\alpha \geqslant 0} d_\alpha w^\alpha = \sum_{\alpha_j, \dots, \alpha_n = 0} d_{\alpha_1, \dots, \alpha_n} w_1^{\alpha_1} \cdots w_n^{\alpha_n}, \quad (49)$$

whose coefficients are determined by the formula

$$d_\alpha = \sum_{|\beta| \leqslant 2|\alpha|} \frac{(-1)^{|\beta|}}{\alpha! \beta!} \frac{\partial^{|\alpha| + |\beta|}}{\partial w^\alpha \partial z^\beta} \left[\Phi(w, z) g^\beta(w, z) \frac{\partial(F)}{\partial(z)} \right] \bigg|_{(0,0)}, \quad (50)$$

where $g^\beta = g_1^{\beta_1} \cdots g_n^{\beta_n}$, $g_j(w, z) = F_j(w, z) - z_j$, $j = 1, \dots, n$.

REMARK 2. The coefficients (50) can be represented in integral form (see (52)).

For $n = 1$ formulas (48) and (50) can be transformed to a form more convenient for computations.

PROPOSITION 4. *Suppose that $F(w, z)$, $\Phi(w, z)$ and $h(w)$ are holomorphic functions of the variables $w = (w_1, \dots, w_m) \in \mathbf{C}^m$ and $z \in \mathbf{C}^1$ in a neighborhood of the point $(0, 0) = (0, \dots, 0; 0) \in \mathbf{C}^{m+1}$, with $F(0, 0) = 0$ and $F'_z(0, 0) = 1$. Then the function $\Phi(w, \varphi(w))$, where $z = \varphi(w) = \varphi(w_1, \dots, w_m)$ is an implicit function determined by the equation $F(w, z) = 0$ in a neighborhood of $(0, 0)$, can be represented by the function series*

$$\Phi(w, \varphi(w)) = \Phi(w, h(w)) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^{k-1}}{\partial z^{k-1}} \left[\Phi'_z(w, z) \theta^k(w, z) \right] \Big|_{z = h(w)}, \quad (48')$$

where $\theta(w, z) = F(w, z) - z + h(w)$, *and also by a power series of the form* (49) *whose coefficients are determined by*

$$d_\alpha = \frac{1}{\alpha!}\frac{\partial^{|\alpha|}\Phi(0,0)}{\partial w^\alpha} + \sum_{k=1}^{2|\alpha|}\frac{(-1)^k}{k!\alpha!}\frac{\partial^{k+|\alpha|-1}}{\partial w^\alpha \partial z^{k-1}}\left[\Phi_z' \cdot g^k\right]\Big|_{(0,0)}, \qquad (50')$$

where $g(w, z) = F(w, z) - z$.

PROOF. Under the assumptions $F_j(0,0) = 0$ and (46) it follows that the Taylor series of the function $g_j(w, z) = F_j(w, z) - z_j$ ($j = 1,\dots, n$) does not contain a free term nor terms in the first powers of the variables $z_1,\dots, z_n$. Consequently, there exist real numbers $\varepsilon > 0$ and $\delta > 0$ such that $|g_j(w, z)| < \varepsilon$, $j = 1,\dots, n$, for any $w \in \bar{v}_\delta$ and $z \in \overline{U}_\varepsilon = \{z \in \mathbf{C}^n\colon |z_j| \leqslant \varepsilon, j = 1,\dots, n\}$. Since the system of functions $z = (z_1,\dots, z_n)$ has a unique simple zero (the point $0 = (0,\dots, 0)$) and the inequalities $|g_j(w, \zeta)| < |\zeta| = \varepsilon$ hold for $w \in \overline{V}_\delta$ and $\zeta \in \Gamma_\varepsilon$, it follows from Theorem 8 in §4 that for any $w \in \overline{V}_\delta$ there is a unique point $z = \varphi(w) \in U_\varepsilon = \{z \in \mathbf{C}^n\colon |z_j| < \varepsilon, j = 1,\dots, n\}$ satisfying (45), and the equality (47) holds by (40).

Moreover, the numbers ε and δ can be chosen in such a way that the inequalities $|h_j(w)| < \varepsilon/2$ and $|\theta_j(w, \zeta)| = |g_j(w, \zeta) - h_j(w)| < \varepsilon/2$, i.e., the inequalities $|\zeta_j - h_j(w)|/|\theta_j(w, \zeta)| < 1$, $j = 1,\dots, n$, hold for $w \in \overline{V}_\delta$ and $\zeta \in \Gamma_\varepsilon$. If we expand the fraction

$$\begin{aligned}\frac{1}{F} = \frac{1}{\prod_{j=1}^n F_j} &= \frac{1}{\prod_{j=1}^n\left[\zeta_j - h_i(w)\right]\left[1 - \theta_j(w, \zeta)/\left(\zeta_j - h_j(w)\right)\right]}\\ &= \frac{1}{[\zeta - h(w)][1 - \theta(w, \zeta)/(\zeta - h(w))]}\end{aligned}$$

under the integral sign in (47) in a multiple geometric series, which converges uniformly and absolutely on $\overline{V}_\delta \times \Gamma_\varepsilon$, and integrate termwise, we get the expansion

$$\Phi(w, \varphi(w)) = \sum_{\beta \geqslant 0}\frac{(-1)^{|\beta|}}{(2\pi i)^{|n|}}\int_{\Gamma_\varepsilon}\frac{\Phi(w, \zeta)\theta^\beta(w, \zeta)(\partial(F)/\partial(\zeta))}{[\zeta - h(w)]^{\beta + I}}\,d\zeta, \qquad (51)$$

where $\beta + I = (\beta_1 + 1,\dots, \beta_n + 1)$, or (48).

To represent $\Phi(w, \varphi(w))$ as a power series we apply the Taylor expansion with coefficients in integral form ([5], [202]) to (51) for $h(w) = 0$. This gives the series (49) with coefficients

$$d_\alpha = \frac{1}{(2\pi i)^m} \int_{\gamma_\delta} \frac{\Delta(\omega, \varphi(\omega))\, d\omega}{\omega^{\alpha+I}}$$
$$= \sum_{\beta \geq 0} \frac{(-1)^{|\beta|}}{(2\pi i)^{m+n}} \int_{\gamma_\delta \times \Gamma_\varepsilon} \frac{\Phi(\omega, \zeta) g^\beta(\omega, \zeta)(\partial(F)/\partial(\zeta))\, d\omega \wedge d\zeta}{\omega^{\alpha+I} \zeta^{\beta+I}}, \tag{52}$$

where $\gamma_\delta = \{\omega \in \mathbf{C}^m \colon |\omega_1| = \cdots = |\omega_m| = \delta\}$ and $d\omega = d\omega_1 \wedge \cdots \wedge d\omega_m$.

Taking account of the facts that $g_j(0,0) = 0$, that the Taylor series of the difference $g_j(w, z) - g_j(w, 0)$ does not have terms of order zero or one, and that $\int_{\gamma_\delta \times \Gamma_\delta} \omega^\alpha \zeta^\beta \, d\omega \wedge d\zeta \neq 0$ only when $\beta_1 = \cdots = \beta_n = \alpha_1 = \cdots = \alpha_m = -1$, we easily show that the terms of (52) are equal to zero when $|\beta| > 2|\alpha|$. Thus, (52) can be written in the form (50).

For $n = 1$ formula (48) can be represented in the form

$$\Phi(w, \varphi(w)) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial z^k} \left[\Phi(w, z) \cdot \theta^k(w, z) \left(1 + \frac{\partial \theta}{\partial z}\right) \right] \bigg|_{z=h(w)}$$
$$= \Phi(w, h(w))$$
$$+ \left[\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^{k-1}}{\partial z^{k-1}} (\Phi \cdot \theta^k) - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^{k-1}}{\partial z^{k-1}} \left(\Phi \frac{\partial \theta^k}{\partial z} \right) \right] \bigg|_{z=h(w)},$$

or (48′). In exactly the same way, (50′) follows from (50).

5.2. ***Local inversion of a holomorphic mapping in*** $\mathbf{C}^n$. The following is a special case of the preceding problem. Let

$$w = f(z) = f(z_1, \ldots, z_n), \qquad f = (f_1, \ldots, f_n), \tag{53}$$

be a holomorphic mapping in a neighborhood of the point $0 \in \mathbf{C}^n$, with

$$f_j(0) = 0, \quad \frac{\partial f_i(0)}{\partial z_k} = \delta_{jk}, \qquad j, k = 1, \ldots, n. \tag{54}$$

It is required to find the expansion of a function $\Phi(z)$ holomorphic in a neighborhood of 0 in a series of powers of the functions (53) and, in particular, a power series expansion of the mapping

$$z = \varphi(w) = \varphi(w_1, \ldots, w_n), \qquad \varphi = (\varphi_1, \ldots, \varphi_n) \tag{55}$$

inverse to the mapping (53) in a neighborhood of 0. We have

THEOREM 5 ([169], [241]). *Under the above assumptions the function* $\Phi(\varphi(w))$ *can be represented as the function series*

$$\Phi(\varphi(w)) = \sum_{\beta\geqslant 0} \frac{(-1)^{|\beta|}}{\beta!} \cdot \frac{\partial^{|\beta|}}{\partial z^{\beta}} \left[\Phi(z)\theta^{\beta}(z)\frac{\partial(f)}{\partial(z)}\right]_{z=w}, \tag{56}$$

where $\theta^{\beta} = \theta_1^{\beta_1}\cdots\theta_n^{\beta_n}$, $\theta_j(z) = f_j(z) - z_j$, *as well as by the power series*

$$\Phi(\varphi(w)) = \sum_{\alpha\geqslant 0} d_{\alpha}w^{\alpha} = \sum_{\alpha_1,\ldots,\alpha_n=0}^{\infty} d_{\alpha_1\cdots\alpha_n}w^{\alpha_1}\cdots w^{\alpha_n}, \tag{57}$$

with coefficients

$$d_{\alpha} = \sum_{|\beta|\leqslant|\alpha|} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \frac{\partial^{|\alpha|+|\beta|}}{\partial z^{\alpha+\beta}} \left[\Phi(z)\theta^{\beta}(z)\frac{\partial(f)}{\partial(z)}\right]\Bigg|_0. \tag{58}$$

PROOF. If $m = n$, $h_j(w) = w_j$, and $F_j(w, z) = f_j(z) - w_j$ in Theorem 2, then (56) follows. Applying the Taylor expansion to (56) and taking into account that $\theta_j(0) = \partial\theta_j(0)/\partial z_k = 0$, $j, k = 1,\ldots, n$, we get the series (57) with the coefficients in (58).

