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АЛЕКСЕЙ ГЕОРГИЕВИЧ ПОСТНИКОВ

ВВЕДЕНИЕ В АНАЛИТИЧЕСКУЮ ТЕОРИЮ ЧИСЕЛ

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ABSTRACT. This book is intended as a mid-level view of analytic number theory, somewhere between a textbook and the latest monographs. The author has attempted to give as broad a picture as possible of the problems of analytic number theory, while avoiding specialization and those topics already sufficiently well treated in the literature. This is the meaning of the title, *Introduction to analytic number theory*. The deep results in analytic number theory involve, of course, the use of well-developed machinery. However, it is advisable for a young scholar to have, in addition to a mastery of powerful tools, a supply of problems to which he can apply these strong techniques. In this the author has tried to be of assistance to his young colleagues. This book is directed to scholars, teachers, and graduate students interested in number theory and its connections with other branches of science.

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Contents

Preface to the American edition

The present translation incorporates an Appendix prepared by P.D.T.A. Elliott. In addition to notes on the text, it includes a supplementary bibliography of 55 items.

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This book is dedicated
to the memory of
Vyacheslav Vasil'evich Stepanov

Preface

The literature on analytic number theory is very rich and naturally reflects the historical development of this science. As an introduction to analytic number theory one can look to the supplementary material (problems and solutions) in Vinogradov's textbook, *Elements of number theory* [148]. The monograph *Number theory* [13] by Borevich and Shafarevich discusses the foundations of the method of Dirichlet generating series (applied to algebraic number theory). The book *Elementary methods in analytic number theory* [60] by Gel'fond and Linnik can be regarded as an elementary reader on the subject. One can find books in Russian dealing with Diophantine approximation (Cassels [17]), the approximation of algebraic and transcendental numbers (Gel'fond [59]), the method of trigonometric sums (Vinogradov [149]), the distribution of primes and the theory of Dirichlet L-functions (Ingham [74], Chudakov [20], Titchmarsh [139], and Prachar [118]), probabilistic number theory (Kubilius [87]), and so on. Nevertheless, there are still some fundamental questions in analytic number theory that have not been adequately discussed in the literature in any systematic way—these are questions directly or indirectly connected with the concept of a numerical semigroup, in other words, with general additive number theory. It is to this aspect of number theory that the present book is devoted.

It is assumed that the reader is familiar with the fundamentals of analysis, number theory, and probability theory. The necessary analysis is covered by Fikhtengol'ts [48], [49], [50], the number theory by Vinogradov [148] or Bukhshtab [16], and the probability theory by the first eight chapters of Gnedenko [61].

The author thanks G. A. Freĭman for his simple proof of the Hardy-Ramanujan asymptotic formula. He is also grateful to K. Yu. Bulota for taking on the difficult task of editing this book; his constructive criticism was most helpful. Finally, the author expresses his sincere gratitude to all who helped in the preparation of this volume.

Standard Notation

To avoid being repetitious let us agree on the meaning of the following notation (unless stated otherwise):

γ	Euler's constant,
$\chi_4(n)$	the nonprincipal character modulo 4,
$\varphi(n)$	Euler's function,
$\tau(n)$	the number of natural divisors of n,
$\sigma(n)$	the sum of the natural divisors of n,
$\nu(n)$	the number of distinct prime divisors of n,
$r(n)$	the number of representations of n as a sum of two squares,
$\alpha_p(n)$	the exponent of the prime p in the canonical factorization of n,
$\mu(n)$	the Möbius function,
$\Lambda(n)$	von Mangoldt's function, $\Lambda(n) = \begin{cases} \ln p, & n = p^\alpha, \\ 0, & n \neq p^\alpha, \end{cases}$
(x)	the distance from the real number x to the nearest integer,
[x]	the integral part of the real number x,
$\{x\}$	the fractional part of x ($\{x\} = x - [x]$),
$\Gamma(s)$	the gamma-function,
$\zeta(s)$	the zeta-function,
$\pi(x)$	the number of primes not exceeding x.

