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Introduction to the Spectral Theory of Polynomial Operator Pencils

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Providence, Rhode Island

АЛЕКСАНДР СЕМЕНОВИЧ МАРКУС

ВВЕДЕНИЕ В СПЕКТРАЛЬНУЮ ТЕОРИЮ
ПОЛИНОМИАЛЬНЫХ ОПЕРАТОРНЫХ ПУЧКОВ

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ABSTRACT. This monograph contains an exposition of the foundations of the spectral theory of polynomial operator pencils acting in a Hilbert space. The main attention is given to the fundamental results of Keldysh on multiple completeness of the eigenvectors and associated vectors of a pencil and on the asymptotic behavior of its eigenvalues, as well as to generalizations of these results. Various theorems on spectral factorization of pencils are presented which are a development of known results of M. G. Kreĭn and Heinz Langer. A large portion of the book involves the theory of selfadjoint pencils, which has numerous applications. This book is intended for mathematicians, workers in mechanics, and theoretical physicists interested in spectral theory and its applications.

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APPENDIX*

On the Eigenvalues and Eigenfunctions of Certain Classes of Nonselfadjoint Equations

M. V. KELDYSH

1. In an appropriate Hilbert space, all the equations considered below can be reduced to the form

$$y = L(\lambda)y + f, \qquad L(\lambda) = K_0 + \lambda K_1 + \cdots + \lambda^n K_n, \tag{1}$$

where y and f are elements of the Hilbert space, λ is a complex parameter, and the K_i are compact operators.*

A compact operator $R(\lambda)$ is the resolvent of $L(\lambda)$ if $(E+R)(E-L) = E$. If the resolvent exists for some $\lambda = \lambda_0$, it is a meromorphic function of λ on the whole plane. We say that y is an eigenelement for the eigenvalue $\lambda = c$, and that $y_1, \dots, y_k$ are elements associated with it (or associated elements) if

$$y = L(c)y, \quad y_k = L(c)y_k + \frac{1}{1!}\frac{\partial L(c)}{\partial c}y_{k-1} + \cdots + \frac{1}{k!}\frac{\partial^k L(c)}{\partial c^k}y. \tag{2}$$

Note that if y is an eigenelement and $y_1, \dots, y_k$ are elements associated with it, then $y(t) = e^{ct}(y_k + y_{k-1}t/1! + \cdots + yt^k/k!)$ is a solution of the equation $y = K_0 y + K_1 \partial y/\partial t + \cdots + K_n \partial^n y/\partial t^n$.

If $\lambda = c$ is a pole of the resolvent $R(\lambda)$, then the principal part of the resolvent is a sum of terms of the form (by yz we understand the operator $Af = (f, z)y$)

$$\frac{y^{(i)}z^{(i)}}{(\lambda - c)^{m_i}} + \frac{y^{(i)}z_1^{(i)} + y_1^{(i)}z^{(i)}}{(\lambda - c)^{m_i - 1}} + \cdots + \frac{y^{(i)}z_{m_i-1}^{(i)} + y_1^{(i)}z_{m_i-2}^{(i)} + \cdots + y_{m_i-1}^{(i)}z^{(i)}}{\lambda - c}, \tag{3}$$

where $y^{(i)}$ and $y_1^{(i)}, \dots, y_{m_i-1}^{(i)}$ are an eigenfunction and functions associated with it for equation (1), while $z^{(i)}$ and $z_1^{(i)}, \dots, z_{m_i-1}^{(i)}$ are an eigenfunction

*This appendix, appearing here under the direction of the Translations Committee of the American Mathematical Society, is a translation of [150] (MR 12, 835).

and functions associated with it for $\lambda = \bar{c}$ for the adjoint equation $z = L^*(\lambda)z$, where $L^*(\lambda)$ is the adjoint of $L(\bar{\lambda})$.

For each eigenvalue, the eigenfunctions and associated functions that appear in the expression (3) for a pole of the resolvent can always be chosen so that, for $k = 0, \dots, m_i - 1$ and $l = 0, \dots, m_j - 1$, all the eigenfunctions and associated functions of equation (1) satisfy

$$\left(\sum_{\nu=0}^{n-1} Y_{\nu k}^{(i)} c_j^{\nu}, z_l^{(j)}\right) + \left(\sum_{\nu=1}^{n-1} \binom{\nu}{1} Y_{\nu k}^{(i)} c_j^{\nu-1}, z_{l-1}^{(j)}\right) + \cdots + \left(\sum_{\nu=l}^{n-1} \binom{\nu}{l} Y_{\nu k}^{(i)} c_j^{\nu-l}, z^{(j)}\right) = \delta_{ij}\delta_{k,m_j-l-1}, \tag{4}$$

where

$$(E - L(\lambda))\left(\frac{y^{(i)}}{(\lambda - c)^{k+1}} + \frac{y_1^{(i)}}{(\lambda - c)^k} + \cdots + \frac{y_k^{(i)}}{\lambda - c}\right) = \sum_{\nu=0}^{n-1} Y_{\nu k}^{(i)} \lambda^{\nu}. \tag{5}$$

The number N of eigenfunctions and associated functions corresponding to a pole of the resolvent is called the *multiplicity* of the eigenvalue. Note that the trace of the principal part of the operator $(\partial L/\partial\lambda)R(\lambda)$ for the pole $\lambda = c$ is equal to $N/(c - \lambda)$.

Starting from the eigenelements and associated elements satisfying (4), we construct n systems of eigenelements and associated elements

$$u_{\nu k}^{(i)} = \left[\frac{d^{\nu}}{dt^{\nu}} e^{c_i t}\left(y_k^{(i)} + y_{k-1}^{(i)} \frac{t}{1!} + \cdots + y^{(i)} \frac{t^k}{k!}\right)\right]_{t=0}, \qquad \nu = 0, 1, \dots, n-1.$$

Definition. A system of eigenelements and associated elements of equation (1) is said to be *n-fold complete* if every system of n elements $f_1, \dots, f_n$ of the Hilbert space can be represented as a limit of linear combinations $f_{\nu}^{(N)} = \sum a_{kN}^{(i)} u_{\nu-1,k}^{(i)}$, $\nu = 1, \dots, n$, with coefficients not depending on ν.

In particular, for $n = 1$ we have completeness in the usual sense.

We remark that, if we set

$$\sum_{\nu=0}^{n-1} Z_{\nu k}^{(i)} \lambda^{\nu} = L^*(\lambda)\left(\frac{z^{(i)}}{(\lambda - \bar{c}_i)^{k+1}} + \frac{z_1^{(i)}}{(\lambda - \bar{c}_i)^k} + \cdots + \frac{z_k^{(i)}}{\lambda - \bar{c}_i}\right),$$

then

$$\sum_{\nu=1}^{n-1} (Z_{\nu k}^{(i)}, u_{\nu l}^{(j)}) = \delta_{ij}\delta_{k,m_j-l+1}.$$

If the expansion

$$f_\nu = \sum_{i=1}^{\infty} \sum_{k=0}^{m_i-1} a_k^{(i)} u_{\nu-1,k}^{(i)}, \qquad \nu = 1, 2, \dots, n, \tag{6}$$

converges, then the coefficients are determined by

$$a_k^{(i)} = \sum_{\nu=0}^{n-1} (f_{\nu+1}, Z_{\nu,m_i-k+1}^{(i)}).$$

Hence, in particular, the expansions of the form (6) are unique.

2. We call an operator H *complete* if the system of its eigenfunctions $x = \lambda H x$, $\lambda \neq \infty$, is complete.

THEOREM 1. *Let H be a complete selfadjoint operator, some power H^m of which has finite absolute norm, let A be an arbitrary compact operator, and let $B_1, \dots, B_{n-1}$ be bounded operators. Then the system of eigenfunctions of the equation*

$$y = (A + \lambda H B_1 + \cdots + \lambda^{n-1} H B_{n-1} + \lambda^n H) y \tag{7}$$

(*and also the adjoint equation*) *is n-fold complete. The eigenvalues of* (7) *asymptotically approach the rays* $\arg \lambda = k\pi/n$.

Note that under the conditions of Theorem 1 the expansion (6) does not necessarily converge.

Suppose the operator H is positive. Let $\varphi(x)$ denote the number of eigenvalues of H not exceeding x, and let $\varphi_n(x) = \varphi(x^n)$. Denote by $\psi_\nu(x)$ the number of eigenvalues of equation (7) situated in the angle $(2\nu - 1)\pi/n < \arg \lambda < (2\nu + 1)\pi/n$ and not exceeding x in modulus.

THEOREM 2. *If in the conditions of Theorem* 1 *the operator is positive, then the eigenvalues of* (7) *asymptotically approach the rays* $\arg \lambda = 2\nu\pi/n$, *and for any* $\varepsilon > 0$ *there exist arbitrarily large values of x for which*

$$(1 - \varepsilon)\varphi_n(x) \leq \psi_\nu(x) \leq (1 + \varepsilon)\varphi_n(x).$$

Imposing restrictions on $\varphi(x)$, we can obtain a more precise proposition.

THEOREM 3. *If, in the conditions of Theorem* 2, $\varphi_n(x)/\omega(x) \to 1$ *as* $x \to \infty$, *where $\omega(x)$ is an increasing continuously differentiable function satisfying* $\alpha\omega(x) < x\omega'(x) < \beta\omega(x)$, $\beta < \alpha + 1$, *then* $\psi_\nu(x)/\varphi_n(x) \to 1$.

The proof is based on the following Tauberian theorem.

THEOREM 4. *Let $\varphi(x)$ and $\psi(x)$ be positive increasing functions, defined for $x > 0$, with $\varphi(x) \to \infty$ and $\alpha\varphi(x) < x\varphi'(x) < \beta\varphi(x)$, $\beta < \alpha + 1$, and define*

$$f(x) = \int_0^\infty \frac{d\varphi(\xi)}{(\xi + x)^{m+1}}, \qquad g(x) = \int_0^\infty \frac{d\psi(\xi)}{(\xi + x)^{m+1}},$$

*where m is the integer part of β. If $f(x)/g(x) \to 1$ as $x \to \infty$, then $\varphi(x)/\psi(x)$
$\to 1$ as $x \to \infty$.*

3. The results of §2 can be used to establish completeness of the system of eigenfunctions, and also to obtain asymptotic expressions for the eigenvalues of large classes of nonselfadjoint differential equations.

As an example we consider elliptic partial differential equations

$$M_\lambda(u) = \sum_{i,k=1}^{m} p_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{m} q_i \frac{\partial u}{\partial x_i} + (r_0 + \lambda r_1 + \cdots + \lambda^n r_n)u = 0 \quad (8)$$

in a domain D, satisfying Lyapunov's conditions, whose coefficients are continuously differentiable functions of $x_1, \ldots, x_m$, and where the p_{ik} and r_n are real with $r_n > \alpha$ and $\sum_{i,k} p_{ik}\xi_i\xi_k > \beta \sum_i \xi_i^2$ in D for $\alpha, \beta > 0$.

THEOREM 5. *For the equation $M_\lambda(u) = 0$ with the boundary condition $u = 0$ or $\sum_{i,k} p_{ik}(\partial u/\partial x_i)\cos(N, x_k) - \sigma u = 0$, where σ is a continuous complex-valued function of a point of the boundary of the domain and N is the normal to the boundary of D, the following assertions are true:*

1) *The system of eigenfunctions and associated functions is n-fold complete.*

2) *The eigenvalues lying in the angle $(2\nu - 1)\pi/n \leq \arg\lambda < (2\nu + 1)\pi/n$ have the asymptotic expressions*

$$\lambda_k^{(\nu)} \sim \left\{\frac{\sigma_m}{2^m \pi^m}\int_D \frac{dx_1 dx_2 \cdots dx_n}{\sqrt{\Delta(x_1, x_2, \ldots, x_n)}}\right\}^{-2/nm} k^{2/mn} e^{2\nu\pi i/n},$$

where σ_m is the volume of the m-dimensional unit ball and $\Delta(x_1, \ldots, x_m)$ is the determinant of $|p_{ik}|$.