COROLLARY 6. *If* $\Phi(z) \equiv z_j$, $j = 1,\ldots, n$, *then* (56) *as well as* (57) *and* (58) *give formulas for inverting the mapping* (53).

THEOREM 7 ([88], [173], [243]). *If the components of the mapping* (52) *have the form*

$$w_j = f_j(z) = z_j\psi_j(z), \quad \psi_j(0) \neq 0, \qquad j = 1,\ldots, n, \tag{59}$$

then the following expansion is valid:

$$\Phi(\varphi(w)) = \sum_{\beta\geqslant 0} \frac{w^{\beta}}{\beta!} \frac{\partial^{|\beta|}}{\partial z^{\beta}} \left[\frac{\Phi(z)\cdot(\partial(f)/\partial(z))}{\psi^{\beta}(z)}\right]_{z=0}. \tag{60}$$

PROOF. The system (59) can be written in the form

$$z - \frac{w_j}{\psi_j(z)} = 0, \qquad j = 1,\ldots, n.$$

Then (60) follows from (48) with $\theta_j(w, z) = -(w_j/\psi_j(z))$ and $h_j(w) = 0$.

5.3. ***Expression of the power series coefficients of implicit functions in terms of the power series coefficients of the original functions. Inversion formulas for a system of power series.*** Suppose that the functions (44) are given in the form of power series

$$F_j(w, z) = z_j + {\sum_{\mu,\nu}}' a_{j\mu\nu}w^{\mu}z^{\nu}, \qquad j = 1,\ldots, n. \tag{61}$$

It is assumed here and below that the multi-indices $\mu = (\mu_1, \ldots, \mu_m)$ and $\nu = (\nu_1, \ldots, \nu_n)$ in $\Sigma'_{\mu,\nu}$, $\Pi'_{\mu,\nu}$ and $\{\ \}'_{\mu,\nu}$ range through all possible tuples of nonnegative integers such that either $|\mu| + |\nu| \geqslant 2$ or $\nu = 0 = (0, \ldots, 0)$, $|\mu| = 1$. Let

$$\Phi(w, z) = \sum_{(\alpha,\beta)\geqslant(0,0)} b_{\alpha,\beta} w^\alpha z^\beta. \tag{62}$$

THEOREM 8.[9] *The coefficients of the power series expansion* (49) *of* $\Phi(w, \varphi(w))$, *where* $z = \varphi(w)$ *is an implicit vector-valued function determined by* (45), *can be expressed in terms of the coefficients of* (61) *and* (62) *by the formula*

$$d_\alpha = \sum_{(\eta,\tau),S} (-1)^{|p(S)|} \cdot b_{\eta\tau} \cdot D(S) \prod_{j=1}^{n} \left[\rho(S_j) - 1\right]! \cdot {\prod_{\mu,\nu}}' \frac{1}{s_{j\mu\nu}!} (a_{j\mu\nu})^{s_{j\mu\nu}}, \tag{63}$$

where $S = (S_1, \ldots, S_n)$, $S_j = \{s_{j\mu\nu}\}'_{\mu,\nu}$ *is a collection of nonnegative integers* $s_{j\mu\nu}$, *only finitely many of which are nonzero,*

$$p(S) = (\rho(S_1), \ldots, \rho(S_r)), \qquad \rho(S_j) = {\sum_{\mu,\nu}}' s_{j\mu\nu},$$

$$D(S) = \det\left\|\delta_{kj}\rho(S_j) - \sigma_k(S_j)\right\|_{j,k=1,\ldots,n},$$

where $\sigma_k(S_j) = \Sigma'_{\mu,\nu} \nu_k s_{j\mu\nu}$, *and* δ_{kj} *is the Kronecker symbol*; *the summation in* $\Sigma_{(\eta,\tau),S}$ *is over all multi-indices* $\eta = (\eta_1, \ldots, \eta_m)$ *and* $\tau = (\tau_1, \ldots, \tau_n)$, *and all systems* $S = (S_1, \ldots, S_n)$ *satisfying the conditions*

$$\begin{aligned} &\eta_k + \sum_{j=1}^{n} \lambda_k(S_j) = \alpha_k, \qquad k = 1, \ldots, m, \\ &\tau_k + \sum_{j=1}^{n} \sigma_k(S_j) = \rho(S_k), \qquad k = 1, \ldots, n, \end{aligned} \tag{64}$$

where $\lambda_k(S_j) = \Sigma'_{\mu,\nu} \mu_k s_{j\mu\nu}$. *Moreover, if* S *contains an* S_j *for which* $\rho(S_j) = 0$ (*i.e.*, $s_{j\mu\nu} = 0$ *for all* $s_{j\mu\nu} \in S_j$), *then the product*

$$\left[\rho(S_j) - 1\right]! \prod_{\mu,\nu} \frac{1}{s_{j\mu\nu}!} a_{j\mu\nu}^{s_{j\mu\nu}}$$

must be replaced by 1 *in the corresponding term of* (63), *and* $\delta_{kj} \cdot \rho(S_j) - \sigma_k(S_j)$ *by* δ_{kj} *in* $D(S)$.

([9]) Theorem 8 was obtained by the author jointly with V. A. Bolotov. The proof given here is due to the author. Bolotov obtained (63) from (58).

REMARK 3. If $\Phi(w, z) \equiv z_j$, $j = 1, \dots, n$, then (63) yields formulas for the power series coefficients of the implicit functions determined by (45). In this case the summation over η and τ disappears in (63), and (64) takes the form

$$\begin{aligned} \sum_{j=1}^{n} \lambda_k(S_j) &= \alpha_k, \qquad k = 1, \dots, m, \\ \delta_{jk} + \sum_{j=1}^{n} \sigma_k(S_j) &= \rho(S_k), \qquad k = 1, \dots, n. \end{aligned} \tag{65}$$

PROOF. Starting with the integral representation (47) and the first of the equalities (52), we get the following formula for the coefficients d_α in (49):

$$d_\alpha = \frac{1}{(2\pi i)^{m+n}} \int_{\gamma_\delta \times \Gamma_\varepsilon} \frac{\Phi(\omega, \zeta)(\partial(F)/\partial(\zeta))\, d\omega \wedge d\zeta}{F(\omega, \zeta) \cdot \omega^{\alpha+I}}. \tag{66}$$

For suitable choices of $\varepsilon > 0$ and $\delta > 0$ the function

$$\frac{\Phi(\varphi, \zeta) \cdot (\partial(F)/\partial(\zeta))}{F(\omega, \zeta)} = \Phi(\omega, \zeta) \frac{\partial(\ln F(\omega, \zeta))}{\partial(\zeta)} \tag{67}$$

is holomorphic on the $(m + n)$-circular compact set $\overline{V}_\delta \times \Gamma_\varepsilon \subset \mathbf{C}^{m+n}$, and can be expanded on this set in an absolutely and uniformly convergent multiple Laurent series. It follows from (66) that the coefficient d_α is equal to the coefficient of $\omega^\alpha \zeta^{-I} = \omega_1^{\alpha_1} \cdots \omega_m^{\alpha_m} \zeta_1^{-1} \cdots \zeta_n^{-1}$ in this series. The Laurent series expansion of the function (67) can be found directly. Taking into account that the inequalities $|g_j(\omega, \zeta)| < |\zeta_j| = \varepsilon$, $j = 1, \dots, n$, hold for $\omega \in \overline{V}_\delta$ and $\zeta \in \Gamma$, we get the expansions

$$\begin{aligned} \ln F_j(\omega, \zeta) &= \ln \zeta_j + \ln\left(1 + \frac{g_j(\omega, \zeta)}{\zeta_j}\right) \\ &= \ln \zeta_j + \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p!\zeta_j^p} [g_j(\omega, \zeta)]^p, \qquad j = 1, \dots, n. \end{aligned} \tag{68}$$

Substituting

$$g_j(\omega, \zeta) = \sum_{\mu, \nu}{}' a_{j\mu\nu} \omega^\mu \zeta^\nu$$

into (68), and then the resulting expansions and the series (62) into (67), we get the desired Laurent series after appropriate transformations, and this gives us (63).

As a consequence of the theorems in §5.1 we obtain formulas for inverting systems of power series.

Suppose that the functions of the system (53) and $\Phi(z)$ are given in the form of power series

$$w_j = f_j(z) = z_j + \sum_{|\alpha|\geqslant 2} a_{j\alpha} z^\alpha, \qquad j = 1, \ldots, n, \tag{69}$$

$$\Phi(z) = \sum_{\alpha \geqslant 0} b_\alpha z^\alpha, \tag{70}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$. We have

THEOREM 9 ([43], [196], [225]). *The expansion coefficients in the power series* (57) *for the function* $\Phi(\varphi(w))$, *where* $z = \varphi(w)$ *is the inverse of the mapping* (53), *can be expressed in terms of the coefficients of the power series* (69) *and* (70) *by the formula*

$$d_\alpha = \sum_{\tau, S} (-1)^{|\rho(S)|} b_\tau \cdot \Delta(S) \prod_{j=1}^{n} \frac{[\rho(S_j) + \alpha_j - 1]!}{\alpha_j!} \prod_{|\nu|\geqslant 2} \frac{1}{s_{j\nu}!} (a_{j\nu})^{s_{j\nu}}, \tag{71}$$

where $S = (S_1, \ldots, S_n)$, $S_j = \{s_{j\nu}\}_{|\nu|\geqslant 2}$ *is a collection of nonnegative integers* $s_{j\nu}$, $\nu = (\nu_1, \ldots, \nu_n)$, *of which only finitely many are nonzero*, $\rho(S) = (\rho(S_1), \ldots, \rho(S_n))$,

$$\rho(S_j) = \sum_{|\nu|\geqslant 2} \nu_{j\nu}, \qquad j = 1, \ldots, n,$$

$$\Delta(S) = \det\left\| \delta_{kj}[\alpha_j + \rho(S_j)] - \sigma_k(S_j) \right\|_{j,k=1,\ldots,n},$$

and $\sigma_k(S_j) = \sum_{|\nu|\geqslant 2} \nu_k s_{j\nu}$, $j, k = 1, \ldots, n$; *the summation in* $\sum_{\tau, S}$ *is over all possible multi-indices* $\tau = (\tau_1, \ldots, \tau_n)$, $\tau_j \geqslant 0$, *and all collections* S, $\rho(S) \geqslant 0$, *satisfying the conditions* $\tau_k + \sum_{j=1}^n \sigma_k(S_j) = \alpha_k + \rho(S_k)$, $k = 1, \ldots, n$. *Moreover, if* $\rho(S_j) = 0$ *and* $\alpha_j = 0$, *then the product*

$$[\rho(S_j) + \alpha_j - 1]! \prod_{|\nu|\geqslant 2} \frac{1}{s_{j\nu}!} (a_{j\nu})^{s_{j\nu}}$$

must be replaced by 1 *in the corresponding term of* (71), *and* $\delta_{kj}[\alpha_j + \rho(S_j)] - \sigma_k(S_j)$ *must be replaced by* δ_{kj} *in the determinant* $\Delta(S)$.