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Appendix

Professor Postnikov's book represents an interesting personal selection of a number of topics from analytic number theory. At the editor's request I append here some further references, including a discussion of more recent advances in areas close to those which he considers.

Chapter 2. §2.4. Linnik [*42*], [*43*]* and A. I. Vinogradov [*55*] applied estimates for trigonometric sums involving powers g^x of an integer g to show, in particular, that every sufficiently large even integer may be expressed as the sum of two primes and a bounded number of powers of 2. For a refined treatment of these and related results, see Gallagher [*31*].

§2.6. This section is modelled on a Selberg version of the first elementary proof of the Prime Number Theorem. For an account of the latter see Elliott [*17*], Chapter 19. For a discussion of appropriate analogues of Tchebyshev's classical bounds for $\pi(x)$, within the format of generalized integers, see Diamond [*8*].

§2.7. Hardy and Ramanujan gave an asymptotic expansion for $p(n)$ in terms of familiar (hyperbolic) functions. By slightly modifying their choice of representing function, Rademacher replaced their asymptotic series by one that converged. An account of this may be found in Rademacher [*46*].

Chapter 3. §3.3. The theory of almost periodic functions in analysis was invented by H. Bohr. An account of this theory, together with various generalisations in analysis, may be found in Besicovitch [*3*]. For discussions of generalisations of the notion of almost periodic functions appropriate to the study of arithmetic functions in number theory, see Schwarz [*50*] and Mauclaire [*44*].

*References cited in italic will be found in the bibliography at the end of this appendix.

For $\alpha \geq 1$ we define a *seminorm* $\| \ \|_\alpha$ on complex-valued arithmetic functions f by

$$\|f\|_\alpha = \limsup_{x\to\infty}\left(x^{-1}\sum_{n\leq x}|f(n)|^\alpha\right)^{1/\alpha}.$$

Those functions for which this seminorm is finite form a linear space L^α over the complex numbers. Factoring out the functions with a seminorm zero then gives a Banach space.

We say that f is *almost periodic* B^α if for each $\varepsilon > 0$ there is a trigonometric polynomial

$$P_\varepsilon(t) = \sum_{j=1}^{k} a_j e^{2\pi i\beta_j t}$$

with real β_j, so that $\|f - P_\varepsilon\| < \varepsilon$. If the polynomial P_ε may be chosen periodic in t, then we say that f is *limit-periodic* B^α.

For any arithmetic function f, the Fourier-Bohr spectrum is the set of real β (mod 1) such that

$$\limsup_{x\to\infty} x^{-1}\left|\sum_{n\leq x} f(n)e^{-2\pi in\beta}\right| > 0.$$

A complete characterisation of multiplicative functions g which, for a given $\alpha \geq 1$, are almost periodic B^α with a nonempty spectrum was given by Daboussi [*4*]:

It is necessary and sufficient that there be a Dirichlet character χ so that the series

$$\sum p^{-1}(\chi(p)g(p)-1), \qquad \sum_{|g(p)|\leq 2}\frac{|\chi(p)g(p)-1|^2}{p},$$

$$\sum_{|g(p)|>2}\frac{|g(p)|^\alpha}{p}, \qquad \sum_{p,k\geq 2}\frac{|g(p^k)|^\alpha}{p^k},$$

taken over the prime numbers, converge. When this condition is satisfied the function g is actually limit-periodic B^α.

For those multiplicative functions g satisfying $|g(n)| \leq 1$, Daboussi showed that the Fourier-Bohr spectrum cannot contain irrationals (mod 1). The restriction on the size of g was later weakened by Daboussi and Delange [7] to membership of the class L^α for some $\alpha > 1$.

Various results of this and related types may be found in the survey of Schwarz [*50*]. See also Indlekofer [*35*].

§§**3.4 and 3.5.** For an application of Novoselov's construction, see Babu [*2*]. On the whole, the approach of Novoselov has not been widely adopted.

The axiom of choice readily guarantees the existence of a finitely additive measure on the natural integers, which assigns to every infinite sequence with an asymptotic density that density as measure. However, it remains a problem to relate such a measure to a specific arithmetic function.