Note that for $n = 1$ assertion 2) was established by Carleman in [64].

For ordinary differential equations, in some cases more precise theorems can be established.

Consider the equation

$$M_\lambda(u) = u^{(m)} + p_1(x, \lambda)u^{(m-1)} + \cdots + p_{m-1}(x, \lambda)u' + [p_m(x, \lambda) + \lambda^n]u = 0, \quad (9)$$

where $p_k(x, \lambda)$ is a polynomial in λ of degree less than kn/m.

Under certain restrictions on the boundary conditions we can establish n-fold completeness of the system of eigenfunctions of equation (9), and, for $m \leq n$, conditions for convergence of the n-fold expansions.

For definiteness we indicate only two cases of boundary conditions:

$$y^{(k)}(0) = y^{(k)}(1), \qquad k = 0, 1, \ldots, m-1, \tag{10}$$

and, for $0 < l < n$,

$$\begin{aligned} &\sum_{k=0}^{m-1} \alpha_k^{(nu)} y^{(k)}(0) = 0, \qquad \nu = 1, \ldots, l; \\ &\sum_{k=0}^{m-1} \beta_k^{(\nu)} y^{(k)}(1) = 0, \qquad \nu = l+1, \ldots, m. \end{aligned} \tag{11}$$

THEOREM 6.** *The system of eigenfunctions of equation* (9) *under the boundary conditions* (10) *or* (11) *is n-fold complete.*

For $m > n$ the expansions

$$y_\nu = \sum_{t=1}^{\infty} \sum_k a_k^{(i)} u_{\nu-1,k}^{(i)}, \qquad \nu = 1, 2, \ldots, n, \tag{12}$$

are uniformly convergent for n arbitrary m-times differentiable functions satisfying the boundary conditions.

For $n = m$ there is a subsequence of partial sums of (12) *which converges uniformly.*

Note that for $n = 1$ this result was obtained by Birkhoff [370]. For $n = m$ Tamarkin [335], [336] established a number of propositions on the expansion of a function in a series of eigenfunctions.

** *Added in translation by A. S. Markus.* In [155] it is shown that the second assertion of this theorem is false for the boundary conditions (11) when $n = 1$ and $l \neq m - l$.

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Brief Comments on the Literature

§**1.** Lemmas 1.7–1.10 are taken from [236].

§**3.** See also [103], Chapter V, §7. Lemmas 3.4 and 3.5 are taken from [235].

§**4.** The proof of Theorem 4.2 is taken from [153]. Keldysh posed the question of the sufficiency of a condition for the order of the operator H in Theorem 4.2 to be finite (see [153], p. 111, and [249]). A positive answer to this question along with precise (in a certain sense) restrictions on H can be found in [249].

Of the numerous results on completeness of the root vectors of a nonselfadjoint operator we mention [71], Chapter XI, §§6 and 9, [72], Chapter XIX, §5, [103], Chapter V, [153], [179], [210], [211], [246], [247], and [275].

§**5.** Lemma 5.6 is taken from [233] (see also [148], Theorem III.1.2). In place of the term "p-subordinate" the term "subordinate with order p" is also used [188].

§**6.** See [103] and [278] about bases in spaces.

In this section we follow the paper [233]. Theorem 6.12 was proved earlier in [148] (Theorem III.4.1) under the condition that the operator B is subordinate to the operator G^p. The reference [148] was preceded by the article [221], where it was established under somewhat stronger restrictions that the root vectors of A form a Bari basis with parentheses (see [177] and [220]).

It was shown in [148] (Theorem III.5.1) that the conditions in Theorem 6.12 cannot be weakened.

Diverse results on convergence of expansions in root vectors of an operator close to a selfadjoint operator were obtained in [25]–[27], [68], [84], [85], and [147]. Perturbation theory for spectral operators was investigated in [54], [72], Chapters XIX and XX, [173], and [321]. A significant number of sources have been devoted to the study of the basis property for the root vectors of dissipative operators and contractions (see [103], Chapter VI, and [278], Lectures VI–X).

§**7.** Lemmas 7.2 and 7.3 are taken from [235].

§**8.** Here we follow [235]. The proof of Theorem 8.4 (under the conditions (8.31) and (8.32)) obtained by Keldysh [150] was based on his Tauberian theorem [151] (see also [103], Chapter V, §11, and [234] and [252]).

§**9.** We follow [235]. Some applications of the results in this section to the determination of spectral asymptotics for differential and pseudodifferential operators are indicated in [62], [171], [172], and [207]. The references [231], [232], [234], and [263] are connected with the contents of §9.

§**10.** See the survey [58] and the monograph [138] about the spectral asymptotics of selfadjoint differential operators. Birkhoff [56] found the asymptotic behavior of the eigenvalues of nonselfadjoint ordinary differential operators for regular boundary conditions (see also [72], Chapter XIX, §4, and [277], Chapter II, §4). The first theorems on the asymptotic behavior of the spectrum of a nonselfadjoint elliptic differential operator were obtained by Carleman [64]; see [24], [170], [232], [234], [244], [263], [264], [269], [270], [285], [312], [318], [339], and [357] for subsequent results.

Birkhoff [56] began the investigation of completeness of root vectors and convergence of expansions in them for nonselfadjoint ordinary differential operators (see also [277], Chapter II, §§5 and 9). The first theorems on completeness of the root functions for nonselfadjoint elliptic differential operators are due to Keldysh [150] and Browder [63] (see also [71], Theorem XIV.6.28). Various results on completeness and convergence of expansions in root functions of differential operators are contained in [23], [25], [26], [28], [29], [59], [72], [77], [84], [85], [115], [132]–[134], [148], [153]–[155], [173], [192], [210]–[212], [221], [233], [265], [275]–[277], [307], [308], [321], [324], [325], [327], and [341].

§**11.** The concept of a chain of an eigenvector and associated vectors and the concept of the multiplicity of an eigenvalue are due to Keldysh [150]. Various properties of the multiplicity of an eigenvalue of a holomorphic operator-valued function were established in [60], [79], [80], [113], [152], [219], [238], [240], [332], and [367]. The definition of a generating polynomial (root function) was introduced independently by S. G. Kreĭn and Trofimov [340] and by Matsaev and Palant [284].

§**12.** Linearization of a polynomial pencil was considered as far back as Keldysh [152]. See [274] about Theorem 12.9. A more general result relating to holomorphic operator-valued functions is contained in [103], Theorem I.5.1, and [309], Theorem VI.14.

See [97], [125], [139], [230], and [337] about different methods for linearizing holomorphic operator-valued functions.

§**13.** A general approach to the study of completeness of derived chains corresponding to various boundary value problems was developed in [290], [292], [293], [295], [296], [299], [300], and [303]. These references were preceded by [186] and [90]–[92], where the concept of completeness for boundary value problems on a half-line were studied, and by [347], which dealt with the concept of completeness for boundary value problems on a finite interval.

§**14.** Lemmas 14.7 and 14.8 were established in [284].

§**15.** Theorems 15.4 and 15.5 were proved by Keldysh in [150] and [152].

Theorems 15.6 and 15.7 were also established by Keldysh [150] under certain additional restrictions (see Remark 15.9), but his proofs were not published. Here we follow [235]. Theorem 15.13 was proved in [234]. Various analogues and generalizations of the Keldysh theorems on asymptotics were obtained in [44], [45], [234], [236], [250]–[252], and [302].

§**16.** Theorem 16.3 was established in [150] and [152].

§**17.** Theorem 17.3 and Lemma 17.4 were obtained in [150] and [152].

Theorem 17.10 was proved in [223]. Various results on convergence of multiple expansions for Keldysh pencils were established in [34], [36], [37], [73], [351], and [362]. The basis property was investigated in [291] and [301] for derived chains corresponding to various boundary value problems.

§**18.** Further results on multiple completeness for Keldysh pencils and pencils close to them were obtained in [32], [34], [38], [137], [248], [272], [273], [283], [284], [289], [290], [292], [297], [299], [300], and [354] (see also the survey [303]).

§**19.** The contents of this section are taken from [293] (a more general result is obtained there). Diverse results on completeness, multiple completeness, and completeness (multiple completeness) with finite defect for a part of the eigenvectors and associated vectors of a pencil were established in [90]–[92], [135], [137], [267], [268], [289], [294]–[296], [298]–[300], and [303]. The problem of minimality (multiple minimality) of a part of the eigenvectors and associated vectors was considered in [167], [304], [305], [352], and [353].

§**20.** Theorem 20.8 was proved in [351] under certain additional assumptions (see also [73]). Convergence of multiple expansions for pencils with unbounded coefficients was considered in [291], [301], and [362].

§**21.** Keldysh [150] obtained the first results on multiple completeness and on the asymptotic behavior of the spectrum for pencils generated by elliptic differential operators (for $m = 1$ and under certain additional restrictions he established the assertions of Theorem 21.1 on multiple completeness and on the principal term of the spectral asymptotics).

There are various assertions about the distribution of the spectrum, (multiple) completeness, and convergence of (multiple) expansions for pencils generated by differential operators in [25], [75], [76], [78], [93], [94], [130], [131], [218], [282], [326], [335], [336], [344], and [351].

Boundary value problems for differential equations containing a spectral parameter in the boundary conditions have been studied in [25], [26], [88], [127], [169], [175], [253], [282], [319], [320], [328], [335], [336], and [356].

§**22.** Factorization of a pencil and operator roots were first used by M. G. Kreĭn and Langer in [186], where Lemmas 22.9 and 22.10 were obtained (for a quadratic pencil).

See [66] and [108] about canonical factorization.

§**23.** Theorem 23.3 (known as the Masani lemma or the basic factorization lemma) first appeared (in a somewhat different form) in [245]. Diverse variants and generalizations of it are found in many papers. Here we follow [96], Chapter I, Lemma 5.1.

The results in subsections 4 and 5 were obtained (by other methods) in [224], [225], and [289]; see also [87], [116], [117], [136], [161], [163], and [346].

§**24.** Theorem 24.2 was proved in [109] (for any Wiener operator-valued function). We present another proof. Lemma 24.4 was also established in [109].

§**25.** Theorem 25.3 was established in [108].

§**26.** The proof given for the Toeplitz-Hausdorff theorem was taken from [306]. The set $W(A)$ is sometimes called the numerical range or the

Hausdorff set of A. The term "numerical range" ("numerical domain") is also used (see [137] and [281]) for the set $R(A)$ which we have called the root domain of an operator-valued function $A(\lambda)$.

Theorem 26.6 is contained in [350], and Theorem 26.7 is in [241] (see also [137] and [281]).

The condition (26.8) was first considered in [333], where it was shown that under this condition a Hölder matrix-valued function $A(t)$ ($t \in \Gamma_0$) admits a factorization of the form $A(t) = t^k A_+(t)A_-(t)$.

The results in subsections 5 and 6 were established in [226] and [227].

§**27.** The results in this section were obtained in [228] (Theorem 27.6 was established there in a less precise formulation). That paper was preceded by [349], where a weakened variant of Theorem 27.5 was proved (without restrictions on the size of the matrices under consideration).

The monographs [107], [149], and [193] are devoted to matrix pencils.

§**28.** The connection in Theorem 28.2 between operator roots of a pencil and special invariant subspaces of the linearizer is one of the important elements of the Kreĭn-Langer method (see [186], §5).

Theorem 28.2 was established in [198], Lemma 28.6 in [237], and Theorem 28.8 in [254] and [256]. Generalizations of Theorems 28.2 and 28.8 to divisors of arbitrary order were obtained in [142] and [202]. Analogous results were proved for holomorphic operator-valued functions in [230] and [337].