REMARK 4. If $\Phi(z) \equiv z_j$, $j = 1, \ldots, n$, then (57) and (71) give formulas for inverting the system of power series (69).

5.4. ***A case of isolating a single-valued regular branch of a system of implicit functions*** **[241]**. We consider without proof a case of isolating a single-valued branch of an implicit vector-valued function determined by the system (45) in a neighborhood of the point $(0,0) \in \mathbf{C}^{m+n}_{(w,z)}$ when

$$\left.\frac{\partial(F)}{\partial(z)}\right|_{(0,0)} = 0.$$

Suppose that the functions of the system (44) can be represented in a neighborhood of $(0,0)$ in the form

$$F_j(w, z) = z_j \mathcal{P}_j(w, z) + \theta_j(w, z), \qquad j = 1, \ldots, n, \tag{72}$$

where $z_j \cdot \mathcal{P}_j(w, z)$ is the homogeneous polynomial of lowest degree in the expansion of F_j at $(0,0)$ in a series of homogeneous polynomials, $\mathcal{P}_j(w,0) \not\equiv 0$; let $w^{\lambda(j)} = w_1^{\lambda_{j1}} \cdots w_m^{\lambda_{jm}}$ be the leading term of $\mathcal{P}_j(w,0)$ in the lexicographical order of its terms (the conditions $\alpha_1 = \lambda_{j,1}, \ldots, \alpha_{k-1} = \lambda_{j,k-1}$ hold for some $k, 1 \leqslant k \leqslant m$, for any term $a_{j\alpha} w^\alpha$ other than $w^{\lambda(j)}$). The coefficient of $w^{\lambda(j)}$ is assumed to be 1. Let

$$\begin{gathered} g_j(w, z) = F_j(w, z) - z_j w^{\lambda(j)}, \qquad j = 1, \ldots, n, \\ \lambda(\beta + I) = (\lambda_{11}(\beta_1 + 1) + \cdots + \lambda_{n1}(\beta_n + 1), \ldots, \lambda_{1m}(\beta_1 + 1) \\ + \cdots + \lambda_{nm}(\beta_n + 1)), \\ \gamma_\delta = \{\omega \in \mathbf{C}^m : |w_j| = \delta, j = 1, \ldots, m\}, \\ \Gamma_\varepsilon = \{\zeta \in \mathbf{C}^n : |\zeta_1| = \cdots = |\zeta_n| = \varepsilon\}, \end{gathered}$$

where δ and ε are sufficiently small positive numbers. Under these assumptions we have

THEOREM 10 [241]. *For the existence of a system of holomorphic functions*

$$z_j = \varphi_j(w) = \varphi_j(w_1, \ldots, w_m), \qquad j = 1, \ldots, n, \tag{73}$$

satisfying the system (45) *in a neighborhoood of the point* $(0,0) \in \mathbf{C}^{m+n}_{(w,z)}$ *and the conditions*

$$\varphi_j(0) = 0, \quad \frac{\partial \varphi_j(0)}{\partial w_k} = 0, \qquad j, k = 1, \ldots, n, \tag{74}$$

it is necessary and sufficient that

$$\sum_{\beta \geqslant 0} \frac{(-1)^{|\beta|}}{(2\pi i)^{m+n}} \int_{\gamma_\delta \times \Gamma_\varepsilon} \frac{\zeta_j g^\beta(w, \zeta)(\partial(F)/\partial(\zeta))\, d\omega \wedge d\zeta}{\zeta^{\beta + I} \omega^{\lambda(\beta + I) + \alpha + I}} = 0 \tag{75}$$

for all integer multi-indices $\alpha = (\alpha_1, \ldots, \alpha_m)$, $-\infty < \alpha_j < \infty$, *such that* $|\alpha| \geqslant 2$ *and* $\alpha_j < 0$ *for at least one* $j \in \{1, \ldots, m\}$. *Under the conditions* (75) *the* *expansion coefficients in the power series*

$$\Phi(w, \varphi(w)) = \sum_{\alpha \geqslant 0} d_\alpha w^\alpha = \sum_{\alpha_1, \ldots, \alpha_n = 0}^{\infty} d_{\alpha_1, \ldots, \alpha_n} w_1^{\alpha_1} \cdots w_m^{\alpha_m}, \tag{76}$$

where $\Phi(w, z)$ *is a homomorphic function in a neighborhood of the point* $(0,0) \in \mathbf{C}^{m+n}_{(w,z)}$ *and* $\varphi = (\varphi_1, \ldots, \varphi_n)$ *is the system* (73), *are determined by*

$$d_\alpha = \sum_{\beta \geqslant 0} \frac{(-1)^{|\beta|}}{\beta![\lambda(\beta + I) + \alpha]!} \frac{\partial^{|\beta| + |\lambda(\beta + I) + \alpha|}}{\partial z^\beta \partial w^{\alpha + \lambda(\beta + I)}} \left[\Phi \cdot g^\beta \frac{\partial(F)}{\partial(z)}\right]\Bigg|_{(0,0)}. \tag{77}$$

(*There are only finitely many nonzero terms in* (75) *and* (77).)

COROLLARY 11. *Under the conditions of Theorem* 10 *and the conditions* (75), *a single-valued branch* (73) *of the implicit vector-valued function determined by the system* (45) *is represented by power series of the form* (76), *where the coefficients are determined by* (77) *with* $\Phi(w, z) = z_j$.

COROLLARY 12. *If* $m = 1$ *in the conditions of Theorem* 10, *then the conditions* (71) *always hold, i.e., there exists an implicit vector-valued function satisfying* (45) *and* (74).

5.5. ***Examples.*** 1. Find an implicit function $z = \varphi(w)$ determined in a neighborhood of the point $(0, 0) \in \mathbf{C}^2$ by the equation

$$z - w + z^p w^q = 0.$$

Setting $m = 1$, $\Phi(w, z) = z$, and $g(w, z) = -w + z^p w^q$ in (49) and (50), we get

$$z = \varphi(w) = \sum_{n=1}^{\infty} c_n w^n,$$

where

$$c_n = \sum_{k=0}^{2n} \frac{1}{k!n!} \frac{\partial^{n+k-1}}{\partial z^{k-1} \partial w^n} \left[(w - z^p w^q)^k\right]\Big|_{(0,0)}$$

$$= \begin{cases} \dfrac{(-1)^r (pr)!}{r!(pr - r + 1)!} & \text{if } n = pr + qr - r + 1, r = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$z = \varphi(w) = \sum_{r=0}^{\infty} \frac{(-1)^r (pr)!}{r!(pr - r - 1)!} w^{(p+q-1)r+1}.$$

2. Find regular branches of the implicit function determined by the equation $z^3 - 3wz + w^3 = 0$ in a neighborhood of the point $(0, 0) \in \mathbf{C}^2$. According to

Corollaries 11 and 12, a single-valued branch of the desired implicit function tangent to the line $z = 0$ is represented by the series

$$z = \sum_{n=2}^{\infty} c_n w^n,$$

where

$$c_n = \sum_{k \geqslant 0} \frac{1}{(2\pi i)^2} \int_{\substack{|\zeta|=\varepsilon \\ |w|=\sigma}} \frac{\zeta(\zeta^3 + w^3)^k (w - \zeta^2)\, dw \wedge d\zeta}{3^k \zeta^k w^{k+n+2}};$$

that is,

$$z = \sum_{r=0}^{\infty} \frac{(3r)!}{3^{3r+1} r!(2r+1)!} w^{3r+2} = \frac{w^2}{3} + \frac{w^5}{3^4} + \frac{w^8}{3^6} + \frac{4w^{11}}{3^9} + \cdots .$$

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Table M.
Integral Representations of Numbers

Suppose that α is an arbitrary complex number, m and k are positive integers, and $n, n_1, \ldots, n_k = 0, \pm 1, \pm 2, \ldots$, and let $(\alpha)_n = \alpha(\alpha - 1) \cdots (\alpha - n + 1), (\alpha)_0 = 1$.

I. The binomial coefficient.

a). *The one-dimensional case.*

Definition. 1) Arbitrary case:

$$\binom{\alpha}{n} = \begin{cases} 1, & n = 0, \\ \dfrac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}, & n = 1, 2, 3, \ldots, \\ 0, & n = -1, -2, -3, \ldots; \end{cases}$$

2) $\alpha = m$:

$$\binom{m}{n} = \begin{cases} 1, & n = 0, \\ \dfrac{m!}{n!(m - n)!}, & n = 1, 2, 3, \ldots, m, \\ 0, & n = -1, -2, -3, \ldots \text{ or } n > m; \end{cases}$$

3) $\alpha = -m$:

$$\binom{-m}{n} = (-1)^n \binom{m + n - 1}{n}, \qquad n = 0, \pm 1, \pm 2, \ldots;$$

4) $\alpha = -1/2$:

$$(-1)^n \binom{-1/2}{n} = 4^{-n} \binom{2n}{n} = \frac{(2n - 1)!!}{(2n)!!}.$$

Integral representation, the formula $M_1 = M_1(w)$.