It is natural to employ the limit as $N \to \infty$ of the frequencies

$$\nu_N(n; n \in A) = \frac{1}{N} \sum_{n \leq N, n \in A} 1$$

which appear in §3.3. This affords an immediate realisation of the idea of Kac that, suitably interpreted, divisibility of an integer by differing primes represents independent events. A similar philosophy underlies the first theorem of §3.4. Unfortunately, when considering multiplicative functions, for example, this approach is fruitful only if the function becomes rapidly smaller on the larger primes. Replacing the asymptotic frequencies by the use of a particular ν_N allows only the approximate independence of divisibility by primes. The study of arithmetic functions within the aesthetic of the classical theory of probability, with its emphasis on sums and products of independent random variables, therefore involves a careful balance between the convenience of a measure, with respect to which appropriate events are independent, and the loss of generality for the class of functions which may be considered. This dilemma is succinctly summed up by Galambos [*30*], in his recent review of a paper by Nanopoulos [*45*].

For works related to §3.4 see Schwarz and Spilker [*51*] and Mauclaire [*44*], where the introduction of Riesz representations forms an attractive feature. For works which allow a larger class of functions to be considered, but involve a less obvious construction of an appropriate measure, see Kubilius [*37*] (which is an improved version of reference [87] in Postnikov's bibliography) and Elliott [*17*].

Chapter 4. §4.2. It is interesting to note that Erdős and Shapiro [*28*] proved that the frequencies

$$\nu_N\left(n; \sum_{m \leq n} \frac{\phi(m)}{m} - \frac{6n}{\pi^2} \leq u\right)$$

possess a continuous limiting distribution as $N \to \infty$.

§4.3. This result of Wirsing prompted much research. An important feature is the consideration of $f(n) \log n$, rather than $f(n)$ directly.

§§4.5 and 4.6. For an extensive discussion of the theorems of Delange, Wirsing and Halász, see Elliott [*17*], Chapter 6. The mean-value theorem of Delange has the following generalisation.

Let $a > 1$. A multiplicative function g belongs to the space L^α and has a nonzero mean value

$$\lim_{x\to\infty} x^{-1} \sum_{n\le x} g(n)$$

if and only if the series

$$\sum_{|g(p)|\le 3/2} p^{-1}|g(p)-1|^2, \qquad \sum_{|g(p)|>3/2} p^{-1}|g(p)|^\alpha,$$

$$\sum_{p,k\ge 2}\sum p^{-k}|g(p^k)|^\alpha, \qquad \sum p^{-1}(g(p)-1),$$

taken over the prime numbers, converge, and if for each prime p,

$$\sum_{k=1}^{\infty} p^{-k} g(p^k) \ne -1.$$

When these conditions are satisfied, the limit

$$\lim_{x\to\infty} x^{-1} \sum_{n\le x} |g(n)|^\alpha$$

actually exists.

For $\alpha = 2$ this was first proved by Elliott [*15*]. Alternative proofs of the necessity of the conditions on the primes were given by Daboussi and Delange [*6*], and of their sufficiency by Schwarz and Spilker [*52*]. Generalisations to $\alpha > 1$ were given independently by Elliott [*18*] and Daboussi [*5*], using quite different methods.

Not surprisingly, the consideration of multiplicative functions g, with limiting mean-value zero, causes more difficulty.

Suppose that g belongs to L^α for some $\alpha > 1$. Then in order that

$$\lim_{x\to\infty} x^{-1} \sum_{n\le x} g(n) = 0$$

it is necessary and sufficient that one of the following four conditions be satisfied:

(i) *One of the series*

$$\sum_{||g(p)|-1|\le 1/2} p^{-1}|1-|g(p)||^2, \qquad \sum_{||g(p)|-1|>1/2} p^{-1}|1-|g(p)||^\alpha$$

diverges.

(ii) *The condition* (i) *fails, but for each real value of t the series*

$$\sum p^{-1}(|g(p)| - \operatorname{Re} g(p)p^{it})$$

diverges.

(iii) *The conditions* (i) *and* (ii) *fail, but there is a real t so that the series in condition* (ii) *converges and*

$$\sum_{k=1}^{\infty} g(p^k)p^{-k(1+it)} = -1$$

for some prime p.