§**29.** The contents of this section were taken from [237]. The starting point for this paper was the concept, introduced by M. G. Kreĭn and Heinz Langer [186], of a complete pair of operator roots. In view of Remark 29.14, Theorems 29.6 and 29.8 follow from results in [186] when $n = 2$.

Generalizations of the results in §29 to divisors of arbitrary order were established in [101] and [142]. Various theorems on factorization of pencils were obtained in [50], [53], [98]–[102], [104]–[107], [110]–[112], [136], [254], [257], [261], [262], [313], [317], [337], [363], [366], and [367].

§**30.** Theorem 30.4 and Corollary 30.5 were proved in [226], Theorems 30.6 and 30.11 in [350], and Corollary 30.8 in [198] and [350]. These references were preceded by the papers [239] and [343].

Diverse results on Riesz bases are contained in [52], [103], and [278].

§**31.** The main properties of quadratic hyperbolic pencils (strongly damped pencils) were established by Kreĭn and Langer ([186], [196]; see also §35). The corresponding results for matrix pencils were obtained earlier in [69].

Lemma 31.2 was proved in [191]. Theorem 31.3 was established for quadratic matrix pencils in [69], for quadratic operator pencils in [186] and [196], and for the general case in [241]. Theorems 31.5 and 31.10 were obtained in [199], Theorems 31.11, 31.14, and 31.16 in [241], Theorem 31.17 in [51] and [237], Theorem 31.18 in [237], Theorem 31.19 in [141], and the results in subsections 8–10 in [241].

Theorem 31.27 was proved under the additional condition $B > 0$ in [182] and [186]. Plus and minus eigenvectors and eigenvalues were first considered in these articles (the terms "eigenvectors (eigenvalues) of the first and second kinds" were used there).

The term "spectral zone" was used in [241] instead of the term "root zone" taken here.

Theorem 31.19 was carried over to weakly hyperbolic pencils in [230]. Other results on decomposition of pencils into linear and nonlinear factors were established in [101], [255], and [258]–[260]. See also [281] and [287] about hyperbolic pencils and operator-valued functions.

The class of quadratic pencils, as the most important class for applications as well as the simplest class, has attracted the attention of many authors ([43], [74], [114], [116]–[118], [164], [165], [167], [168], [182], [186], [204], [266], [280], [296], [304], [305], [326], [360], and [361]). We mention separately references which consider the pencil $\lambda I - A - \lambda^2 B$, where A and B are selfadjoint compact operators ("S. G. Kreĭn pencils"): [36], [41], [42], [46], [78], [86], [87], [119], [120], [161], [163], [182], [186], [187], [203], [204], [224], [225], [241], [272], [273], [289], [342], [343], [358], and [369].

§**32.** Variational methods for nonlinear spectral problems were studied in [1]–[22], [69], [122], [123], [191], [196], [208], [314], [315], [342], [343], and [359] (see also the bibliography of [22]).

Theorems 32.8, 32.9, and 32.16 are apparently new results in this section.

The articles [144], [145], and [338] are devoted to approximation methods for determining the eigenvalues and eigenvectors of operator pencils.

§**33.** The results in this section were obtained by V. I. Matsaev and the author [369]. Corollary 33.7 answers a question posed by Kopachevskiĭ.

§**34.** Quadratic elliptic pencils (weakly damped pencils) were first studied by M. G. Kreĭn and Heinz Langer in [182] and [186]. Factorization of nonnegative operator-valued functions has numerous applications in analysis and probability theory, and a number of papers have been devoted to it (see [67], [316], and [334]).

Theorem 34.3 was proved by Matsaev (see [40]). Theorem 34.5 was established by Ginzburg [95], and the proof given is due to Matsaev (see also [348]). Lemma 34.12 was obtained in [316] (without the requirement that $A(\lambda_0)$ be invertible).

§**35.** There is a good introduction to the theory of operators acting in Kreĭn spaces in [182], and a systematic exposition of this theory in [48] and [365].

Lemma 35.2 was established by Phillips [286].

Theorem 35.3 was proved by Langer [194], [195]. Theorem 35.4 is contained in [197]. The simplified presentations of the proofs of Theorems 35.3 and 35.4 given here were constructed by T. Ya. Azizov on the basis of [61] and [197]. Theorem 35.6 was established in [194] and [196], but it was obtained earlier under certain additional assumptions in [186].

A generalization of Theorem 35.6 to weakly hyperbolic pencils of arbitrary order was given in [200], [229], and [230]. Further developments of the Kreĭn-Langer method [186] are contained in [107], [124], [164], [165], [198]–[202], and [366].

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Bibliography

1. Yu. Sh. Abramov, *On the theory of nonlinear eigenvalue problems*, Dokl. Akad. Nauk SSSR **212** (1973), 11–14; English transl. in Soviet Math. Dokl. **14** (1973).

2. ——, *Variational principles for nonlinear eigenvalue problems*, Funktsional. Anal. i Prilo-zhen. **7** (1973), no. 4, 76–77; English transl. in Functional Anal. Appl. **7** (1973).

3. ——, *Variational properties of the eigenvalues of certain problems that are nonlinear with respect to the parameter*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. **9** (1973), 23–39. (Russian)

4. ——, *A class of nonlinear eigenvalue problems*, Mat. Zametki **15** (1974), 907–913; English transl. in Math. Notes **15** (1974).

5. ——, *Analogue of Newton's method for certain classes of nonlinear eigenvalue problems*, Mat. Issled. **9** (1974), no. 3(33), 15–21. (Russian)

6. ——, *Elliptic operators that depend nonlinearly on a parameter*, Differentsial'nye Uravneniya **10** (1974), 1525–1526; English transl. in Differential Equations **10** (1974).

7. ——, *Inequalities for the eigenvalues of some nonlinear spectral problems*, Izv. Vyssh. Uchebn. Zaved. Mat. **1977**, no. 12(187), 3–6; English transl. in Soviet Math. (Iz. VUZ) **21** (1977).

8. ——, *Some questions on the spectral theory of the equations $Tx = \lambda Sx$*, Izv. Vyssh. Uchebn. Zaved. Mat. **1977**, no. 1(176), 3–13; English transl. in Soviet Math. (Iz. VUZ) **21** (1977).

9. ——, *A duality theorem for a nonlinear programming problem with Rayleigh functionals*, Third All-Union Sem. Numerical Methods of Nonlinear Programming, Abstracts of Reports, Kharkov, 1979, pp. 18–19. (Russian)

10. ——, *A class of nonlinear programming problems with Rayleigh functionals*, First All-Union Conf. System-Modelling of Socio-Economic Processes, Abstracts of Reports, Voronezh, 1980, pp. 104–105. (Russian)

11. ——, *Duality in extremal problems generated by spectral problems for operator pencils*, Dokl. Akad. Nauk SSSR **255** (1980), 777–780; English transl. in Soviet Math. Dokl. **22** (1980).

12. ——, *Duality in spectral optimization, and numerical ranges of a family of selfadjoint operators*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **107** (1982), 189–192; English transl. in J. Soviet Math. **36** (1987), no. 3.

13. ——, *Numerical ranges, zones, and spectra of families of selfadjoint operators*, Dokl. Akad. Nauk SSSR **257** (1981), 1033–1037; English transl. in Soviet Math. Dokl. **23** (1981).

14. ——, *Solvability of extremal problems generated by operator pencils*, Optimizatsiya Vyp. 27(44) (1981), 143–152. (Russian)

15. ——, *Extremal problems generated by spectral problems*, Uspekhi Mat. Nauk **36** (1981), no. 5(221), 161–162; English transl. in Russian Math. Surveys **36** (1981).

16. ——, *On a property of operator matrices and its determinant*, Voprosy Vychisl. i Prikl. Mat. (Tashkent) Vyp. 66 (1981), 137–141. (Russian)

17. ——, *The minimax relation for a class of nonconvex functions generated by operator pencils*, Dokl. Akad. Nauk UzSSR **1981**, no. 7, 11–13. (Russian)

18. ——, *On optimal plans for spectral optimization problems*, Dokl. Akad. Nauk SSSR **261** (1981), 1033–1036; English transl. in Soviet Math. Dokl. **24** (1981).

19. ——, *Methods of spectral optimization and their application in oscillations of conservative and strongly damped systems*, Fifth All-Union Congr. Theoret. Appl. Mech., Abstracts of Reports, Alma-Ata, 1981, p. 7. (Russian)

20. ——, *Zones and spectra of families of abstract Toeplitz operators*, Teor. Funktsiĭ, Funktsional. Anal. i Prilozhen. Vyp. **39** (1983), 3–6. (Russian)

21. ——, *Extremal problems generated by operator pencils*, Sibirsk. Mat. Zh. **24** (1983), no. 1, 3–20; English transl. in Siberian Math. J. **24** (1983).

22. ——, *Variational methods in the theory of operator pencils. Spectral optimization*, Izdat. Leningrad. Gos. Univ., Leningrad, 1983. (Russian)

23. Shmuel Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math. **15** (1962), 119–147.

24. ——, *Lectures on elliptic boundary value problems*, Van Nostrand, Princeton, N.J., 1965.

25. M. S. Agranovich, *On summability of series in root vectors of nonselfadjoint elliptic operators*, Funktsional. Anal. i Prilozhen. **10** (1976), no. 3, 1–12; English transl. in Functional Anal. Appl. **10** (1976).

26. ——, *Spectral properties of diffraction problems*, Appendix to N. N. Voĭtovich, B. Z. Katsenelenbaum, and A. N. Sivov, *The generalized method of eigenoscillations in diffraction theory*, "Nauka", Moscow, 1977, pp. 289–416. (Russian)

27. ——, *On the convergence of series in the root vectors of almost selfadjoint operators*, Trudy Moskov. Mat. Obshch. **41** (1980), 163–180; English transl. in Trans. Moscow Math. Soc. **1982**, no. 1(41).

28. ——, *On elliptic pseudodifferential operators on a closed curve*, Trudy. Moskov. Mat. Obshch. **47** (1984), 22–67; English transl. in Trans. Moscow Math. Soc. **1985**.

29. M. S. Agranovich and B. A. Amosov, *Matrix elliptic pseudodifferential operators on a closed curve*, Funktsional. Anal. i Prilozhen. **15** (1981), no. 3, 79–81; English transl. in Functional Anal. Appl. **15** (1981).

30. N. I. Akhiezer and I. M. Glazman, *The theory of linear operators in Hilbert space*, 2nd rev. ed., "Nauka", Moscow, 1966; English transl. of 1st ed., Vols. I, II, Ungar, New York, 1961.

31. A. M. Akhmedov, *Some completeness theorems for rational operator pencils*, Questions in Mathematical Cybernetics and Applied Mathematics, Vyp. 1 (Dzh. È. Allakhverdiev et al., editors), "Èlm," Baku, 1975, pp. 11–17. (Russian)

32. Dzh. È. Allakhverdiev, *On completeness of the system of eigenelements and associated elements for nonselfadjoint operators close to normal ones*, Dokl. Akad. Nauk SSSR **115** (1957), 207–210. (Russian)

33. ——, *On completeness of the system of eigenelements and associated elements for operators which are rational functions of a parameter*, Dokl. Akad. Nauk SSSR **159** (1964), 951–954; English transl. in Soviet Math. Dokl. **5** (1964).

34. ——, *On completeness of the system of eigenelements and associated elements for nonselfadjoint operators*, Dokl. Akad. Nauk SSSR **160** (1965), 503–506; English transl. in Soviet Math. Dokl. **6** (1965).

35. ——, *On completeness of the system of eigenelements and associated elements for a class of nonselfadjoint operators dependent on a parameter λ*, Dokl. Akad. Nauk SSSR **160** (1965), 1231–1234; English transl. in Soviet Math. Dokl. **6** (1965).

36. ——, *On multiply complete systems and nonselfadjoint operators dependent on a parameter* λ, Dokl. Akad. Nauk SSSR **166** (1966), 11–14; English transl. in Soviet Math. Dokl. 7 (1966).