1)

$$\binom{\alpha}{n} = \operatorname*{res}_w (1 + w)^\alpha w^{-n-1} = \frac{1}{2\pi i} \int_{|w| = \rho} (1 + w)^\alpha w^{-n-1} \, dw,$$

$$0 < \rho < 1;$$

2)

$$\binom{m}{n} = \operatorname*{res}_w (1+w)^m w^{-n-1} = \frac{1}{2\pi i}\int_{|w|=\rho}(1+w)^m w^{-n-1}\,dw, \qquad 0<\rho<\infty,$$

or, the identity $\binom{m}{n} = \binom{m}{m-n}$, a frequently used representation is

$$\binom{m}{n} = \operatorname*{res}_w (1+w)^m w^{-m+n-1} = \frac{1}{2\pi i}\int_{|w|=\rho}(1+w)^m w^{-m+n-1}\,dw, \qquad 0<\rho<\infty;$$

3)

$$\binom{m+n-1}{n} = \operatorname*{res}_w (1-w)^{-m} w^{-n-1} = \frac{1}{2\pi i}\int_{|w|=\rho}(1-w)^{-m} w^{-n-1}\,dw, \qquad 0<\rho<1,$$

or, the identity $\binom{m+n-1}{n} = \binom{m+n-1}{m-1}$, a frequently used representation is ($n = 0, 1, \dots$),

$$\binom{m+n-1}{n} = \operatorname*{res}_w (1-w)^{-n-1} w^{-m} = \frac{1}{2\pi i}\int_{|w|=\rho}(1-w)^{-n-1} w^{-m}\,dw, \qquad 0<\rho<1;$$

4)

$$\binom{2n}{n} = \operatorname*{res}_w (1-4w)^{-1/2} w^{-n-1} = \frac{1}{2\pi i}\int_{|w|=\rho}(1-4w)^{-1/2} w^{-n-1}\,dw, \qquad 0<\rho<1.$$

b) *Multidimensional case.*

Definition. 1) Arbitrary case:

$$\binom{\alpha}{n_1,\dots,n_k} = \begin{cases} 1, & n_1=n_2=\cdots=n_k=0, \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-n_1-\cdots-n_k+1)}{n_1!n_2!\cdots n_k!}, & \\ n_1,\dots,n_k=0,1,\dots,\ 1\leqslant n_1+\cdots+n_k, & \\ 0, & \text{otherwise;} \end{cases}$$

2) $\alpha = m$:

$$\binom{m}{n_1,\dots,n_k} = \begin{cases} 1, & n_1 = n_2 = \cdots = n_k = 0, \\ \dfrac{m!}{n_1! \cdots n_k!(m - n_1 - \cdots - n_k)!}, & \\ n_1, n_2, \dots, n_k = 0, 1, 2, \dots, \ 1 \leqslant n_1 + \cdots + n_k \leqslant m, & \\ 0, & \text{otherwise;} \end{cases}$$

3) $\alpha = -m$:

$$\binom{-m}{n_1,\dots,n_k} = (-1)^{n_1+\cdots+n_k}\binom{m + n_1 + \cdots + n_k - 1}{n_1,\dots,n_k},$$

$$n_1, n_2, \dots, n_k = 0, \pm 1, \pm 2, \dots.$$

Integral representation, the formula $M_1 = M_1(w_1,\dots,w_k)$:

1)

$$\binom{\alpha}{n_1,\dots,n_k} = \operatorname*{res}_{w_1,w_2,\dots,w_k} (1 + w_1 + \cdots + w_k)^{\alpha} w_1^{-n_1-1} \cdots w_k^{-n_k-1}$$

$$= \frac{1}{(2\pi i)^k}\int_{\Gamma(\rho)} (1 + w_1 + \cdots + w_k)^{\alpha} w_1^{-n_1-1} \cdots w_k^{-n_k-1}\, dw,$$

where $\Gamma(\rho) = \{w = (w_1,\dots,w_k);\ |w_i| = \rho_i,\ 0 < \rho_i < 1,\ i = 1,\dots,k\}$, the skeleton of a polydisk;

2)

$$\binom{m}{n_1,\dots,n_k} = \operatorname*{res}_{w_1,\dots,w_k} (1 + w_1 + \cdots + w_k)^{m} w_1^{-n_1-1} \cdots w_k^{-n_k-1}$$

$$= \frac{1}{(2\pi i)^k}\int_{\Gamma(\rho)} (1 + w_1 + \cdots + w_k)^{m} w_1^{-n_1-1} \cdots w_k^{-n_k-1}\, dw,$$

where $\Gamma(\rho) = \{w = (w_1,\dots,w_k);\ |w_i| = \rho_i,\ 0 < \rho_i < \infty,\ i = 1,\dots,k\}$, the skeleton of a polydisk;

3)

$$\binom{m + n_1 + \cdots + n_k - 1}{n_1,\dots,n_k}$$

$$= \operatorname*{res}_{w_1,\dots,w_k} (1 - w_1 - \cdots - w_k)^{-m} w_1^{-n_1-1} \cdots w_k^{-n_k-1}$$

$$= \frac{1}{(2\pi i)^k}\int_{\Gamma(\rho)} (1 - w_1 - \cdots - w_k)^{-m} w_1^{-n_1-1} \cdots w_k^{-n_k-1}\, dw,$$

where $\Gamma(\rho) = \{w = (w_1,\dots,w_k);\ |w_i| = \rho_i,\ 0 < \rho_j < 1,\ i = 1,\dots,k\}$, the skeleton of a polydisk.

II. The expansion coefficients of the logarithmic function $-\ln(1-w)$.
Definition and integral representation, the formula $M_2 = M_2(w)$:

$$\left.\begin{array}{ll} 0, & n=0, \\ 1/n, & n=1,2,3,\dots \end{array}\right\} = \operatorname*{res}_w(-\ln(1-w))w^{-n-1}$$
$$= -\frac{1}{2\pi i}\int_{|w|=\rho} \ln(1-w)w^{-n-1}\,dw, \qquad 0<\rho<1.$$

III. The expansion coefficients of the exponential function $\exp(\alpha w)$.
Definition and integral representation, the formula $M_3 = M_3(w)$:

$$\frac{\alpha^n}{n!} = \operatorname*{res}_w(e^{\alpha w}w^{-n-1}) = \frac{1}{2\pi i}\int_{|w|=\rho} e^{\alpha w}w^{-n-1}\,dw,$$
$$0<\rho<\infty,\ n=0,1,2,\dots.$$

IV. The (ordinary) Euler numbers E_n, $n = 0,1,2,\dots$.
Definition and integral representation, the formula $M_4 = M_4(w)$:

$$E_n = n!\operatorname*{res}_w \cosh^{-1}(w)w^{-n-1} = n!\frac{1}{2\pi i}\int_{|w|=\rho} \cosh^{-1}(w)w^{-n-1}\,dw,$$
$$0<\rho<\infty,\ n=0,1,2,\dots.$$

V. The Bernoulli numbers B_n, $n = 0,1,2,\dots$.
Definition and integral representation, the formula $M_5 = M_5(w)$:

$$B_n = n!\operatorname*{res}_w(e^w-1)^{-1}w^{-n} = \frac{1}{2\pi i}\int_{|w|=\rho}(e^w-1)^{-1}w^{-n}\,dw,$$
$$0<\rho<\infty,\ n=0,1,2,\dots.$$

VI. The Stirling numbers $s_1(m,n)$ **of the first kind,** $n = 0,1,2,\dots$.
Definition:

$$(w)_m = \sum_{n=0}^{\infty} s_1(m,n)w^n, \qquad s_1(0,0)=1.$$

Integral representation, the formula $M_6 = M_6(w)$:

$$s_1(m,n) = \operatorname*{res}_w(w)_m w^{-n-1} = \frac{m!}{n!}\operatorname*{res}_w[\ln(1+w)]^n w^{-m-1}$$
$$= \frac{m!}{n!}\frac{1}{2\pi i}\int_{|w|=\rho}[\ln(1+w)]^n w^{-m-1}\,dw, \qquad 0<\rho<1,\ n=0,1,2,\dots.$$

VII. The Stirling numbers $s_2(m, n)$ of the second kind, $n = 0, 1, 2, \ldots$.
Definition:

$$w^m = \sum_{n=0}^{\infty} s_2(m, n)(w)_n, \qquad s_2(0,0) = 1.$$

Integral representation, the formula $M_7 = M_7(w)$:

$$s_2(m, n) = \frac{m!}{n!} \operatorname*{res}_w (e^w - 1)^n w^{-m-1} = \frac{m!}{n!} \frac{1}{2\pi i} \int_{|w|=\rho} (e^w - 1)^n w^{-m-1}\, dw,$$

$$0 < \rho < \infty, m = 0, 1, 2, \ldots.$$

VIII. The gamma function $\Gamma(z)$.
Definition (integral representation), the formula $M_8 = M_8(s)$:

$$\Gamma(z) = \int_0^{\infty} e^{-s} s^{z-1}\, ds, \qquad \operatorname{Re} z \geqslant 0;$$

a particular case is

$$n! = \Gamma(n+1) = \int_0^{\infty} e^{-s} s^n\, ds, \qquad n = 0, 1, 2, \ldots.$$

IX. The beta function $B(u, v)$.
Definition (integral representation), the formula $M_9 = M_9(t)$:

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v+1)} = \int_0^1 t^{u-1}(1-t)^{v-1}\, dt, \quad \operatorname{Re} u > 0, \quad \operatorname{Re} v > 0;$$

a particular case is

$$\frac{n!m!}{(n+m+1)!} = B(n+1, m+1) = \int_0^1 t^n (1-t)^m\, dt, \qquad n, m = 0, 1, 2, \ldots.$$

X. The generalized Stirling numbers $s_2^{(\alpha)}(m, n, k)$ of the second kind, $n = 0, 1, \ldots$, and k is a positive integer ([129], [208]).
Definition (integral representation), the formula $M_{10} = M_{10}(w)$:

$$s_2^{(\alpha)}(m, n, k) = \frac{m!}{n!} \operatorname*{res}_w e^{\alpha w} \frac{(e^{kw} - 1)^n}{w^{m+1}}$$

$$= \frac{m!}{n!} \frac{1}{2\pi i} \int_{|w|=\rho} e^{\alpha w}(e^{kw} - 1)^n w^{-m-1}\, dw,$$

$$0 < \rho < \infty, n = 0, 1, 2, \ldots.$$

XI. The generalized Bernoulli numbers $B_n^{(m)}$, $n = 0, 1, 2, \ldots$ ([54], Chapter 15).

Definition (integral representation), the formula $M_{11} = M_{11}(w)$:

$$B_n^{(m)} = n! \operatorname*{res}_w (e^w - 1)^{-m} w^{-n+m-1}$$
$$= n! \frac{1}{2\pi i} \int_{|w|=\rho} (e^w - 1)^{-m} w^{-n+m-1}\, dw, \qquad 0 < \rho < \infty, n = 0, 1, 2, \ldots.$$

XII. The Euler numbers $E_n^{(m)}$ of mth order, $n = 0, 1, 2, \ldots$ [34].

Definition (integral representation), the formula $M_{12} = M_{12}(w)$:

$$E_n^{(m)} = n! \operatorname*{res}_w (\cosh w)^{-m} w^{-n-1} = n! \frac{1}{2\pi i} \int_{|w|=\rho} (\cosh w)^{-m} w^{-n-1}\, dw,$$
$$0 < \rho < \infty, n = 0, 1, 2, \ldots.$$

XIII. The generalized Euler numbers $A(n, k)$, $n, k = 0, 1, 2, \ldots$ [41].

Definition (integral representation), the formula $M_{13} = M_{13}(w, z)$:

$$A(n, k) = n! \operatorname*{res}_{w,z} \frac{(z-1)^{n+1}}{(ze^{-w} - 1)} w^{-n-1} z^{-k-1}$$
$$= \frac{n!}{(2\pi i)^2} \int_{\substack{|w|=\rho_1 \\ |z|=\rho_2}} \frac{(z-1)^{n+1}\, dw \wedge dz}{(ze^{-w} - 1) w^{n+1} z^{k+1}},$$
$$0 < \rho_1 < \infty, 0 < \rho_2 < 1, n, k = 0, 1, 2, \ldots$$

Bibliography

[1] Morton Abramson and W. O. J. Moser, *Combinations, successions and the n-kings problem*, Math. Mag. **39** (1966), 269–273.