(iv) *The conditions* (i), (ii), *and* (iii) *fail, but*

$$\sum_{p\leq x} \frac{1-|g(p)|}{p} \to \infty$$

as $x \to \infty$.

This theorem was proved by Elliott [*18*].

For multiplicative functions g which assume only nonnegative real values, the following result of Elliott [*20*] may be compared with that of Wirsing in §4.3.

Suppose that a finite limiting mean-value

$$\lim_{x\to\infty} x^{-1} \sum_{n\leq x} g(n)$$

exists. Then for each δ, $0 < \delta < 1$, *so does*

$$\lim_{x\to\infty} x^{-1} \sum_{n\leq x} g(n)^{\delta}.$$

Moreover, these new limits are all zero, unless the series $\sum p^{-1}(g(p)^{1/2}-1)^2$, *taken over the prime numbers, converges.*

As shown in that same reference, this result has an interesting application to Ramanujan's modular coefficient function $\tau(n)$.

In his work on multiplicative functions Wirsing used approximate integral equations. For real-valued functions satisfying $|g(n)| \leq 1$ he was able to establish the existence of a limiting mean-value, so settling an old conjecture of Erdős and Wintner. This same result can be obtained by the method of Halász (ref. [63] of Postnikov), using Dirichlet series. For a proof of that result along lines quite different from those of Wirsing or Halász, see Hildebrand [*34*].

§4.4. Other proofs of the Turán-Kubilius inequality may be found in Elliott [*12*], [*16*]. In the notation of Postnikov, let λ_N denote the supremum of

$$\frac{1}{ND^2(N)} \sum_{n=1}^{N} |f(n) - A(N)|^2,$$

taken over all complex-valued additive functions not identically zero on the interval $1 \le n \le N$. The Turán-Kubilius inequality guarantees a uniform bound for λ_N.

The asymptotic best value $3/2 + o(1)$ for the constant was obtained by Kubilius (announced in a 1981 Budapest conference on number theory), [38], [39], and (with a different method) by Hildebrand [32]. In fact Kubilius' method led to the sharper estimate $\lambda_N = 3/2 + O((\log N)^{-1/2})$. This he has recently improved [40], by replacing the exponent $-1/2$ in the error term with -1.

The Turán-Kubilius inequality may be viewed within the format of functional analysis, so that it has a dual:

$$\sum_{p \le N} p \left| \sum_{\substack{n=1 \\ n \cong 0 (\operatorname{mod} p)}}^{N} a_n - \frac{1}{p} \sum_{n=1}^{N} a_n \right|^2 + \sum_{\substack{p^k \le N \\ k \ge 2}} p^k \left| \sum_{\substack{n=1 \\ n \cong 0 (\operatorname{mod} p^k)}}^{N} a_n \right|^2 < C_1 \sum_{n=1}^{N} |a_n|^2$$

valid for all complex a_n. Here $n \cong 0 \pmod{p^k}$ denotes that p^k divides n, but p^{k+1} does not. The use of a dual Turán-Kubilius inequality to study arithmetic functions seems to have begun with Elliott [14]. Explicitly or implicity such a notion underlies many of the recent developments mentioned in the commentary on Chapter 3 and on the present chapter. An extensive overview of these matters, including the relation of the Turán-Kubilius inequality to a certain Hermitian operator, and various generalisations of it involving differing norms, is given in Elliott [22]. See also Elliott [13] for a connection with the large sieve.

Generalisations of the Turán-Kubilius inequality on the integers and involving powers other than squares may be found in Elliott [19], [22], [23]. See also Ruzsa [47], [48] and Hildebrand [33]. Besides this, one may give generalisations in which the argument n in the additive function $f(n)$ is replaced by other sequences of arithmetic interest, such as the shifted primes $p+1$, or polynomials in n, as in Elliott [17], Chapter 4, and Alladi [1].

§**4.7.** Another derivation of Lévy's criterion for an infinite convolution to be continuous may be found in Elliott [17], Lemma 1.22. It goes by way of a concentration function estimate, from the theory of probability, due to Kolmogorov and Rogozin.