37. ——, *On nonselfadjoint operators rationally dependent on a spectral parameter*, Dokl. Akad. Nauk SSSR **186** (1969), 743–746; English transl. in Soviet Math. Dokl. **10** (1969).

38. ——, *An estimate of the resolvent and completeness theorems for operators dependent on a spectral parameter*, Izv. Akad. Nauk. Azerbaĭdzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk **1974**, no. 6, 3–36. (Russian)

39. Dzh. È. Allakhverdiev and È. È. Gasanov, *A completeness theorem for the system of eigenelements and associated elements of rational operator pencils in a Banach space*, Izv. Akad. Nauk Azerbaĭdzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk **1974**, no. 5, 54–66. (Russian)

40. D. Z. Arov, *Stable dissipative linear stationary dynamical scattering systems*, J. Operator Theory **2** (1979), 95–126. (Russian)

41. N. G. Askerov, S. G. Kreĭn, and G. I. Laptev, *On a class of nonselfadjoint boundary value problems*, Dokl. Akad. Nauk SSSR **155** (1964), 499–502; English transl. in Soviet Math. Dokl. **5** (1964).

42. N. K. Askerov, S. G. Kreĭn, and G. I. Laptev, *The problem of oscillations of a viscous fluid and the operator equations connected with it*, Funktsional. Anal. i Prilozhen. **2** (1968), no. 2, 21–31; English transl. in Functional Anal. Appl. **2** (1968).

43. R. Kh. Attiya [Attiya-Khasanein], *Completeness of a part of the eigenvectors and associated vectors of a quadratic polynomial operator period*, Azerbaĭdzhan. Gos. Univ. Uchen. Zap. Ser. Fiz.-Mat. Nauk **1976**, no. 5, 43–47. (Russian)

44. V. A. Avakyan, *The asymptotic distribution of the spectrum of a linear pencil perturbed by an analytic operator-valued function*, Funktsional. Anal. i Prilozhen **12** (1978), no. 2, 66–67; English transl. in Functional Anal. Appl. **12** (1978).

45. ——, *The asymptotic distribution of the spectrum of an abstract operator-valued function, and some of its applications*, Sibirsk. Mat. Zh. **25** (1984), no. 1, 3–18; English transl. in Siberian Math. J. **25** (1984).

46. T. Ya. Azizov, *On completeness and the basis property for the eigenvectors and associated vectors of J-selfadjoint operators of class $\mathscr{K}(H)$*, Dokl. Akad. Nauk SSSR **253** (1980), 1033–1035; English transl. in Soviet Math. Dokl. **22** (1980).

47. T. Ya. Azizov and I. S. Iokhvidov, *Linear operators on spaces with an indefinite metric and their applications*, Itogi Nauki i Tekhniki: Mat. Anal., vol. 17, VINITI, Moscow, 1979, pp. 113–205; English transl. in J. Soviet Math. **15** (1981), no. 4.

48. ——, *Foundations of the theory of linear operators in spaces with an indefinite metric*, "Nauka", Moscow, 1986. (Russian)

49. A. M. Babaev, *On completeness of the system of eigenvectors and associated vectors of a certain class of compact operators on a Banach space*, Izv. Akad. Nauk Azerbaĭdzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk **1968**, no. 6, 10–15. (Russian)

50. A. I. Balinskiĭ and V. S. Zayachkovskiĭ, *On criteria for factorization of operator pencils on a Banach space*, Mat. Metody i Fiz.-Mekh. Polya Vyp. 16 (1982), 14–19. (Russian)

51. A. I. Balinskiĭ and L. M. Zoriĭ, *A method of investigating the spectrum of polynomial pencils of selfadjoint operators on a Hilbert space*, Dopovidī Akad. Nauk Ukraīn. RSR. Ser. A **1972**, 485–488. (Ukrainian)

52. N. K. Bari, *Biorthogonal systems and bases in a Hilbert space*, Moskov. Gos. Univ. Uchen. Zap. Vyp. 148, (1951), Mat. 4, 69–107. (Russian)

53. Harm Bart, Israel Gohberg, and Marinus A. Kaashoek, *Minimal factorization of matrix and operator functions*, Birkhäuser, 1979.

54. A. G. Baskakov, *Methods of abstract harmonic analysis in the theory of perturbations of linear operators*, Sibirsk. Mat. Zh. **24** (1983), no. 1, 21–39; English transl. in Siberian Math. J. **24** (1983).

55. Yu. M. Berezanskiĭ, *Expansion in eigenfunctions of selfadjoint operators*, "Naukova Dumka", Kiev, 1965; English transl., Amer. Math. Soc., Providence, R.I., 1968.

56. George D. Birkhoff, *Boundary value and expansion problems of ordinary linear differential equations*, Trans. Amer. Math. Soc. **9** (1908), 373–395.

57. M. Sh. Birman and M. Z. Solomyak, *On estimates of singular numbers of integral operators*. II, Vestnik Leningrad. Univ. **1967**, no. 13 (Ser. Mat. Mekh. Astr. vyp. 3), 21–28. (Russian)

58. ——, *Asymptotic properties of the spectrum for differential equations*, Itogi Nauki i Tekhniki: Mat. Anal., vol. 14, VINITI, Moscow, 1977, pp. 5–58; English transl. in J. Soviet Math. **12** (1979), no. 3.

59. S. K. Bloshanskaya, *On a necessary condition for the basis property*, Differentsial'nye Uravneniya **17** (1981), 778–788; English transl. in Differential Equations **17** (1981).

60. H. den Boer and G. Ph. A. Thijsse, *Semistability of sums of partial multiplicities under additive perturbation*, Integral Equations Operator Theory **3** (1980), 23–42.

61. J. Bognár, *A proof of the spectral theorem for J-positive operators*, Acta Sci. Math. (Szeged) **45** (1973), 75–80.

62. K. Kh. Boĭmatov and A. G. Kostyuchenko, *The distribution of the eigenvalues of the equation $Au = \lambda Bu$ in the whole space*, Dokl. Akad. Nauk SSSR **277** (1984), 1292–1295; English transl. in Soviet Math. Dokl. **30** (1984).

63. Felix E. Browder, *On the eigenfunctions and eigenvalues of the general linear elliptic differential operator*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 433–439.

64. T. Carleman, *Über die asymptotische Verteilung der Eigenwerte partieller Differentialgleichungen*, Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl. **88** (1936), 119–132.

65. Henri Cartan, *Calcul différentiel*, Hermann, Paris, 1967; English transl., Hermann, Paris, and Houghton-Mifflin, Boston, Mass., 1971; *Formes différentielles*, Hermann, Paris, 1967; English transl., Hermann, Paris, and Houghton-Mifflin, Boston, Mass., 1970.

66. Kevin F. Clancey and Israel Gohberg, *Factorization of matrix functions and singular integral operators*, Birkhäuser, 1981.

67. Allen Devinatz, *The factorization of operator valued functions*, Ann. of Math. (2) **73** (1961), 458–495.

68. V. G. Dolgolaptev, *On the basis property for the root vectors of weakly perturbed operators*, Mat. Zametki **34** (1983), 867–872; English transl. in Math. Notes **34** (1983).

69. R. J. Duffin, *A minimax theory for overdamped networks*, J. Rational Mech. Anal. **4** (1955), 221–233.

70. Nelson Dunford and Jacob T. Schwartz, *Linear operators*. Vol. I, Interscience, 1958.

71. ——, *Linear operators*. Vol. II, Interscience, 1963.

72. ——, *Linear operators*. Vol. III, Interscience, 1971.

73. R. M. Dzhabar-zade, *Expansion in eigenelements and associated elements of an operator polynomially dependent on λ*, Azerbaĭdzhan. Gos. Univ. Uchen. Zap. Ser. Fiz.-Mat. i Khim. Nauk **1964**, no. 3, 75–81. (Russian)

74. ——, *On completeness of the system of eigenelements and associated elements of operators quadratically dependent on a spectral parameter*, Izv. Akad. Nauk Azerbaĭdzhan. SSR Ser. Fiz.-Tekhn. i Mat. Nauk **1977**, no. 1, 41–45. (Russian)

75. M. G. Dzhavadov, *On completeness of a certain part of the eigenfunctions of a nonselfadjoint differential operator*, Dokl. Akad. Nauk SSSR **159** (1964), 723–725; English transl. in Soviet Math. Dokl. **5** (1964).

76. ——, *On m-fold completeness of half the eigenfunctions and associated functions of an ordinary differential operator of order $2m$*, Dokl. Akad. Nauk SSSR **160** (1965), 754–757; English transl. in Soviet Math. Dokl. **6** (1965).

77. M. M. Dzhrbashyan, *The basis property for biorthogonal systems generated by boundary value problems for differential operators of fractional order*, Application of Methods of the Theory of Functions and Functional Analysis to Problems in Mathematical Physics (Proc.

Seventh Soviet-Czechoslovak Sem.), Erevan. Gos. Univ., Erevan, 1982, pp. 103–111. (Russian) R. Zh. Mat. **1984**, 5Б776.

78. J. Eisenfeld, *Quadratic eigenvalue problems*, J. Math. Anal. Appl. **23** (1968), 58–70.

79. V. M. Eni, *On stability of a root number of an analytic operator-valued function and on perturbations of its eigenvalues and eigenvectors*, Dokl. Akad. Nauk SSSR **173** (1967), 1251–1254; English transl. in Soviet Math. Dokl. **8** (1967).

80. ——, *On the multiplicity of an eigenvalue of an operator pencil*, Mat. Issled. **4** (1969), no. 2(12), 32–41. (Russian)

81. M. A. Evgrafov, *Analytic functions*, 2nd ed., "Nauka", Moscow, 1968; English transl. of 1st ed., Saunders, Philadelphia, Pa., 1966.

82. I. A. Fel'dman, *On some projection methods for solving the radiant energy transfer equation*, Mat. Issled. **7** (1972), no. 4(26), 228–236. (Russian)

83. G. M. Fikhtengol'ts, *A course in differential and integral calculus*. Vol. 3, 5th ed., Fizmatgiz, Moscow, 1970; German transl., VEB Deutscher Verlag Wiss., Berlin, 1972.

84. L. F. Fridlender, *On some spectral properties of very weak nonselfadjoint perturbations of selfadjoint operators*, Trudy Moskov. Mat. Obshch. **41** (1980), 181–216; English transl. in Trans. Moscow Math. Soc. **1982**, no. 1(41).

85. ——, *Sur le spectre de la perturbation faible d'un opérateur auto-adjoint*, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), 465–468.

86. Avner Friedman, *Nonlinear eigenvalue problems*, Advances in Differential and Integral Equations (Conf., Madison, Wisc., 1968; In Memoriam: R. E. Langer), Studies Appl. Math., vol. 5, SIAM, Philadelphia, Pa., 1969, pp. 9–13.

87. Avner Friedman and Marvin Shinbrot, *Nonlinear eigenvalue problems*, Acta Math. **121** (1968), 77–125.

88. Charles T. Fulton, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A **77** (1977), 293–308.

89. L. G. Garadzhaev, *On the problem of normal oscillations of a heavy viscous fluid in a container*, Sibirsk. Mat. Zh. **25** (1984), no. 2, 213–215. (Russian)

90. M. G. Gasymov, *On the theory of polynomial operator pencils*, Dokl. Akad. Nauk SSSR **199** (1971), 747–750; English transl. in Soviet Math. Dokl. **12** (1971).

91. ——, *On multiple completeness of part of the eigenvectors and associated vectors of polynomial operator pencils*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. **6** (1971), 131–147. (Russian)

92. ——, *Multiple completeness with a finite-dimensional defect of part of the eigenvectors and associated vectors of operator pencils*, Functional Analysis, Theory of Functions, and Their Applications, Vyp. 3, Part 1, Dagestan. Gos. Univ., Makhachkala, 1976, pp. 55–62. (Russian) R.Zh. Mat. **1978**, 1Б501.