[2] ______, *Arrays with fixed row and column sums*, Discrete Math. **6** (1973), 1–14.

[3] L. A. Aĭzenberg, *Integral representations of functions holomorphic in n-circular domains ("Extension" of Szegö kernels)*, Mat. Sb. **65(107)** (1964), 104–143. (Russian)

[4] ______, *Extension of integral representations with Szegö kernels and quasi-kernels for n-circular domains*, Properties of Holomorphic Functions of Several Complex Variables, Inst. Fiz. Sibirsk. Otdel. Akad. Nauk SSSR, Krasnoyarsk, 1973, pp. 3–34. (Russian)

[5] L. A. Aĭzenberg and B. S. Zinov'ev, *Elementary properties and integral representations of holomorphic functions of several complex variables*, Krasnoyarsk. Gos. Univ., Krasnoyarsk, 1975. (Russian)

[6] P. S. Aleksandrov, *Combinatorial topology*, OGIZ, Moscow, 1947; English transl., Vols. 1, 2, 3, Graylock Press, Albany, N. Y., 1956, 1957, 1960.

[7] George E. Andrews, *On the foundations of combinatorial theory*. V, Studies in Appl. Math. **50** (1971), 345–375.

[8] ______, *On the q-analog of Kummer's theorem and applications*, Duke Math. J. **40** (1973), 525–528.

[9] ______, *Applications of basic hypergeometric functions*, SIAM Rev. **16** (1974), 441–484.

[10] ______, *Identities in combinatorics*. I, Discrete Math. **11** (1975), 97–106.

[11] T. L. Austin, *The enumeration of point labelled chromatic graphs and trees*, Canad. J. Math. **12** (1960), 535–545.

[12] A. K. Avramenok and G. P. Egorychev, *Integral representation and asymptotic estimation of solutions of certain enumeration problems*, Combinatorial and Asymptotic Analysis, vyp. 1, Krasnoyarsk. Gos. Univ., Krasnoyarsk, 1975, pp. 109–122. (Russian)

[13] S. Bachmuth, *Induced automorphisms of free groups and free metabelian groups*, Trans. Amer. Math. Soc. **122** (1966), 1–17.

[14] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Univ. Press, 1935.

[15] B. M. Bajšanski and R. Bojanić, *On two sums*, Bull. Soc. Math. Phys. Macédoine **5** (1954), 34–36 (1955). (Serbo-Croatian; English summary).

[16] A. Erdélyi et al. [Bateman Manuscript Project], *Higher transcendental functions*. Vol. I, McGraw-Hill, 1953.

[17] ______, *Higher transcendental functions*. Vol. III, McGraw-Hill, 1955.

[18] E. F. Beckenbach (Editor), *Applied combinatorial mathematics*, Wiley, 1964.

[19] L. W. Beineke and R. E. Pippert, *The number of labeled k-dimensional trees*, J. Combinatorial Theory **6** (1969), 200–205.

[20] E. T. Bell, *Euler algebra*, Trans. Amer. Math. Soc. **25** (1923), 135–154.

[21] ______, *Algebraic arithmetic*, Amer. Math. Soc., Providence, R. I., 1927.

[22] ______, *Postulational bases for the umbral calculus*, Amer. J. Math. **62** (1940), 717–724.

[23] C. Berge, *Principes de combinatoire*, Dunod, Paris, 1968; English transl., Academic Press, 1971.

[24] C. N. Bhaskaranada, *A problem in arrangement*, Math. Student **36** (1968), 237–239.

[25] Z. W. Birnbaum and Ronald Pyke, *On some distributions related to the statistic* D_n^+, Ann. Math. Statist. **29** (1958), 179–187.

[26] M. T. L. Bizley, *A note on some elementary derangement and allied problems*, J. Inst. Actuaries Studients' Soc. **16** (1960/62), 147–151.

[27] ______, *A generalization of Nanjundiah's identity*, Amer. Math. Monthly **77** (1970), 863–865.

[28] Salomon Bochner and William Ted Martin, *Several complex variables*, Princeton Univ. Press, Princeton, N. J., 1948.

[29] L. A. Bokut', *A basis for free polynilpotent Lie algebras*, Algebra i Logika **2** (1963), no. 4, 13–19. (Russian)

[30] M. A. Borodin, *Certain integral representations of functions holomorphic in doubly circular domains*, Sibirsk. Mat. Zh. **10** (1969), 287–296; English transl. in Siberian Math. J. **10** (1969).

[31] N. Bourbaki, *Algèbre*, Chaps. 4–6, Actualités Sci. Indust., nos. 1102, 1179, Hermann, Paris, 1950, 1952.

[32] Hans Bremermann, *Distributions, complex variables, and Fourier transforms*, Addison-Wesley, 1965.

[33] Renato Caccioppoli, *Residui di integrali doppi e intersezioni di curve analitiche*, Ann. Mat. Pura Appl. (4) **29** (1949), 1–14.

[34] L. Carlitz, *Recurrences for the Bernoulli and Euler numbers*, J. Reine Angew. Math. **214/215** (1964), 184–191.

[35] ______, *Multiple sums and generating functions*, Collect. Math. **17** (1965), 281–296.

[36] ______, *Generating functions*, Fibonacci Quart. **7** (1969), 359–393.

[37] ______, *Bernoulli numbers*, Fibonacci Quart. **6** (1968), 71–85.

[38] ______, *A binomial identity related to ballots and trees*, J. Combinatorial Theory Ser. A **14** (1973), 261–263.

[39] ______, *An application of MacMahon's master theorem*, SIAM J. Appl. Math. **26** (1974), 431–436.

[40] ______, *Some q-expansion formulas*, Glasnik Mat. Ser. III **8** (**28**) (1973), 205–214.

[41] L. Carlitz et al., *Asymptotic properties of Eulerian numbers*, Z. Wahrsch. Verw. Gebiete **23** (1972), 47–54.

[42] Henri Cartan, *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Hermann, Paris, 1961; English transl., Hermann, Paris, and Addison-Wesley, Reading, Mass., 1963.

[43] A. Cayley, *Note sur une formule pour la reversion des séries*, J. Reine Angew. Math. **52** (1856), 276–284.

[44] Bruce Chalmers, *Fonctions de plusieurs variables complexes. Sur le comportement, à la frontière, de la fonction-noyau de Bergman*, C. R. Acad. Sci. Paris Sér. A-B **266** (1968), A1132-A1134.

[45] N. G. Chebotarëv, *Theory of algebraic functions*, OGIZ, Moscow, 1948. (Russian)

[46] Laurent Chottin, *Une démonstration combinatoire de la formule de Lagrange à deux variables*, Discrete Math. **13** (1975), 215–224.

[47] C. A. Church, Jr., *On the enumeration of certain triangular arrays*, Fibonacci Quart. **8** (1970), 235–241.

[48] P. C. Consul and M. J. Kemp, *On an interesting property of a combinatorial function*, Canad. Math. Bull. **13** (1970), 533–536.

[49] P. C. Consul and L. R. Shenton, *Use of Lagrange expansion for generating discrete generalized probability distributions*, SIAM J. Appl. Math. **23** (1972), 239–249.

[50] ______, *On the multivariate generalization of the family of discrete Lagrange distributions*, Multivariate Statistical Inference (Proc. Res. Sem., Dalhousie Univ., Halifax, 1972), North-Holland, 1973, pp. 13–23.

[51] Miklós Csörgö, *Exact probability distribution functions of some Rényi type statistics*, Proc. Amer. Math. Soc. **16** (1965), 1158–1167.

[52] Dragoš M. Cvetković and Slobodan K. Simić, *On enumeration of certain types of sequences*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 412–460 (1973), 159–164.

[53] G. Darboux, *Mémoire sur l'approximation des fonctions de très grands nombres, et sur une classe étendue dem développements en série*, J. Math. Pures Appl. (3) **4** (1878), 5–56, 377–416.

[54] F. N. David and D. E. Barton, *Combinatorial chance*, Hafner, New York, 1962.

[55] Peter Doubilet, *On the foundations of combinatorial theory*. VII: *Symmetric functions through the theory of distribution and occupancy*, Studies in Appl. Math. **51** (1972), 377–396.

[56] Peter Doubilet, Gian-Carlo Rota and Richard Stanley, *On the foundations of combinatorial theory*. VI: *The idea of generating function*, Proc. Sixth Berkeley Sympos. Math. Statist and Probab. (1970/71), Vol. II, Univ. of California Press, Berkeley, Calif., 1972, pp. 267–318.

[57] Robert J. Douglas, *Tournaments that admit exactly one Hamiltonian circuit*, Proc. London Math. Soc. (3) **21** (1970), 716–730.

[58] Freeman J. Dyson, *Statistical theory of the energy levels of complex systems*, I, II, III, J. Mathematical Phys. **3** (1962), 140–156, 157–165, 166–175.

[58a] J. Gunson, *Proof of a conjecture by Dyson in the statistical theory of energy levels*, J. Math. Phys. **3** (1962), 752–753.

[58b] Kenneth G. Wilson, *Proof of a conjecture by Dyson*, J. Math. Physics **3** (1962), 1040–1043.

[59] A. M. Èfros and A. M. Danilevskiĭ, *Operational calculus and contour integrals*, Gos. Nauchno-Tekhn. Izdat. Ukrain., Kharkov, 1937. (Russian)

[60] G. P. Egorychev, *A method for classifying and obtaining numerical identities in combinatorics*, Seminar "Algebraic Systems", No. 3: Imbedding of Groups, Algorithmic Questions, Inst. Fiz. Sibirsk. Otdel. Akad. Nauk SSSR, Krasnoyarsk, 1970, pp. 94–115. (Russian)

[61] ______, *The interrelation between symbolic methods in combinatorics and the apparatus of generating functions*, ibid., pp. 116–130. (Russian)

[62] ______, *The ranks of the factors in the lower central series of a free solvable group*, Sibirsk. Mat. Zh. **13** (1972), 708–713; English transl. in Siberian Math. J. **13** (1972).