A direct proof that the divergence of the series $\sum p^{-1}$ $(f(p) \ne 0)$ guarantees the continuity of the limit law for $\nu_N(n; f(n) \le z)$, when that exists, was indicated by Szüsz in [53] and a letter to the present author. It employs Wiener's criterion for characteristic functions, and Tchebyshev's inequality in the theory of probability. A detailed account may be found in Elliott [17], [21]. At present there does not seem to be any truly short proof of this result.

There is a generalisation of this last result, due to Erdős [*26*], Elliott and Ryavec [*24*], as in Elliott [*17*], which shows that suitably interpreted it remains true even if the frequencies $\nu_N(n; f(n) \le z)$ do not possess a limiting distribution as $N \to \infty$.

§**4.8.** Erdős [*27*] proved that there is an absolute constant c so that the inequality

$$\nu_N\left(n; a < \frac{\phi(n)}{n} \le a\left(1 + \frac{1}{t}\right)\right) \le \frac{c}{\log t}$$

holds uniformly for all real $a > 0$, $2 \le t \le N$. This concentration function estimate is better than the one employed in the text by Postnikov, and leads (see Elliott [*17*], Chapter 5) to the sharper estimate

$$\Phi_N(x) = \Phi(x) + O\left(\frac{\log\log N}{\log N \log\log\log N}\right).$$

It was conjectured by Erdős that this error term should be $O((\log N)^{-1})$.

The elementary but ingenious method of Erdős yields an upper bound of the form $c_1(\log(x/\varepsilon))^{-1}$ for $\Phi(x + \varepsilon x) - \Phi(x)$, provided $0 < \varepsilon < x < 1$. Diamond and Rhoads [*11*] improved this to an estimate for the modulus of continuity of $\Phi(x)$, with a bound $c_2(\log(1/\varepsilon))^{-1}$ to hold uniformly in $x > 0$, $0 < \varepsilon < 1$. For this they employ characteristic functions.

§**4.9.** For other applications to number theory of the theory of products of random variables, see Elliott [*17*], Chapter 7. This includes results of Galambos [*29*] concerning the convergence of frequencies $\nu_N(n; g(n) \le z)$ to a symmetric limit law, and of Levin, Timofeev, and Tuliaganov [*41*] characterizing those multiplicative functions g for which the frequencies $\nu_N(n; g(n) - \alpha(N) \le z\beta(N))$ converge when the renormalising constants $\alpha(N), \beta(N) > 0$ are suitably chosen.

The value distribution of positive-valued arithmetic functions h may be studied in terms of frequencies $\nu_N(n; \log h(n) - \alpha(N) \le z\beta(N))$. For those which grow rapidly there is another perspective. Let $N(y)$ denote the number of integers n for which $h(n) \le y$, and characterize those h for which $\delta = \lim y^{-1}N(y)$, $y \to \infty$, exists and is positive. For example, when h is Euler's totient function, $\delta = \varsigma(2)\varsigma(3)/\varsigma(6)$, in terms of the Riemann zeta function. This is already asserted in Erdős [*25*]. For a sharpening of this result see Diamond [*9*]. A detailed account of a general theory for multiplicative functions h, employing the method of Halász (ref. [63] of Postnikov), is given by Diamond and Erdős [*10*].

§**4.10.** The important pioneering results stated in this section have been improved or generalised by many authors, and form part of an extensive and rapidly growing area known as Probabilistic Number Theory. For an account of such matters up until 1980 see Elliott [*17*]; see also Elliott [*21*] (especially the Supplement), Babu [*2*], and Alladi [*1*]. There are, moreover, many interesting recent developments.

§4.11. It is interesting to compare the treatment of the integral equation (7) in Postnikov's account of the method of Levin and Faĭnleĭb, with the proof of Lemma 4 which he gives in §4.3 when establishing Wirsing's theorem on mean values.

§4.12. For further results concerning the Riemann zeta function and divisor problems see Ivić [*36*], and the remarks of Heath-Brown in the new edition of the well-known book of Titchmarsh [*54*].

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