93. M. G. Gasymov and M. G. Dzhavadov, *Multiple completeness of part of the eigenfunctions and associated functions of differential operator pencils*, Dokl. Akad. Nauk SSSR **203** (1972), 1235–1237; English transl. in Soviet Math. Dokl. **13** (1972).

94. M. G. Gasymov and A. M. Magerramov, *On multiple completeness of the system of eigenfunctions and associated functions of a certain class of differential operators*, Akad. Nauk Azerbaĭdzhan. SSR Dokl. **30** (1974), no. 3, 9–12. (Russian)

95. Yu. P. Ginzburg, *On divisors and minorants of operator-valued functions of bounded type*, Mat. Issled. **2** (1967), no. 4(6), 47–72; English transl. in Amer. Math. Soc. Transl. (2) **103** (1973).

96. I. Ts. Gokhberg [Israel Gohberg] and I. A. Fel'dman, *Convolution equations and projection methods for their solution*, "Nauka", Moscow, 1971; English transl., Amer. Math. Soc., Providence, R.I., 1974.

97. I. Gohberg, M. A. Kaashoek, and D. C. Lay, *Equivalence, linearization, and decomposition of holomorphic operator functions*, J. Functional Anal. **28** (1978), 102–144.

98. I. Gohberg, M. A. Kaashoek, L. Lerer, and L. Rodman, *Common multiples and common divisors of matrix polynomials.* I: *Spectral method*, Indiana Univ. Math. J. **30** (1981), 321–356.

99. ——, *Common multiples and common divisors of matrix polynomials.* II: *Vandermonde and resultant matrices*, Linear and Multilinear Algebra **12** (1982/83), 159–203.

100. I. Gohberg, M. A. Kaashoek, and L. Rodman, *Spectral analysis of families of operator polynomials and a generalized Vandermonde matrix.* I: *The finite-dimensional case*, Topics in Functional Analysis (M. G. Kreĭn Seventieth Birthday Vol.), Academic Press, 1978, pp. 91–128.

101. ——, *Spectral analysis of families of operator polynomials and a generalized Vandermonde matrix.* II: *The infinite-dimensional case*, J. Functional Anal. **30** (1978), 358–389.

102. I. Gohberg, M. A. Kaashoek, and F. van Schagen, *Similarity of operator blocks and canonical forms.* I, Integral Equations Operator Theory **3** (1980), 350–396.

103. I. Ts. Gokhberg [Israel Gohberg] and M. G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators in Hilbert space*, "Nauka", Moscow, 1965; English transl., Amer. Math. Soc., Providence, R.I., 1969.

104. I. Gohberg, P. Lancaster, and L. Rodman, *Spectral analysis of matrix polynomials.* I: *Canonical forms and divisors*, Linear Algebra and Appl. **20** (1978), 1–44.

105. ——, *Spectral analysis of matrix polynomials.* II: *The resolvent form and spectral divisors*, Linear Algebra and Appl. **21** (1978), 65–88.

106. ——, *Representations and divisibility of operator polynomials*, Canad. J. Math. **30** (1978), 1045–1069.

107. ——, *Matrix polynomials*, Academic Press, 1982.

108. I. Ts. Gokhberg [Israel Gohberg] and Yu. Laĭterer [Jürgen Leiterer], *General theorems on the canonical factorization of operator functions with respect to a contour*, Mat. Issled. **7** (1972), no. 3(25), 87–134. (Russian)

109. ——, *The factorization of operator-functions relative to a contour.* II, Math. Nachr. **54** (1972), 41–74. (Russian)

110. I. Gohberg, L. Lerer, and L. Rodman, *Stable factorization of operator polynomials.* I, II, J. Math. Anal. Appl. **74** (1980), 401–431; **75** (1980), 1–40.

111. I. Gohberg and L. Rodman, *On spectral analysis of nonmonic matrix and operator polynomials.* II: *Dependence on the finite spectral data*, Israel J. Math. **30** (1978), 321–334.

112. ——, *On the spectral structure of monic matrix polynomials and the extension problem*, Linear Algebra and Appl. **24** (1979), 157–172.

113. I. Ts. Gokhberg [Israel Gohberg] and E. I. Sigal, *An operator generalization of the logarithmic residue theorem and Rouché's theorem*, Mat. Sb. **84(126)** (1971), 607–629; English transl. in Math. USSR Sb. **13** (1971).

114. A. M. Gomilko, *An upper estimate for the number of eigenvalues of an operator pencil dependent on a parameter*, Izv. Akad. Nauk Azerbaĭdzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk **3** (1982), no. 4, 19–23. (Russian)

115. V. I. Gorbachuk and M. L. Gorbachuk, *Boundary value problems for differential-operator equations*, "Naukova Dumka", Kiev, 1984. (Russian)

116. I. V. Goryuk, *A theorem on completeness of the system of eigenvectors and associated vectors of the operator pencil* $L(\lambda) = \lambda^2 C + \lambda B + E$, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1970**, no. 1, 55–60; English transl. in Moscow Univ. Math. Bull. **25** (1970).

117. ——, *On factorization of a quadratic operator pencil*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1970**, no. 5, 28–35; English transl. in Moscow Univ. Math. Bull. **25** (1970).

118. ——, *On completeness of the system of eigenvectors and associated vectors of a quadratic selfadjoint pencil*, Mat. Issled. **7** (1972), no. 1(23), 193–197. (Russian)

119. W. M. Greenlee, *Double unconditional bases associated with a quadratic characteristic parameter problem*, J. Functional Anal. **15** (1974), 306–339.

120. ——, *A quadratic eigenvalue problem*, Proc. Amer. Math. Soc. **40** (1973), 123–127.

121. P. Grisvard, *Caractérisation de quelques espaces d'interpolation*, Arch. Rational Mech. Anal. **25** (1967), 40–63.

122. K. P. Hadeler, *Mehrparametrige und nichtlineare Eigenwertaufgaben*, Arch. Rational Mech. Anal. **27** (1967), 306–328.

123. ——, *Variationsprinzipien bei nichtlinearen Eigenwertaufgaben*, Arch. Rational Mech. Anal. **30** (1968), 297–307.

124. K. Harbarth and H. Langer, *A factorization theorem for operator pencils*, Integral Equations Operator Theory **2** (1979), 344–364.

125. Georg Heinig, *Linearisierung und Realisierung holomorpher Operatorfunktionen*, Wiss. Z. Tech. Hochschule Karl-Marx-Stadt **22** (1980), 453–459.

126. Einar Hille and Ralph S. Phillips, *Functional analysis and semi-groups*, rev. ed., Amer. Math. Soc., Providence, R.I., 1957.

127. Don B. Hinton, *An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary condition*, Quart. J. Math. Oxford. Ser. (2) **30** (1979), 33–42.

128. Kenneth Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.

129. Lars Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218.

130. V. A. Il'in, *Necessary and sufficient conditions for a subsystem of the eigenfunctions and associated functions of a Keldysh pencil of ordinary differential operators to be a basis*, Dokl. Akad. Nauk SSSR **227** (1976), 796–799; English transl. in Soviet Math. Dokl. **17** (1976).

131. ——, *On properties of a reduced subsystem of eigenfunctions and associated functions of a Keldysh pencil of ordinary differential operators*, Dokl. Akad. Nauk SSSR **230** (1976), 30–33; English transl. in Soviet Math. Dokl. **17** (1976).

132. ——, *Necessary and sufficient conditions for spectral expansions to be bases and to be equiconvergent with a trigonometric series*. I, II, Differentsial'nye Uravneniya **16** (1980), 771–794, 980–1009; English transl. in Differential Equations **16** (1980).

133. ——, *On the unconditional basis property on a closed interval for systems of eigenfunctions and associated functions of a second-order differential operator*, Dokl. Akad. Nauk SSSR **273** (1983), 1048–1053; English transl. in Soviet Math. Dokl. **28** (1983).

134. ——, *On the absolute and uniform convergence of expansions in eigenfunctions and associated functions of a nonselfadjoint elliptic operator*, Dokl. Akad. Nauk SSSR **274** (1984), 19–23; English transl. in Soviet Math. Dokl. **29** (1984).

135. G. A. Isaev, *On completeness of a certain part of the eigenvectors and associated vectors of polynomial operator pencils*, Uspekhi Mat. Nauk **28** (1973), no. 1(169), 241–242. (Russian)

136. ——, *Linear factorization of polynomial operator pencils*, Mat. Zametki **13** (1973), 551–559; English transl. in Math. Notes **13** (1973).

137. ——, *The numerical range of operator pencils and multiple completeness in the Keldysh sense*, Funktsional. Anal. i Prilozhen. **9** (1975), no. 1, 31–34; English transl. in Functional Anal. Appl. **9** (1975).

138. Victor Ivriĭ, *Precise spectral asymptotics for elliptic operators acting in fiberings over manifolds with boundary*, Lecture Notes in Math., vol. 1100, Springer-Verlag, 1984.

139. M. A. Kaashoek, C. V. M. van der Mee, and L. Rodman, *Analytic operator functions with compact spectrum*. I, Integral Equations Operator Theory **4** (1981), 504–547.

140. ——, *Analytic operator functions with compact spectrum*. III, J. Operator Theory **10** (1983), 219–250.

141. V. I. Kabak and A. S. Markus, *On decomposition of a polynomial pencil into linear factors*, Uspekhi Mat. Nauk **30** (1975), no. 4(184), 245–246. (Russian)

142. V. I. Kabak, A. S. Markus, and I. V. Mereutsa, *On the connection between the spectral properties of a polynomial operator pencil and its factors*, Mat. Issled. Vyp. 45 (1977), 29–57. (Russian)

143. L. V. Kantorovich and G. P. Akilov, *Functional analysis*, 2nd rev. ed., "Nauka", Moscow, 1977; English transl., Pergamon Press, 1982.

144. O. Karma, *Asymptotic error estimates for approximate characteristic values of holomorphic Fredholm operator-valued functions*, Zh. Vychisl. Mat. i Mat. Fiz. **11** (1971), 559–568; English transl. in USSR Comput. Math. and Math. Phys. **11** (1971).

145. ——, *Approximation in the eigenvalue problem with holomorphic dependence of the operator on a parameter*. II, Tartu Riikl. Ül. Toimetised Vyp. 633 (1983), 19–28. (Russian)

146. Tosio Kato, *Perturbation theory for linear operators*, Springer-Verlag, 1966.

147. V. È. Katsnel'son, *On conditions for the basis property of the system of root vectors for a certain class of operators*, Funktsional. Anal. i Prilozhen. **1** (1967), no. 2, 39–51; English transl. in Functional Anal. Appl. **1** (1967).

148. ——, *On convergence and summability of series in the root vectors for certain classes of nonselfadjoint operators*, Author's summary of Candidate's dissertation, Khar'kov. Gos. Univ., Kharkov, 1967. (Russian)

149. P. S. Kazīmīrs'kiĭ, *Decomposition of matrix polynomials into factors*, "Naukova Dumka", Kiev, 1981. (Ukrainian)

150. M. V. Keldysh, *On the eigenvalues and eigenfunctions of certain classes of nonselfadjoint equations*, Dokl. Akad. Nauk SSSR **77** (1951), 11–14; English transl. in this volume.

151. ——, *On a Tauberian theorem*, Trudy Mat. Inst. Steklov. **38** (1951), 77–86; English transl. in Amer. Math. Soc. Transl. (2) **102** (1973).

152. ——, *On completeness of the eigenfunctions for certain classes of nonselfadjoint linear operators*, Uspekhi Mat. Nauk **27** (1971), no. 4(160), 15–41; English transl. in Russian Math. Surveys **27** (1971).