[63] ______, *The inversion of combinatorial relations*, Kombinatornyĭ Anal. Vyp. 3 (1974), 10–14. (Russian)

[64] ______, *The method of coefficients and the generating function for the number of nonisomorphic tournaments having a unique Hamiltonian circuit*, Kombinatornyĭ Anal. Vyp. **3** (1974), 5–9. (Russian)

[65] ______, *Inversion of one-dimensional combinatorial relations*, Some Questions on the Theory of Groups and Rings, Inst. Fiz. Sibirsk. Otdel. Akad. Nauk SSSR, Krasnoyarsk, 1973, pp. 110–122. (Russian)

[66] ______, *Application of multiple residues to the computation of Szegö and Bergman kernels for certain n-circular domains*, Some Properties of Holomorphic Functions of Several Complex Variables, Inst. Fiz. Sibirsk. Otdel. Akad. Nauk SSSR, Krasnoyarsk, 1973, pp. 55–76. (Russian)

[67] ______, *Combinatorial sums and the method of generating functions*, Krasnoyarsk. Gos. Univ., Krasnoyarsk, 1974. (Russian)

[68] ______, *Computation of certain combinatorial sums with linear constraints on the summation indices*, Sibirsk. Mat. Zh. **16** (1975), 856–862; English transl. in Sibirian Math. J. **16** (1975).

[69] ______, *Computation of Szegö and Bergman kernels for certain n-circular domains with the help of multidimensional residues*, Izv. Vyssh. Uchebn. Zaved. Matematika **1975**, no. 1(152), 101–103; English transl. in Soviet Math. **19** (1975).

[70] ______, *The method of coefficients (multidimensional case)*, Combinatorial and Asymptotic Analysis, vyp. I, Krasnoyarsk. Gos. Univ., Krasnoyarsk, 1975, pp. 151–161. (Russian)

[71] G. P. Egorychev and A. P. Yuzhakov, *Application of multidimensional residues to the determination of generating functions and the computation of combinatorial sums*, Holomorphic Functions of Several Complex Variables (Sem., Krasnoyarsk, 1972), Inst. Fiz. Sibirsk. Otdel. Akad. Nauk SSSR, Krasnoyarsk, 1972, pp. 47–61. (Russian)

[72] ______, *On the determination of generating functions and combinatorial sums with the help of multidimensional residues*, Sibirsk. Mat. Zh. **15** (1974), 1049–1060; English transl. in Siberian Math. J. **15** (1974).

[73] Ora Engelberg, *Generalizations of the ballot problem*, Z. Wahrsch. Verw. Gebiete **3** (1965), 271–275.

[74] R. C. Entringer, *A combinatorial interpretation of the Euler and Bernoulli numbers*, Nieuw Arch. Wisk. (3) **14** (1966), 241–246.

[75] William Feller, *An introduction to probability theory and its applications*. Vol. 1, 2nd ed., Wiley, 1957.

[76] Jonas Ekman Fjeldstad, *A generalization of Dixon's formula*, Math. Scand. **2** (1954), 46–48.

[77] Géza Freud, *Restglied eines Tauberschen Satzes*, Acta Math. Acad. Sci. Hungar. **2** (1951), 299–308.

[78] B. A. Fuks, *Introduction to the theory of analytic functions of several complex variables*, Fizmatgiz, Moscow, 1962; English transl., Amer. Math. Soc., Providence, R. I., 1963.

[79] ______, *Special chapters in the theory of analytic functions of several complex variables*, Fizmatgiz, Moscow, 1963; English transl., Amer. Math. Soc., Providence, R. I., 1965.

[80] D. B. Fuks, A. T. Fomenko, and V. L. Gutenmakher, *Homotopic topology*, 2nd ed., Moskov. Gos. Univ., Moscow, 1969. (Russian)

[81] A. O. Gel'fond, *Residues and their applications*, "Nauka", Moscow, 1966; English transl., "Mir", Moscow, 1971.

[82] ______, *The calculus of finite differences*, 3rd ed., "Nauka", Moscow, 1967; English transl., Hindustan, Delhi, 1971.

[83] A. O. Gel'fond and Yu. V. Linnik, *Elementary methods in analytic number theory*, Fizmatgiz, Moscow, 1962; English transl., Rand McNally, Chicago, Ill., 1965, and Pergamon Press, 1966.

[84] R. R. Goldberg, *Problem* 4805, Amer. Math. Monthly **65** (1958), 633.

[85] Jay Goldman and Gian-Carlo Rota, *On the foundations of combinatorial theory*. IV: *Finite vector spaces and Eulerian generating functions*, Studies in Appl. Math. **49** (1970), 239–258.

[86] V. L. Goncharov, *On the field of combinatory analysis*, Izv. Akad. Nauk SSSR Ser. Mat. **8** (1944), 3–48; English transl. in Amer. Math. Soc. Transl. (2) **19** (1962).

[87] I. J. Good, *Saddle-point methods for the multinomial distribution*, Ann. Math. Statist. **28** (1957), 861–881.

[88] ______, *Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes*, Proc. Cambridge Philos. Soc. **56** (1960), 367–380.

[89] ______, *Analysis of cumulative sums by multiple contour integration*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 115–122.

[90] ______, *The multivariate saddlepoint method and chi-squared for the multinomial distribution*, Ann. Math. Statist. **32** (1961), 535–548.

[91] ______, *Proofs of some 'binomial' identities by means of MacMahon's 'Master Theorem'*, Proc. Cambridge Philos. Soc. **58** (1962), 161–162.

[92] ______, *A short proof of MacMahon's 'Master Theorem'*, Proc. Cambridge Philos. Soc. **58** (1962), 160.

[93] ______, *The generalization of Lagrange's expansion and the enumeration of trees*, Proc. Cambridge Philos. Soc. **61** (1965), 499–517; correction, ibid. **64** (1968), 489.

[94] Yu. M. Gorchakov, *Basic commutators and a collection process*, Seminar "Algebraic Systems", No. 3, Imbedding of Groups, Algorithmic Questions, Inst. Fiz. Sibirsk. Otdel. Akad. Nauk SSSR, Krasnoyarsk, 1970, pp. 32–70. (Russian)

[95] Yu. M. Gorchakov and G. P. Egorychev, *The ranks of the factors in the lower central series of a free polynilpotent group*, Dokl. Akad. Nauk SSSR **204** (1972), 12–14; English transl. in Soviet Math. Dokl. **13** (1972).

[96] ______, *The ranks of the factors in the lower central series of a free polynilpotent group*, Some Questions in the Theory of Groups and Rings, Inst. Fiz. Sibirsk. Otdel. Akad. Nauk SSSR, Krasnoyarsk, 1973, pp. 91–109. (Russian)

[97] H. W. Gould, *Some generalizations of Vandermonde's convolution*, Amer. Math. Monthly **63** (1956), 84–91.

[98] ______, *Some relations involving the finite harmonic series*, Math. Magazine **34** (1960/61), 317–321.

[99] ______, *Note on a combinatorial identity in the theory of bi-colored graphs*, Fibonacci Quart. **5** (1967), 247–250.

[100] ______, *Note on two binomial coefficient identities of Rosenbaum*, J. Math. Phys. **10** (1969), 49–50.

[101] ______, *Combinatorial identities. A standardized set of tables listing* 500 *binomial coefficient summations*, Henry W. Gould, Morgantown, W. Va., 1972.

[102] ______, *Explicit formulas for Bernoulli numbers*, Amer. Math. Monthly **69** (1972), 44–51.

[103] ______, *The case of the strange binomial identities of Professor Moriarty*, Fibonacci Quart. **10** (1972), 381–391.

[104] ______, *The design of the four binomial identities: Moriarty intervenes*, Fibonacci Quart. **12** (1974), 300–308.

[105] Zakkula Govindarajulu and Yukio Suzuki, *A note on an identity involving binomial coefficients*, Ann. Inst. Statist. Math. **15** (1963), 83–85.

[106] I. S. Gradshtein and I. M. Ryzhik, *Tables of integrals, series, and products*, 4th ed., Fizmatgiz, Moscow, 1963; English transl., Academic Press, 1965.

[107] K. W. Gruenberg, *Residual properties of infinite soluble groups*, Proc. London Math. Soc. (3) **7** (1957), 29–62.

[108] Hansraj Gupta, *A summation problem*, Math. Student **22** (1954), 105–107.

[109] Johann G. Hagen, *Synopsis der hoeheren Mathematik*. Vol. I: *Arithmetische und algebraische Analyse*. Chapter III: "Theorie der combinationen", Verlag Felix L. Dames, Berlin, 1891, pp. 55–68.

[110] Peter Hagis, Jr., *An analytic proof of the formula for* F_n, Fibonacci Quart. **2** (1964), 267–268.

[111] Paul R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1950.

[112] Marshall Hall, Jr., *The theory of groups*, MacMillan, 1959.

[113]______, *A survey of combinatorial analysis*, Some Aspects of Analysis and Probability, Surveys in Appl. Math., Vol 4, Wiley, 1958, pp. 35–104.

[114] B. R. Handa, *Extensions of Vandermonde type convolutions with several summations and their applications*. II, Canad. Math. Bull. **12** (1969), 63–74.

[115] Frank Harary, *Graph theory*, Addison-Wesley, 1969.

[116] G. H. Hardy, *Divergent series*, Clarendon Press, Oxford, 1949.

[117] G. H. Hardy and J. E. Littlewood, *Notes on the theory of series*. XI: *On Tauberian theorems*, Proc. London Math. Soc. (2) **30** (1930), 23–37.

[118] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (2) **17** (1918), 75–115.

[119] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Clarendon Press, Oxford, 1938.

[120] M. L. J. Hautus and D. A. Klarner, *The diagonal of a double power series*, Duke Math. J. **38** (1971), 229–235.

[121] Hermite, *Sur quelques développements en série de fonctions de plusieurs variables*, C. R. Acad. Sci. Paris **60** (1865), 432–440.

[122] H. J. Hietala and B. B. Winter, *Note on a combinatorial identity*, Math. Magazine **38** (1965), 149–151.

[123] D. F. Holt, *Rooks inviolate*, Math. Gazette **58** (1974), 131–134.

[124] Lars Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, N. J., 1966.

[125] L. C. Hsu, *Note on a pair of combinatorial reciprocal identities*, Math. Student **22** (1954), 175–178.

[126] Adolf Hurwitz and R. Courant, *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen*, 3rd ed., Springer-Verlag, 1929.

[127] Rudolph C. Hwa and Vigdor L. Teplitz, *Homology and Feynman integrals*, Benjamin, 1966.

[128] G. I. Ivchenko and Yu. I. Medvedev, *Some higher-dimensional theorems in the classical permutation problem*, Teor. Veroyatnost. i Primenen. **10** (1965), 156–162; English transl. in Theory Probab. Appl. **10** (1965).

