Lattices with Unique Complements

V. N. SALIĬ

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РЕШЕТКИ С ЕДИНСТВЕННЫМИ ДОПОЛНЕНИЯМИ

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ABSTRACT. It is known that the class of uniquely complemented lattices contains all Boolean lattices, but is not limited to these. However at present there is no explicit example of a non-Boolean lattice of this class. Also unanswered is the question of whether this class contains any complete non-Boolean lattices. This book is the first monograph devoted to these classical problems of lattice theory. It contains all of the necessary information on ordered sets and lattices, and requires no specialized knowledge of the reader.

This book is intended for scholars, graduate students, and students studying or interested in general algebra and mathematical logic.

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Preface

Time passes, and one after another the classical problems of lattice theory are being solved. It is already known that the congruence lattices of algebras are precisely the compactly generated lattices (Grätzer and Schmidt, 1963), that the variety of all lattices can be defined by a single identity (McKenzie, 1970), that each finite lattice is isomorphic to a lattice of partitions of a finite set (Pudlák and Tůma, 1977), and that the finite sublattices of a free lattice can be described by the quasi-identities of semidistributivity and the Whitman condition (Nation, 1982). Apparently the algorithmic unsolvability of the word problem for a free modular lattice with four generators will be proved in the near future.

Another problem in this series is the one pertaining to the mysterious class of uniquely complemented lattices.

As early as 1904, while investigating various axiom systems for a Boolean algebra, Huntington suggested that it probably could be defined as a lattice in which each element has one and only one complement. Not long before, German mathematicians (Schröder, Vogt, Lüroth, Korselt, and Dedekind), by a series of more and more convincing counterexamples, finally broke the stubborn resistance of Peirce, who considered all lattices to be distributive, and a new attempt to eliminate distributivity from the list of Boolean postulates could not, of course, remain unnoticed.

At the end of the 1930's there was still neither a proof nor a refutation of Huntington's conjecture, but in the face of many corroborating facts (when additional conditions, such as modularity, atomicity, and De Morgan's laws, were imposed), there were few who doubted its validity. In 1945 there appeared most unexpectedly a theorem of Dilworth: Any lattice is embeddable in a uniquely complemented lattice.

The exceedingly complicated construction actually produced nondistributive uniquely complemented lattices, but they were obtained by a limit process and were so cumbersome that, for example, each such lattice contained a free lattice on a countable number of generators. In the last forty years Dilworth's proof was essentially simplified three times, and in the last version (Adams and Sichler) it seems to be completely accessible. Nevertheless, it is of some concern that at present there is not even one explicit example, outside the class of Boolean algebras, of a uniquely complemented lattice—such lattices have not been encountered in mathematical practice.

A more important observation is that the existing methods for constructing nondistributive uniquely complemented lattices do not preserve completeness. Therefore in the class of complete lattices Huntington's question is still unanswered: Do there exist complete nondistributive uniquely complemented lattices? This is the problem of which we spoke at the outset.

The present book can be briefly characterized as lattice theory focused on one problem. It actually contains all of the information from lattice theory, from the very first definitions, connected in some way with the central problem.

The whole first chapter and parts of the second and third are auxiliary in nature: in them we introduce the basic concepts and prove some results needed for the subsequent development. General algebraic constructions and the corresponding properties are assumed to be known. From among the assertions on the foundations of the theory of ordered sets we take as a postulate the Hausdorff principle and deduce from it the Kuratowski-Zorn lemma, Zermelo's theorem, and the theorem on the existence of a choice function. All of these are constantly and explicitly used in proofs.

Being narrowly focused, the above-mentioned sections can make no pretense to constituting a general introduction to the theory of lattices and Boolean algebras; for this the reader should turn to the books of Skornyakov [1] and Vladimirov [1].

The second chapter can be regarded as a mathematical history of Huntington's problem. Uniquely complemented lattices arose in the investigation of the axiomatics of Boolean algebras (Huntington's theorem). Starting from the definition of a Boolean algebra and Peirce's theorem, we will run through a whole series of propositions establishing distributivity of a uniquely complemented lattice under various conditions. A new criterion for distributivity of such lattices (Theorem 13) enables us to present all such results in a rather simple and uniform way. The text contains many figures; they help to enlist one's geometric intuition in verifying nonobvious calculations. The chapter ends with a proof of Dilworth's theorem and an example, due to Adams and Sichler, of a proper lattice variety containing nondistributive uniquely complemented lattices.

In the third chapter we consider various attempts at progress in solving the main problem. Along with known results (distributivity of compactly generated and, in general, continuous uniquely complemented lattices, the decomposition of a complete uniquely complemented lattice into a direct product of an atomic and an atomless component, and the completion of a Boolean algebra by cuts) we discuss some new approaches, mainly connected with the search for suitable representations of complete lattices. See Theorem 23 on the realization of a complete lattice as the lattice of all P-subalgebras of an elementary unary P-algebra, Theorem 38 on direct decompositions of a complete uniquely complemented lattice by means of regular elements, and the interpretation of properties of complete lattices in the language of lattice transformations of sets (§7). Some important results on complete Boolean algebras can be proved without using distributivity, and then they become theorems for the entire class of complete uniquely complemented lattices (e.g., Theorems 26, 31, and 32).

I would like to thank Professor L. A. Skornyakov for his comments, some of which turned out to be very significant.

This book ends with a list of ten problems with references to the context in which they arose, and with a bibiliography containing items of particular interest to those concerned with uniquely complemented lattices.

As history shows, the definitive solution of difficult algebraic problems does not always have consequences equivalent to the mathematical values achieved in the course of attaining the goal. The above-mentioned problems were not exceptional in this sense, and we can state in advance that neither a proof of distributivity of complete uniquely complemented lattices nor the discovery of a counterexample wil elicit far-reaching changes in lattice theory. But the apparent inaccessibility of the problem under discussion testifies to the lack of suitable methods for attacking it, and therefore we can hope that, as has often happened, the creation of such methods will stimulate new fruitful investigations.

V. Saliĭ

Saratov, May 1982

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Problems

I. Is it true that if for any element x of a uniquely complemented lattice L each element of the o-subset $Z(x) = \{y \in L \colon x \wedge y = x' \wedge y = O\}$ is contained in some maximal element and $Z(x)$ has only a finite number of maximal elements, then L is distributive? (II.4.8)

II. Is the variety of p-modular lattices the smallest of the nondistributive Dilworth varieties? (II.5.3)

III. Give an explicit example of a nondistributive uniquely complemented lattice. (II.5.4)

IV. Is the property of uniqueness of complements preserved under cut-completion? (III.2.3)

V. Is it possible to prove Theorem 28 without using the axiom of choice? (III.5.4)

VI. Is every $\wedge$-continuous uniquely complemented lattice distributive? (III.5.7)

VII. Is a uniquely complemented lattice distributive if it is both $\wedge$-continuous and (dually) $\vee$-continuous? (III.5.7)

VIII. Find an internal characterization of the class of completely collectively complemented lattices. (III.6.3)

IX. Can every association be (isomorphically) embedded in some association of the form $K_L(A)$, where A is a nonempty set and L a completely complemented lattice? (III.7.5)

X. *Does there exist a complete nondistributive uniquely complemented lattice?*

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