153. M. V. Keldysh and V. B. Lidskiĭ, *Questions in the spectral theory of nonselfadjoint operators*, Proc. Fourth All-Union Math. Congr. (Leningrad, 1961), Vol. I, Izdat. Akad. Nauk SSSR, Leningrad, 1963, pp. 101–120. (Russian)

154. G. M. Kesel'man, *On the unconditional convergence of eigenfunction expansions for certain differential operators*, Izv. Vyssh. Uchebn. Zaved. Mat. **1964**, no. 2(39), 82–93. (Russian)

155. A. P. Khromov, *Eigenfunction expansions for ordinary linear differential operators with nonregular splitting boundary conditions*, Mat. Sb. **70(112)** (1966), 310–329. (Russian)

156. N. D. Kopachevskiĭ, *On oscillations of a capillary, viscous, rotating fluid*, Dokl. Akad. Nauk SSSR **219** (1974), 1065–1068; English transl. in Soviet Math. Dokl. **15** (1974).

157. ——, *On the existence of surface waves in the problem of normal oscillations of an ideal fluid rotating in a partially filled container*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 2, 84–85; English transl. in Functional Anal. Appl. **12** (1978).

158. ——, *Normal oscillations of a system of heavy viscous rotating fluids*, Dokl. Akad. Nauk Ukrain. SSR Ser. A **1978**, 586–590. (Russian)

159. ——, *Small motions and normal oscillations of a system of heavy viscous rotating fluids*, Preprint No. 33, Phys.-Tekhn. Inst. Nizkikh Temperatur, Akad. Nauk Ukrain. SSR, Kharkov, 1978. (Russian) R. Zh. Mat. **1978**, 11Б512.

160. ——, *Small motions and characteristic oscillations of an ideal rotating fluid*, Preprint No. 38, Phys.-Tekhn. Inst. Nizkikh Temperatur, Akad. Nauk Ukrain. SSR, Kharkov, 1978. (Russian)

161. ——, *On the basis properties of the system of eigenvectors and associated vectors of the selfadjoint operator pencil* $I - \lambda A - \lambda^{-1}B$, Funktsional. Anal. i Prilozhen. **15** (1981), no. 2, 77–78; English transl. in Functional Anal. Appl. **15** (1981).

162. ——, *On properties of the system of modes of surface waves in a rotating ideal fluid*, Functional Analysis and Applied Mathematics (V. A. Marchenko, editor), "Naukova Dumka", Kiev, 1982, pp. 43–55. (Russian) R. Zh. Mat. **1983**, 8Б554.

163. ——, *On the p-basis property for the system of root vectors of the selfadjoint operator pencil* $I - \lambda A - \lambda^{-1}B$, Functional Analysis and Applied Mathematics (V. A. Marchenko, editor), "Naukova Dumka", Kiev, 1982, pp. 55–70. (Russian) R. Zh. Mat. **1983**, 8Б893.

164. A. G. Kostyuchenko and M. B. Orazov, *On certain properties of the roots of a selfadjoint quadratic pencil*, Funktsional. Anal. i Prilozhen. **9** (1975), no. 4, 28–40; English transl. in Functional Anal. Appl. **9** (1975).

165. ——, *The problem of oscillations of an elastic half-cylinder, and related selfadjoint quadratic pencils*, Trudy Sem. Petrovsk. Vyp. 6 (1981), 97–146; English transl. in J. Soviet Math. **33** (1986), no. 3.

166. A. G. Kostyuchenko and G. V. Radzievskiĭ, *On Abel summation of n-fold expansions*, Sibirsk. Mat. Zh. **15** (1974), 855–870; English transl. in Siberian Math. J. **15** (1974).

167. A. G. Kostyuchenko and A. A. Shkalikov, *Selfadjoint quadratic operator pencils and elliptic problems*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 2, 38–61; English transl. in Functional Anal. Appl. **17** (1983).

168. ——, *On the theory of selfadjoint quadratic operator pencils*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1983**, no. 6, 40–51; English transl. in Moscow Univ. Math. Bull. **38** (1983).

169. A. N. Kozhevnikov, *Different asymptotics of two series of eigenvalues of a certain elliptic boundary value problem*, Mat. Zametki **22** (1977), 699–710; English transl. in Math. Notes **22** (1977).

170. ——, *On the asymptotic behavior of the eigenvalues of elliptic systems*, Funktsional. Anal. i Prilozhen. **11** (1977), no. 4, 82–83; English transl. in Functional Anal. Appl. **11** (1977).

171. ——, *Remainder estimates for eigenvalues and complex powers of the Douglis-Nirenberg elliptic systems*, Comm. Partial Differential Equations **6** (1981), 1111–1136.

172. ——, *On the operator of the linearized steady-state Navier-Stokes problem*, Mat. Sb. **125 (167)** (1984), 3–18; English transl. in Math. USSR Sb. **53** (1986).

173. Henry P. Kramer, *Perturbation of differential operators*, Pacific J. Math. **7** (1957), 1405–1435.

174. M. A. Krasnosel'skii et al., *Integral operators in spaces of summable functions*, "Nauka", Moscow, 1966; English transl., Noordhoff, 1975.

175. A. O. Kravitskiĭ, *On the two-fold expansion in a series of eigenfunctions of a certain nonselfadjoint boundary value problem*, Differentsial'nye Uravneniya **4** (1968), 165–177; English transl. in Differential Equations **4** (1968).

176. M. G. Kreĭn, *On "loaded" integral equations whose distribution functions are not monotonic*, Collection Dedicated to the Memory of D. A. Grave, GITTL, Moscow, 1940, pp. 88–103. (Russian)

177. ——, *On Bari bases in a Hilbert space*, Uspekhi Mat. Nauk **12** (1957), no. 3(75), 333–341. (Russian)

178. ——, *Integral equations on a half-line with kernel dependent on the difference between the arguments*, Uspekhi Mat. Nauk **13** (1958), no. 5(83), 3–120; English transl. in Amer. Math. Soc. Transl. (2) **22** (1962).

179. ——, *Criteria for completeness of the system of root vectors of a dissipative operator*, Uspekhi Mat. Nauk **14** (1959), no. 3(87), 145–152; English transl. in Amer. Math. Soc. Transl. (2) **26** (1963).

180. ——, *Perturbation determinants and a trace formula for unitary and selfadjoint operators*, Dokl. Akad. Nauk SSSR **144** (1962), 268–271; English transl. in Soviet Math. Dokl. **3** (1962).

181. ——, *On some new studies in the theory of perturbations of selfadjoint operators*, First Math. Summer School (Kanev, 1963), Part I, "Naukova Dumka", Kiev, 1964, pp. 103–187; English transl. in M. G. Kreĭn, *Topics in differential and integral equations and operator theory*, Birkhäuser, 1983.

182. ____, *Introduction to the geometry of indefinite J-spaces and the theory of operators on these spaces*, Second Math. Summer School (Katsiveli, 1964), Part I, "Naukova Dumka", Kiev, 1965, pp. 15–92; English transl. in Amer. Math. Soc. Transl. (2) **93** (1970).

183. ____, *Analytic problems and results in the theory of linear operators on Hilbert space*, Proc. Internat. Congr. Math. (Moscow, 1966), "Mir", Moscow, 1968, pp. 189–216; English transl. in Amer. Math. Soc. Transl. (2) **90** (1970).

184. M. G. Kreĭn and G. K. [Heinz] Langer, *The spectral function of a selfadjoint operator in a space with indefinite metric*, Dokl. Akad. Nauk SSSR **152** (1963), 39–42; English transl. in Soviet Math. Dokl. **4** (1963).

185. ____, *On the theory of quadratic pencils of selfadjoint operators*, Dokl. Akad. Nauk SSSR **154** (1964), 1258–1261; English transl. in Soviet Math. Dokl. **5** (1964).

186. ____, *On some mathematical principles in the linear theory of damped oscillations of continua*, Applications of Function Theory in Continuum Mechanics (Proc. Internat. Sympos., Tbilisi, 1963), Vol. 2, "Nauka", Moscow, 1965, pp. 283–322; English transl., Parts I, II, Integral Equations Operator Theory **1** (1978), 364–399, 539–566.

187. S. G. Kreĭn, *On the oscillations of a viscous fluid in a container*, Dokl. Akad. Nauk SSSR **159** (1964), 262–265; English transl. in Soviet Math. Dokl. **5** (1964).

188. ____, *Linear differential equations in Banach space*, "Nauka", Moscow, 1967; English transl., Amer. Math. Soc., Providence, R.I., 1971.

189. S. G. Kreĭn and G. I. Laptev, *On the problem of the motion of a viscous fluid in an open container*, Funktsional. Anal. i Prilozhen. **2** (1968), no. 1, 40–50; English transl. in Functional Anal. Appl. **2** (1968).

190. S. G. Kreĭn and Yu. I. Petunin, *Scales of Banach spaces*, Uspekhi Mat. Nauk **21** (1966), no. 2(128), 89–168; English transl. in Russian Math. Surveys **21** (1966).

191. Rolf Kühne, *Minimaxprinzipe für stark gedämpfte Scharen*, Acta Sci. Math. (Szeged) **29** (1968), 39–68.

192. V. P. Kurdyumov, *On the Riesz basis property of the root vectors of an integral operator with kernel of Green function type*, Differentsial'nye Uravneniya i Vychisl. Mat. (Saratov) Vyp. 6 (1976), Part 2, 25–43. (Russian)

193. Peter Lancaster, *Lambda-matrices and vibrating systems*, Pergamon Press, 1966.

194. Heinz Langer, *Spektraltheorie linearer Operatoren in J-Raumen und einige Anwerdurgen auf die Schar* $L(\lambda) = \lambda^2 I + \lambda B + C$, Habilitationsschrift, Dresden, 1965.

195. ____, *Spektralfunktionen einer Klasse J-selbstadjungierter Operatoren*, Math. Nachr. **33** (1967), 107–120.

196. ____, *Über stark gedämpfte Scharen in Hilbertraum*, J. Math. and Mech. **17** (1967/68), 685–705.

197. ____, *Invariante Teilräume definisierbarer J-selbstadjungierter Operatoren*, Ann. Acad. Sci. Fenn. Ser. A I No. 475 (1971).

198. ____, *Über eine Klasse nichtlinearer Eigenwertprobleme*, Acta Sci. Math. (Szeged) **35** (1973), 73–86.

199. ____, *Über eine Klasse polynomialer Scharen selbstadjungierter Operatoren im Hilbertraum*. I, J. Functional Anal. **12** (1973), 13–29.

200. ____, *Über eine Klasse polynomialer Scharen selbstadjungierter Operatoren im Hilbertraum*. II, J. Functional Anal. **16** (1974), 221–234.

201. ____, *Zur Spektraltheorie polynomialer Scharen selbstadjungierter Operatoren*, Math. Nachr. **65** (1975), 301–319.

202. ____, *Factorization of operator pencils*, Acta Sci. Math. (Szeged) **38** (1976), 83–96.

203. E. A. Larionov, *On bases made up of root vectors of an operator pencil*, Dokl. Akad. Nauk SSSR **206** (1972), 283–286; English transl. in Soviet Math. Dokl. **13** (1972).

204. ____, *On selfadjoint quadratic pencils*, Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), 138–154; English transl. in Math. USSR Izv. **3** (1969).

205. Jürgen Leiterer, *Über analytische Operatorfunktionen einer komplexen Veränderlichen mit stetigen Randwerten*, Math. Nachr. **72** (1976), 247–274.

206. A. F. Leont'ev, *Entire functions. Series of exponentials*, "Nauka", Moscow, 1983. (Russian)

207. S. Z. Levendorskiĭ, *The asymptotic distribution of eigenvalues*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 810–852; English transl. in Math. USSR Izv. **21** (1983).

208. ———, *Spectral asymptotics of nonlinear pencils and operators with nonempty essential spectrum*, Dokl. Akad. Nauk SSSR **272** (1983), 1314–1317; English transl. in Soviet Math. Dokl. **28** (1983).