[129] ______, *Asymptotic representations of finite differences of a power function at an arbitrary point*, Teor. Veroyatnost. i Primenen. **10** (1965), 151–156; English transl. in Theory Probab. Appl. **10** (1965).

[130] A. A. Karatsuba, *Principles of analytic number theory*, "Nauka", Moscow, 1975; Spanish transl., Mir, Moscow, 1979.

[131] Josef Kaucký, *Bemerkung zu einer Arbeit von Viktor Klee*, Časopis Pěst. Mat. **91** (1966), 194–197.

[132] ______, *Combinatorial identities*, Veda, Bratislava, 1975. (Slovak)

[133] John L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955.

[134] Donald E. Knuth, *The art of computer programming*. Vol. 1, Addison-Wesley, 1969.

[135] V. F. Kolchin, *The speed of convergence to limit distributions in a classical problem on balls*, Teor. Veroyatnost. i Primenen. **11** (1966), 144–156; English transl. in Theory Probab. Appl. **11** (1966).

[136] V. F. Kolchin and V. P. Chistyakov, *Combinatorial problems of probability theory*, Itogi Nauki: Teor. Veroyatnost., Mat. Statist., Teoret. Kibernet., vol. 11, VINITI, Moscow, 1974, pp. 5–45; English transl. in J. Soviet Math. **4** (1975), no. 3.

[137] V. F. Kolchin, B. A. Sevast'yanov, and V. P. Chistyakov, *Random allocations*, "Nauka", Moscow, 1976; English transl., Wiley, 1978.

[138] I. V. Konoval'tsev and E. P. Lipatov, *On some properties of plane rooted trees*, Kibernetika **1970**, no. 5, 106–111; English transl. in Cybernetics **6** (1970).

[139] *The Kourovka notebook: Unsolved problems in the theory of groups*, 2nd ed., Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1967; English transl. of 7th (1980) ed., Amer. Math. Soc. Transl. (2) **121** (1983).

[140] V. A. Krechmar, *Problem book on algebra*, 2nd ed., GITTL, Moscow, 1950. (Russian)

[141] Dzh. Kutlumuratov, *On the development of combinatorial methods in mathematics*, "Karakalpakiya", Nukus, 1964. (Russian)

[142] J. Leray, *Le calcul différentiel et intégral sur une variété analytique complexe* (*Problème de Cauchy*, III), Bull. Soc. Math. France **87** (1959), 81–180.

[143] V. M. Levchuk, *Subgroups of the unitriangular group*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 1202–1220; English transl. in Math. USSR Izv. **8** (1974).

[144] Yu. V. Linnik and I. V. Ostrovskiĭ, *Decomposition of random variables and vectors*, "Nauka", Moscow, 1972; English transl., Amer. Math. Soc., Providence, R. I., 1977.

[145] V. A. Liskovets, *On a recursion method for enumerating graphs with labelled vertices*, Dokl. Akad. Nauk SSSR **184** (1969), 1284–1287; English transl. in Soviet Math. Dokl. **10** (1969).

[146] V. N. Lyamin and B. I. Selivanov, *Enumeration problems for simple hypergraphs*, Kombinatornyĭ Anal. Vyp. **3** (1974), 57–67; *Hypertrees with a given number of end vertices and edges*, ibid., 68–71. (Russian)

[147] Percy A. MacMahon, *Combinatory analysis*. Vols. I, II, Cambridge Univ. Press, 1915, 1916.

[148] V. A. Malyshev, *Asymptotic behavior of stationary probabilities for two-dimensional positive random walks*, Sibirsk. Mat. Zh. **14** (1973), 156–169; English transl. in Siberian Math. J. **14** (1973).

[149] ______, *Random walks. The Wiener-Hopf equation in a quadrant of the plane. Galois automorphisms*, Izdat. Moskov. Gos. Univ., Moscow, 1970. (Russian)

[150] ______, *Wiener-Hopf equations and their applications in probability theory*, Itogi Nauki: Teor. Veroyatnost., Mat. Statist., Teoret. Kibernet., Vol. 13, VINITI, Moscow, 1976, pp. 5–35; English transl. in J. Soviet Math. **7** (1977), no. 2.

[151] Marcia Ascher, *A combinatorial identity*, Fibonacci Quart. **12** (1974), 186–188.

[152] Enzo Martinelli, *Sulle estensioni della formula integrale di Cauchy alle funzioni analitiche di più variabili complesse*, Ann. Mat. Pura Appl. (4) **34** (1953), 277–347.

[153] ______, *Contributi alla teoria dei residui per le funzioni di due variabili complesse*, Ann. Mat. Pura Appl. (4) **39** (1955), 335–343.

[154] A. M. Mathai and P. N. Rathie, *Enumeration of almost cubic maps*, J. Combinatorial Theory Ser. B **13** (1972), 83–90.

[155] A. M. Mathai and R. K. Saxera, *Generalized hypergeometric functions with applications in statistics and physical sciences*, Lecture Notes in Math., Vol. 348, Springer-Verlag, 1973.

[156] Elna B. McBride, *Obtaining generating functions*, Springer-Verlag, 1971.

[157] John Milnor, *Singular points of complex hypersurfaces*, Princeton Univ. Press, Princeton, N. J., and Univ. of Tokyo Press, Tokyo, 1968.

[158] S. G. Mohanty and B. R. Handa, *Extensions of Vandermonde type convolutions with several summations and their applications*. I, Canad. Math. Bull **12** (1969), 45–62.

[159] ______, *A generalized Vandermonde-type convolution and associated inverse series relations*, Proc. Cambridge Philos. Soc. **68** (1970), 459–474.

[160] J. W. Moon, *Various proofs of Cayley's formula for counting trees*, A Seminar on Graph Theory (F. Harary and L. Beineke, editors), Holt, Reinhart and Winston, 1967, pp. 70–78.

[161] Ronald Mullin and Gian-Carlo Rota, *On the foundations of combinatorial theory*. III: *Theory of binomial enumeration*, Graph Theory and its Applications (Proc. Adv. Sem., Madison, Wisc., 1969), Academic Press, 1970, pp. 167–213.

[162] R. C. Mullin and R. G. Stanton, *A combinatorial property of spanning forests in connected graphs*, J. Combinatorial Theory **3** (1967), 236–243.

[163] M. A. Naĭmark, *Normed rings*, GITTL, Moscow, 1956; English transl., Noordhoff, 1959.

[164] Raghavan Narasimhan, *Analysis on real and complex manifolds*, North-Holland, 1968.

[165] Eugen Netto, *Lehrbuch der Combinatorik*, 2nd ed., Teubner, Leipzig, 1927; reprint, Chelsea, New York, 1958.

[166] Nguyen-Huu-Bong, *Some combinatorial properties of summation operators*, J. Combinatorial Theory Ser. A **14** (1973), 253–235.

[167] Ivan Niven, *Formal power series*, Amer. Math. Monthly **76** (1969), 871–889.

[168] F. J. Papp, *Another proof of Tepper's inequality*, Math. Magazine **45** (1972), 119–121.

[169] J. K. Percus, *A note on extension of the Lagrange inversion formula*, Comm. Pure Appl. Math. **17** (1964), 137–146.

[170] ______, *Combinatorial methods*, Springer-Verlag, 1971.

[171] F. Pham, *Introduction à l'étude topologique des singularités de Landau*, Mém. Sci. Math., fasc. 164, Gauthier-Villars, Paris, 1967.

[172] M. L. Platonov, *Combinatorial numbers of a class of mappings*, Combinatorial and Asymptotic Analysis, Vyp. 1, Krasnoyarsk. Gos. Univ., Krasnoyarsk, 1975, pp. 81–95. (Russian)

[173] H. Poincaré, *Sur les résidus des intégrales doubles*, Acta Math. **9** (1887), 321–380.

[174] G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen*, Acta Math. **68** (1937), 145–254.

[175] A. G. Postnikov, *Introduction to analytic number theory*, "Nauka", Moscow, 1971. (Russian)

[176] Karl Prachar, *Primzahlverteilung*, Springer-Verlag, 1957.

[177] C. Ramanujacharyulu, *A combinatorial identity*, The Many Facets of Graph Theory (Proc. Conf., Kalamazoo, Mich., 1968), Lecture Notes in Math., Vol. 110, Springer-Verlag, 1969, pp. 271–273.

[178] S. Ramanujan, *Collected papers*, Cambridge Univ. Press, 1927.

[179] P. K. Rashevskiĭ, *Riemannian geometry and tensor analysis*, 3rd ed., "Nauka", Moscow, 1967; German transl. of 1st ed., VEB Deutscher Verlag Wiss., Berlin, 1959.

[180] M. L. Rasulov, *The contour integral method, and its application to the investigation of problems for differential equations*, "Nauka", Moscow, 1964; English transl., Interscience, 1967.

[181] ______, *Applications of the method of the contour integral to the solution of problems for second order parabolic systems*, "Nauka", Moscow, 1975. (Russian)

[182] Alfréd Rényi, *Some remarks on the theory of trees*, Magyar Tud. Akad. Mat. Kutató Int. Közl. 4 (1959), 73–85.

[183] ______, *New methods and results in combinatorial analysis*. I, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **16** (1966), 77–105. (Hungarian)

[184] George de Rham, *Variétés différentiables. Formes, courants, formes harmoniques*, Actualités Sci. Indust., no. 1222, Hermann, Paris, 1955.

[185] E. Riekstiņš, *Asymptotic expansions of integrals*. Vol. I, "Zinatne", Riga, 1974. (Russian)

[186] John Riordan, *An introduction to combinatorial analysis*, Wiley, New York, and Chapman & Hall, London, 1958.

[187] ______, *Combinatorial identities*, Wiley, 1968.

[188] E. G-Rodeja F. [E. Garcia-Rodeja Fernández], *On identities of Jensen, Gould and Carlitz*, Proc. Fifth Annual Reunion Spanish Mathematicians (Valencia, 1964), Publ. Inst. "Jorge Juan" Mat., Madrid, 1967, pp. 11–14. Amer. Math. Monthly **73** (1966), 507–508.

[190] ______, *On lattice paths with diagonal steps*, Canad. Math. Bull. 7 (1964), 470–472.

[191] Gian-Carlo Rota, *On the foundations of combinatorial theory*. I: *Theory of Möbius functions*, Z. Wahrsch. Verw. Gebiete **2** (1964), 340–368.

[192] ______, *Baxter algebras and combinatorial identities*. I, II, Bull. Amer. Math. Soc. **75** (1969), 325–329, 330–334.