209. B. Ya. Levin, *Distribution of zeros of entire functions*, GITTL, Moscow, 1956; English transl., Amer. Math. Soc., Providence, R.I., 1964.

210. V. B. Lidskiĭ, *Conditions for the completeness of the system of root subspaces for nonselfadjoint operators with discrete spectrum*, Trudy Moskov. Mat. Obshch. **8** (1959), 83–120; English transl. in Amer. Math. Soc. Transl. (2) **34** (1963).

211. ———, *Nonselfadjoint operators having a trace*, Dokl. Akad. Nauk SSSR **125** (1959), 485–487; English transl. in Amer. Math. Soc. Transl. (2) **47** (1965).

212. ———, *On summability of series in the principal vectors of nonselfadjoint operators*, Trudy Moskov. Mat. Obshch. **11** (1962), 3–35; English transl. in Amer. Math. Soc. Transl. (2) **40** (1964).

213. J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*. Vol. 1, Dunod, Paris, 1968; English transl., Springer-Verlag, 1972.

214. F. G. Maksudov, *Polynomial operator pencils in the presence of a continuous spectrum*, Izv. Akad. Nauk Azerbaĭdzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk **1974**, no. 5, 35–40. (Russian)

215. ———, *Multiple expansion in the eigenfunctions and associated functions of a quadratic pencil of one-dimensional singular differential operators*, Spectral Theory of Operators (F. G. Maksudov and M. G. Gasymov, editors), "Èlm", Baku, 1977, pp. 125–162. (Russian)

216. ———, *Expansion in generalized eigenvectors and associated vectors of a polynomial operator pencil in the presence of a continuous spectrum*, Spectral Theory of Operators, Vyp. 3, "Èlm", Baku, 1980, pp. 5–33. (Russian)

217. F. G. Maksudov and È. È. Pashaeva, *Spectral analysis of a family of nonselfadjoint differential operators of odd order*, Dokl. Akad. Nauk SSSR **275** (1984), 553–557; English transl. in Soviet Math. Dokl. **29** (1984).

218. K. S. Mamedov, *On the distribution of the spectrum of a polynomial differential-operator pencil and applications*, Akad. Nauk Azerbaĭdzhan. SSR Dokl. **39** (1983), no. 6, 10–13. (Russian)

219. A. S. Markus, *On holomorphic operator-valued functions*, Dokl. Akad. Nauk SSSR **119** (1958), 1099–1102. (Russian)

220. ———, *On a basis of root vectors of a dissipative operator*, Dokl. Akad. Nauk SSSR **132** (1960), 524–527; English transl. in Soviet Math. Dokl. **1** (1960).

221. ———, *On expansion in root vectors of a weakly perturbed selfadjoint operator*, Dokl. Akad. Nauk SSSR **142** (1962), 538–541; English transl. in Soviet Math. Dokl. **3** (1962).

222. ———, *On the spectral theory of polynomial operator pencils in a Banach space*, Sibirsk. Mat. Zh. **8** (1967), 1346–1369; English transl. in Siberian Math. J. **8** (1967).

223. ———, *On the convergence of multiple expansions in the eigenvectors and associated vectors of an operator pencil*, Mat. Issled. **4** (1969), no. 4(14), 57–69; English transl. in Amer. Math. Soc. Transl. (2) **110** (1977).

224. ———, *On the completeness of part of the eigenvectors and associated vectors for certain nonlinear spectral problems*, Funktsional. Anal. i Prilozhen. **5** (1971), no. 4, 78–79; English transl. in Functional Anal. Appl. **5** (1971).

225. ———, *On the completeness of a part of the eigenvectors and associated vectors of an analytic operator-valued function*, Mat. Issled. **9** (1974), no. 3(33), 105–126. (Russian)

226. A. S. Markus and V. I. Matsaev, *On the spectral properties of holomorphic operator-valued functions in Hilbert space*, Mat. Issled. **9** (1974), no. 4(34), 79–91. (Russian)

227. ____, *On the spectral theory of holomorphic operator-valued functions in Hilbert space*, Funktsional. Anal. i Prilozhen. **9** (1975), no. 1, 76–77; English transl. in Functional Anal. Appl. **9** (1975).

228. ____, *Two remarks about factorization of matrix-valued functions*, Mat. Issled. Vyp. 42 (1976), 216–223. (Russian)

229. ____, *On the factorization of a weakly hyperbolic operator pencil*, Funktsional. Anal. i Prilozhen. **10** (1976), no. 1, 81–82; English transl. in Functional Anal. Appl. **10** (1976).

230. ____, *On the spectral factorization of holomorphic operator-valued functions*, Mat. Issled. Vyp. 47 (1978), 71–100; rev. English transl., Selecta Math. Sovietica **4** (1985), 325–354.

231. ____, *On the asymptotic behavior of the spectrum of operators which are close to being normal*, Funktsional. Anal. i Prilozhen. **13** (1979), no. 3, 93–94; English transl. in Functional Anal. Appl. **13** (1979).

232. ____, *Operators generated by sesquilinear forms, and their spectral asymptotics*, Mat. Issled. Vyp. 61 (1981), 86–103. (Russian)

233. ____, *On the convergence of eigenvector expansions for an operator which is close to being selfadjoint*, Mat. Issled. Vyp. 61 (1981), 104–129. (Russian)

234. ____, *Comparison theorems for spectra of linear operators, and spectral asymptotics*, Trudy Moskov. Mat. Obshch. **45** (1982), 133–181; English transl. in Trans. Moscow Math. Soc. **1984**, no. 1(45).

235. ____, *Comparison theorems for spectra of linear operators, and Keldysh asymptotics for pencils*, Preprint, Inst. Math. with Computing Center, Acad Sci. Moldavian SSR, Kishinev, 1983. (Russian)

236. ____, *A comparison theorem for spectra, and spectral asymptotics for a Keldysh pencil*, Mat. Sb. **123 (165)** (1984), 391–406; English transl. in Math. USSR Sb. **51** (1985).

237. A. S. Markus and I. V. Mereutsa, *On a complete collection of roots of the operator equation corresponding to a polynomial operator pencil*, Izv. Akad. Nauk SSSR Ser. Mat. **37** (1973), 1108–1131; English transl. in Math. USSR Izv. **7** (1973).

238. A. S. Markus and E. È. Parilis, *The change of the Jordan structure of a matrix under small perturbations*, Linear Algebra and Appl. **54** (1983), 139–152.

239. A. S. Markus and G. I. Russu, *On a basis of eigenvectors of a selfadjoint polynomial pencil*, Mat. Issled. **6** (1971), no. 1(19), 114–125. (Russian)

240. A. S. Markus and E. I. Sigal, *On the multiplicity of a characteristic number of an analytic operator-valued function*, Mat. Issled. **5** (1970), no. 3(17), 129–147. (Russian)

241. A. S. Markus, V. I. Matsaev, and G. I. Russu, *Some generalizations of the theory of strongly damped pencils to the case of pencils of arbitrary order*, Acta. Sci. Math. (Szeged) **34** (1973), 245–271. (Russian)

242. A. I. Markushevich, *Theory of analytic functions*. Vol. I: *Fundamentals of the theory*, 2nd rev. aug. ed., "Nauka", Moscow, 1967; English transl. of 1st ed., *Theory of functions of a complex variable*. Vols. 1, 2, Prentice-Hall, Englewood Cliffs, N.J., 1965.

243. ____, *Theory of analytic functions*. Vol. II: *Further construction of the theory*, 2nd rev. aug. ed., "Nauka", Moscow, 1968; English transl. of 1st ed., *Theory of functions of a complex variable*. Vols. 2, 3, Prentice-Hall, Englewood Cliffs, N.J., 1965, 1967.

244. Kenji Maruo, *Asymptotic distribution of eigenvalues of non-symmetric operators associated with strongly elliptic sesquilinear forms*, Osaka J. Math. **9** (1972), 547–560.

245. P. Masani, *The Laurent factorization of operator-valued functions*, Proc. London Math. Soc. (3) **6** (1956), 59–69.

246. V. I. Matsaev, *On a class of compact operators*, Dokl. Akad. Nauk SSSR **139** (1961), 548–551; English transl. in Soviet Math. Dokl. **2** (1961).

247. ——, *Some theorems on the completeness of root subspaces of compact operators*, Dokl. Akad. Nauk SSSR **155** (1964), 273–276; English transl. in Soviet Math. Dokl. **5** (1964).

248. V. I. Matsaev and E. Z. Mogul'skiĭ, *Some tests for multiple completeness of the system of eigenvectors and associated vectors of polynomial operator pencils*, Teor. Funktsiĭ, Funktsional. Anal. i Prilozhen. Vyp. 13 (1971), 3–45. (Russian)

249. ——, *On the completeness of weak perturbations of selfadjoint operators*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. **56** (1976), 90–103; English transl. in J. Soviet Math. **14** (1980), no. 2.

250. V. I. Matsaev and Yu. A. Palant, *On the distribution of the spectrum of a polynomial operator pencil*, Akad. Nauk Armyan. SSR Dokl. **42** (1966), 257–261. (Russian)

251. ——, *On the distribution of the spectrum of a rational operator pencil*, Mat. Issled. **4** (1969), no. 2(12), 156–157; English transl. in Amer. Math. Soc. Transl. (2) **110** (1977).

252. ——, *Spectral theory of operator pencils*, Izdat. Donetsk. Gos. Univ., Donetsk, 1975. (Russian)

253. Reinhard Mennicken, *Eigenvalue problems depending nonlinearly on the parameter*, Banach Space Theory and Its Applications (Bucharest, 1981), Lecture Notes in Math., vol. 991, Springer-Verlag, 1983, pp. 156–181.

254. I. V. Mereutsa, *On properties of the roots of an operator equation that correspond to an operator polynomial pencil*, Mat. Issled. **8** (1973), no. 1(27), 96–115. (Russian)

255. ——, *On the decomposition of a polynomial operator pencil into linear factors*, Mat. Issled. **8** (1973), no. 2(28), 102–114. (Russian)

256. ——, *A criterion for the regularity of a root of an operator equation corresponding to a polynomial operator pencil*, Mat. Issled. **10** (1975), no. 3(37), 214–215. (Russian)

257. ——, *On the factorization of a polynomial operator pencil*, Mat. Issled. Vyp. 45 (1977), 115–124. (Russian)

258. ——, *On the decomposition of a polynomial operator pencil into factors*, Mat. Issled. Vyp. 47 (1978), 101–119. (Russian)

259. ——, *On the decomposition of a polynomial operator pencil into linear factors*, Investigations in Functional Analysis and Differential Equations (B. A. Shcherbakov, editor), "Shtiintsa", Kishinev, 1981, pp. 61–64. (Russian)

260. ——, *On the factorization of a polynomial operator pencil*, Mat. Issled. Vyp. 67 (1982), 100–108. (Russian)

261. I. V. Mereutsa and D. I. Taragan, *On some properties of divisors of a holomorphic operator-valued function*, Investigations in Functional Analysis and Differential Equations (B. A. Shcherbakov, editor), "Shtiintsa", Kishinev, 1981, pp. 65–75. (Russian)

262. ——, *On divisors of a holomorphic operator-valued function*, Mat. Issled. Vyp. 61 (1981), 130–140. (Russian)

263. V. A. Mikhaĭlets, *Distribution of the eigenvalues of operators that are close to being selfadjoint*, Funktsional. Anal. i Prilozhen. **15** (1981), no. 1, 78–79; English transl. in Functional Anal. Appl. **15** (1981).

264. ——, *Asymptotics of the spectrum of elliptic operators and boundary conditions*, Dokl. Akad. Nauk SSSR **266** (1982), 1059–1063; English transl. in Soviet Math. Dokl. **26** (1982).