[193] Gian-Carlo Rota, D. Kahaner and A. Odlyzko, *On the foundations of combinatorial theory*. VIII: *Finite operator calculus*, J. Math. Anal. Appl. **42** (1973), 684–760.

[194] K. A. Rybnikov, *Introduction to combinatorial analysis*, Izdat. Moskov. Univ., Moscow, 1972. (Russian)

[195] V. N. Sachkov, *Enumeration problems in combinatorial analysis*, Questions in Cybernetics (Proc. Sem. Combinatorial Math., Moscow, 1971; S. I. Samoĭlenko et al., editors), Nauchn. Sovet. Kompleks. Probl. "Kibernetika" Akad. Nauk SSSR, Moscow, 1973, pp. 146–163. (Russian)*

[196] R. A. Sack, *Generalization of Lagrange's expansion for functions of several implicitly defined variables*, J. Soc. Indust. Appl. Math. **13** (1965), 913–926.

[197] O. V. Sarmanov, B. A. Sevast'yanov, and V. E. Tarakanov, *Some combinatorial identities*, Mat. Zametki **11** (1972), 121–127; English transl. in Math. Notes **11** (1972).

[198] I. J. Schwatt, *An introduction to the operations with series*, Univ. of Pennsylvania Press, Philadelphia, Pa., 1924.

[199] B. I. Selivanov, *Enumeration of homogeneous hypergraphs with a simple cycle structure*, Kombinatornyĭ Anal. vyp. **2** (1972), 60–67. (Russian)

[200] B. A. Sevast'yanov, *Convergence of the distribution of the number of empty boxes to Gaussian and Poisson processes in a classical problem on balls*, Teor. Veroyatnost. i Primenen. **12** (1967), 144–154; English transl. in Theory Probab. Appl. **12** (1967).

[201] B. A. Sevast'yanov and V. P. Chistyakov, *Asymptotic normality in a classical problem on balls*, Teor. Veroyatnost. i Primenen. **9** (1964), 223–237; English transl. in Theory Probab. Appl. **9** (1964).

[202] B. V. Shabat, *Introduction to complex analysis*, "Nauka", Moscow, 1969. (Russian)

[203] J. Sheehan, *An identity*, Amer. Math. Monthly **77** (1970), 168.

**Editor's note*. According to the National Union Catalog (U.S.A.), the main library card-file entry for this item is "Seminar po Kombinatornoĭ Matematike, Moscow, 1971".

[204] F. V. Shirokov and V. A. Signaevskiĭ, *Minimal covers of a finite set*, Dokl. Akad. Nauk SSSR **207** (1972), 1066–1069; English transl. in Soviet Math. Dokl. **13** (1972).

[205] A. I. Shirshov, *On bases of a free Lie algebra*, Algebra i Logika **1** (1962), 14–19. (Russian)

[206] A. L. Shmel'kin, *Free polynilpotent groups*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 91–122; English transl. in Amer. Math. Soc. Transl. (2) **55** (1966).

[207] J. P. Singhal, *A generating function and a binomial identity*, Duke Math. J. **39** (1972), 25–30.

[208] V. P. Sinha and G. K. Dhawan, *On generalized Stirling numbers and polynomials*, Riv. Mat. Univ. Parma (2) **10** (1969), 95–100.

[209] Rosedith Sitgreaves, *Some properties of Stirling numbers of the second kind*, Fibonacci Quart. **8** (1970), 172–181.

[210] Michael Skalsky, *A note on a convolution-type combinatorial identity*, Amer. Math. Monthly **74** (1967), 836–838.

[211] A. I. Skopin, *The factor groups of a certain upper central series of a free group*, Dokl. Akad. Nauk SSSR **74** (1950), 425–428. (Russian)

[212] ______, *Ideals of relations*, Trudy Mat. Inst. Steklov. **80** (1965), 117–128; English transl. in Proc. Steklov Inst. Math. **80** (1965).

[213] Lucy Joan Slater, *Generalized hypergeometric functions*, Cambridge Univ. Press, 1966.

[214] V. G. Sokolov, *An analogue of the Witt formula for free solvable groups*, Algebra i Logika **8** (1969), 367–372; English transl. in Algebra and Logic **8** (1969).

[215] G. Sorani, *Sull'indicatore logaritmico per le funzioni di più variabili complesse*, Rend. Mat. e Appl. (5) **19** (1960), 130–142.

[216] Edwin H. Spanier, *Algebraic topology*, McGraw-Hill, 1966.

[217] H. M. Srivastava, *Remark on a combinatorial identity of F. Y. Wu*, Nordisk Mat. Tidskr. **20** (1972), 85–86.

[218] Richard P. Stanley, *Ordered stuctures and partitions*, Mem. Amer. Math. Soc. No. 119 (1972).

[219] ______, *Generating functions*, M.I.T., Cambridge, Mass., 1975.**

[220] B. S. Stechkin, *Generalized valences*, Mat. Zametki **17** (1975), 433–442; English transl. in Math. Notes **17** (1975).

[221] V. E. Stepanov, *Limit distributions of certain characteristics of random mappings*, Teor. Veroyatnost. i Primenen. **14** (1969), 639–653; English transl. in Theory Probab. Appl. **14** (1969).

[222] ______, *Random graphs*, Questions in Cybernetics (Proc. Sem. Combinatorial Math., Moscow, 1971; S. I. Samoĭlenko et al., editors), Nauchn. Sem. Kompleks. Probl. "Kibernetika" Akad. Nauk SSSR, Moscow, 1973, 164–185. (Russian)***

[223] M. A. Subkhankulov, *Tauberian theorems with a remainder*, "Nauka", Moscow, 1976. (Russian)

[224] J. R. Swenson, *The chromatic polynomial of a complete bipartite graph*, Amer. Math. Monthly **80** (1973), 797–798.

[225] J. J. Sylvester, *On the change of systems of independent variables*, Quart J. Pure Appl. Math. **1** (1857), 42–56, 126–134.

[226] a) Gabor Szegö, *Orthogonal polynomials*, rev. ed., Amer. Math. Soc., Providence, R. I., 1959.

b) Ya. L. Geronimus, Appendix to the Russian transl. of [226] a), Fizmatgiz, Moscow, 1961, pp. 414–494; English transl. in Amer. Math. Soc. Transl. (2) **108** (1977).

[227] Lajos Takács, *A moment problem*, J. Austral. Math. Soc. **5** (1965), 487–490.

[228] ______, *Combinatorial methods in the theory of stochastic processes*, Wiley, 1967.

***Editor's note.* See also the paper with the same author and title in *Studies in Combinatorics* (MAA Studies in Math., 17, Math. Assoc. America, Washington, D. C., 1978, pp. 100–141).

***See the footnote to [195].

[229] ______, *On an identity of Shih-Chieh Chu*, Acta Sci. Math. (Szeged) **34** (1973), 383–391.

[230] C. Teleman, *Elements of topology and differentiable manifolds*, Ed. Didact. şi Pedagog., Bucharest, 1964 (Romanian); German transl., VEB Deutscher Verlag Wiss., Berlin, 1968.

[231] W. T. Tutte, *On elementary calculus and the Good formula*, J. Combinatorial Theory (B) **18** (1975), 97–137.

[232] I. I. Viktorova and V. P. Chistyakov, *Asymptotic normality in a problem on balls with arbitrary probabilities of falling in boxes*, Teor. Veroyatnost. i Primenen. **10** (1965), 162–167; English transl. in Theory Probab. Appl. **10** (1965).

[233] N. Ya. Vilenkin, *Combinatorics*, "Nauka", Moscow, 1969; English transl., Academic Press, 1971.

[234] I. M. Vinogradov, *Trigonometric sums in number theory*, rev. ed., "Nauka", Moscow, 1971; English transl., Statist. Publ. Soc., Calcutta, 1978.

[235] V. S. Vladimirov, *Methods in the theory of functions of several complex variables*, "Nauka", Moscow, 1964; English transl., MIT Press, Cambridge, Mass., 1966.

[236] E. T. Whittaker and G. N. Watson, *A course of modern analysis*. Vol. I, 4th ed., Cambridge Univ. Press, 1927.

[237] A. P. Yuzhakov, *On the theory of residues of functions of two complex variables*, Moskov. Oblast. Ped. Inst. Uchen. Zap. **110** (Mat. 7) (1962), 153–162. (Russian)

[238] ______, *On the residues of functions of several complex variables*, Izv. Vyssh. Uchebn. Zaved. Matematika **1964**, no. 5(42), 149–161. (Russian)

[239] ______, *On the residues of rational functions of two variables*, Holomorphic Functions of Several Complex Variables (Sem., Krasnoyarsk, 1972), Inst. Fiz. Sibirsk. Otdel. Akad. Nauk SSSR, Krasnoyarsk, 1972, pp. 153–163. (Russian)

[240] ______, *Principles of the theory of multidimensional residues*, Krasnoyarsk. Gos. Univ., Krasnoyarsk, 1975. (Russian)

[241] ______, *On an application of the multiple logarithmic residue to the expansion of implicit functions in power series*, Mat. Sb. **97**(139) (1975), 177–192; English transl. in Math. USSR Sb. **26** (1975).

[242] ______, *On the representation of implicit functions as function series*, Combinatorial and Asymptotic Analysis, vyp. 1, Krasnoyarsk. Gos. Univ., Krasnoyarsk, 1975, pp. 179–183. (Russian)

[243] A. P. Yuzhakov and A. V. Kuprikov, *On the logarithmic residue in* $\mathbf{C}^n$, Some Properties of Holomorphic Functions of Several Complex Variables, Inst. Fiz. Sibirsk. Otdel. Akad. Nauk SSSR, Krasnoyarsk, 1973, pp. 181–191. (Russian)

[244] B. S. Zinov'ev, *On reproducing kernels for n-circular domains and linear convexity*, Candidate's Dissertation, Ural State Univ., Sverdlovsk, 1971. (Russian)

[245] ______, *On reproducing kernels for multiply circular domains of holomorphy*, Sibirsk. Mat. Zh. **15** (1974), 35–48; English transl. in Soviet Math. J. **15** (1974).

[246] B. S. Zinov'ev and G. P. Svistkova, *Finding the reproducing Bergman and Szegö kernels for a certain class of multiply circular domains*, Some Properties of Holomorphic Functions of Several Complex Variables, Inst. Fiz. Sibirsk. Otdel. Akad. Nauk SSSR, Krasnoyarsk, 1973, pp. 209–214. (Russian)

[247] T. Hearne and C. Wagner, *Minimal covers of finite sets*, Discrete Math. **5** (1973), 247–251.

[248] R. Borges, *On the principle of inclusion and exclusion*, Period. Math. Hungar. **3** (1973), 149–156.

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