265. V. P. Mikhaĭlov, *On Riesz bases in* $\mathscr{L}_2(0,1)$, Dokl. Akad. Nauk SSSR **144** (1962), 981–984; English transl. in Soviet Math. Dokl. **3** (1962).

266. A. I. Miloslavskiĭ, *Spectral properties of a class of quadratic operator pencils*, Funktsional. Anal. i Prilozhen. **15** (1981), no. 2, 81–82; English transl. in Functional Anal. Appl. **15** (1981).

267. S. S. Mirzoev, *Two-fold completeness of a part of the eigenvectors and associated vectors of polynomial operator pencils of fourth order*, Izv. Akad. Nauk Azerbaĭdzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk **1974**, no. 6, 37–42. (Russian)

268. ——, *On multiple completeness of the root vectors of polynomial operator pencils corresponding to boundary value problems on a half-line*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 2, 84–85; English transl. in Functional Anal. Appl. **17** (1983).

269. Sigeru Mizohata, *Sur les propriétés asymptotiques des valeurs propres pour les opérateurs elliptiques*, Proc. Japan. Acad. **41** (1965), 104–108.

270. ——, *Sur les propriétés asymptotiques des valeurs propres pour les opérateurs elliptiques*, J. Math. Kyoto Univ. **4** (1964/65), 399–428.

271. E. Z. Mogul'skiĭ, *A theorem on completeness of the system of eigenvectors and associated vectors of a rational operator pencil*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. **3** (1968), 427–442. (Russian)

272. ——, *On completeness of the system of eigenvectors and associated vectors of an operator pencil*, Dokl. Akad. Nauk SSSR **183** (1968), 775–777; English transl. in Soviet Math. Dokl. **9** (1968).

273. ——, *On multiple completeness of the system of eigenvectors and associated vectors of a linear fractional operator pencil*, Bull. Akad. Shtiintse RSS Moldoven. **1975**, no. 2, 3–10. (Russian)

274. P. H. Müller, *Zu einer Spektralbetrachtung von Atkinson und Sz.-Nagy*, Acta Sci. Math. (Szeged) **17** (1956), 195–197.

275. M. A. Naĭmark, *On some criteria for completeness of the system of eigenvectors and associated vectors of a linear operator in Hilbert space*, Dokl. Akad. Nauk SSSR **98** (1954), 727–730. (Russian)

276. ——, *Spectral analysis of nonselfadjoint operators*, Uspekhi Mat. Nauk **11** (1956), no. 6(72), 183–202; English transl. in Amer. Math. Soc. Transl. (2) **20** (1962).

277. ——, *Linear differential operators*, 2nd ed., "Nauka", Moscow, 1969; English transl. of 1st ed., Parts I, II, Ungar, New York, 1967, 1968.

278. N. K. Nikol'skiĭ, *Treatise on the shift operator*, "Nauka", Moscow, 1980; English transl., Springer-Verlag, 1985.

279. I. A. Novosel'skiĭ, *Tests for completeness of the system of root vectors of a linear operator in a space with two norms*, Dokl. Akad. Nauk SSSR **167** (1966), 747–750; English transl. in Soviet Math. Dokl. **7** (1966).

280. M. B. Orazov, *On completeness of the root vectors of a selfadjoint quadratic pencil*, Izv. Akad. Nauk Turkmen. SSR Ser. Fiz.-Tekhn. Khim. i Geol. Nauk **1980**, no. 5, 7–13. (Russian)

281. M. B. Orazov and G. V. Radzievskiĭ, *Theorems on completeness and the basis property for the eigenvectors of a hyperbolic operator-valued function*, Sibirsk. Mat. Zh. **16** (1975), 572–587; English transl. in Siberian Math. J. **16** (1975).

282. M. B. Orazov and A. A. Shkalikov, *On the n-fold basis property of the eigenfunctions of certain regular boundary value problems*, Sibirsk. Mat. Zh. **17** (1976), 627–639; English transl. in Siberian Math. J. **17** (1976).

283. Yu. A. Palant, *On a test for completeness of the system of eigenvectors and associated vectors of a polynomial operator pencil*, Dokl. Akad. Nauk SSSR **141** (1961), 558–560; English transl. in Soviet Math. Dokl. **2** (1961).

284. ——, *On a method for getting tests for multiple completeness of the system of eigenvectors and associated vectors of a polynomial operator pencil*, Vestnik Khar'kov. Univ. No. 53, Ser. Mekh.-Mat. Vyp. (4) 34 (1970), 3–13. (Russian)

285. Pham The Lai, *Comportement asymptotique du noyau de la résolvante et des valeurs propres d'un opérateur elliptique non nécessairement auto-adjoint*, Israel J. Math. **23** (1976), 221–250.

286. R. S. Phillips, *The extension of dual subspaces invariant under an algebra*, Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), Jerusalem Academic Press, Jerusalem, and Pergamon Press, Oxford, 1961, pp. 366–398.

287. B. M. Podlevskiĭ, *On selfadjoint polynomial operator pencils that are spectrally equivalent to selfadjoint operators*, Ukrain. Mat. Zh. **36** (1984), 660–662; English transl. in Ukrainian Math. J. **36** (1984).

288. I. I. Privalov, *Boundary properties of analytic functions*, 2nd ed., GITTL, Moscow, 1950; German transl., VEB Deutscher Verlag Wiss., Berlin, 1956.

289. G. V. Radzievskiĭ, *Multiple completeness of the root vectors of a Keldysh pencil perturbed by an operator-valued function analytic in the disk*, Mat. Sb. **91 (133)** (1973), 310–335; English transl. in Math. USSR Sb. **20** (1973).

290. ——, *On a method for proving completeness of the root vectors of operator-valued functions*, Dokl. Akad. Nauk SSSR **214** (1974), 291–294; English transl. in Soviet Math. Dokl. **15** (1974).

291. ——, *On the basis property of derived chains*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 1182–1218; English transl. in Math. USSR Izv. **9** (1975).

292. ——, *On completeness of derived chains*, Mat. Sb. **100 (142)** (1976), 37–58; English transl. in Math. USSR Sb. **29** (1976).

293. ——, *On some criteria for multiple completeness of the root vectors of operator-valued functions analytic in an angle*, Ukrain. Mat. Zh. **28** (1976), 203–212; English transl. in Ukrainian Math. J. **28** (1976).

294. ——, *On the problem of the completeness of the root vectors corresponding to two spectral series of a Keldysh pencil*, Ukrain. Mat. Zh. **28** (1976), 413–418; English transl. in Ukrainian Math. J. **28** (1976).

295. ——, *Completeness theorems for a part of the root vectors of a Keldysh pencil*, Dokl. Akad. Nauk Ukrain. SSR Ser. A **1976**, 597–600. (Russian)

296. ——, *A quadratic pencil of operators*, Preprint 76-24, Inst. Math. Acad. Sci. Ukrainian SSR, Kiev, 1976. (Russian) MR **56** # 16418.

297. ——, *Completeness of the root vectors of a Keldysh pencil which is perturbed by an operator-valued function* $S(\lambda)$ *with* $S(\infty) = 0$, Mat. Zametki **21** (1977), 391–398; English transl. in Math. Notes **21** (1977).

298. ——, *On completeness of the set of root vectors of the operator pencil* $L(\lambda) = I - \lambda^{-k}B - \lambda^{n}A$, Uspekhi Mat. Nauk **34** (1979), no. 1(205), 241–242; English transl. in Russian Math. Surveys **34** (1979).

299. ——, *On completeness of the derived chains corresponding to boundary value problems on a finite segment*, Ukrain. Mat. Zh. **31** (1979), 279–288; English transl. in Ukrainian Math. J. **31** (1979).

300. ——, *On completeness of the derived chains corresponding to boundary value problems on a half-line*, Ukrain. Mat. Zh. **31** (1979), 407–416; English transl. in Ukrainian Math. J. **31** (1979).

301. ——, *On bases consisting of derived chains corresponding to boundary value problems*, Dokl. Akad. Nauk SSSR **251** (1980), 283–287; English transl. in Soviet Math. Dokl. **21** (1980).

302. ——, *Asymptotics of the distribution of the characteristic numbers of operator-valued functions analytic in an angle*, Mat. Sb. **112 (154)** (1980), 396–420; English transl. in Math. USSR Sb. **40** (1981).

303. ——, *The problem of completeness of the root vectors in the spectral theory of operator-valued functions*, Uspekhi Mat. Nauk **37** (1982), no. 2(224), 81–145; English transl. in Russian Math. Surveys **37** (1982).

304. ——, *On a method for proving the minimality and the basis property of eigenvectors*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 1, 24–30; English transl. in Functional Anal. Appl. **17** (1983).

305. ——, *A quadratic pencil of operators (equivalence of part of the root vectors)*, Preprint No. 84.32, Inst. Math. Acad. Sci. Ukrainian SSR, Kiev, 1984. (Russian) MR **86b:**47029.

306. R. Raghavendran, *Toeplitz-Hausdorff theorem on numerical ranges*, Proc. Amer. Math. Soc. **20** (1969), 284–285.

307. Alexander G. Ramm, *Theory and applications of some new classes of integral equations*, Springer-Verlag, 1980.

308. M. L. Rasulov, *The contour integral method and its application to the investigation of problems for differential equations*, "Nauka", Moscow, 1964; English transl., North-Holland, Amsterdam, and Interscience, New York, 1967.

309. Michael Reed and Barry Simon, *Methods of modern mathematical physics*. Vol. I, Academic Press, 1972.

310. ——, *Methods of modern mathematical physics*. Vol. IV, Academic Press, 1978.

311. Frédéric [Frigyes] Riesz and Béla Sz.-Nagy, *Leçons d'analyse fonctionelle*, 4th ed., Akad. Kiadó, Budapest, and Gauthier-Villars, Paris, 1965; English transl. of 2nd ed., *Functional analysis*, Ungar, New York, 1955.

312. D. Robert, *Propriétés spectrales d'opérateurs pseudo-différentiels*, Comm. Partial Differential Equations **3** (1978), 755–826.

313. Leiba Rodman, *On existence of common multiples of monic operator polynomials*, Integral Equations Operator Theory **1** (1978), 400–414.

314. E. H. Rogers, *A minimax theory for overdamped systems*, Arch. Rational Mech. Anal. **16** (1964), 89–96.

315. ——, *Variational properties of nonlinear spectra*, Arch. Rational Mech. Anal. **18** (1968), 479–490.

316. Marvin Rosenblum and James Rovnyak, *The factorization problem for nonnegative operator valued functions*, Bull. Amer. Math. Soc. **77** (1971), 287–318.

317. Brian Rowley, *Wiener-Hopf factorization of operator polynomials*, Integral Equations Operator Theory **3** (1980), 437–462.

318. G. V. Rozenblyum, *Spectral asymptotics of normal operators*, Funktsional. Anal. i Prilozhen. **16** (1982), no. 2, 82–83; English transl. in Functional Anal. Appl. **16** (1982).

319. E. M. Russakovskiĭ, *An operator treatment of a boundary value problem with a spectral parameter appearing polynomially in the boundary conditions*, Funktsional. Anal. i Prilozhen. **9** (1975), no. 4, 91–92; English transl. in Functional Anal. Appl. **9** (1975).

320. Albert Schneider, *A note on eigenvalue problems with eigenvalue parameter in the boundary conditions*, Math. Z. **136** (1974), 163–167.

321. J. Schwartz, *Perturbations of spectral operators, and applications*. I: *Bounded perturbations*, Pacific J. Math. **4** (1954), 415–458.

322. Marvin Shinbrot, *Note on a nonlinear eigenvalue problem*, Proc. Amer. Math. Soc. **14** (1963), 552–558.

323. ——, *A nonlinear eigenvalue problem*. II, Arch. Rational Mech. Anal. **15** (1964), 368–376.

324. A. A. Shkalikov, *On completeness of the eigenfunctions and associated functions of an ordinary differential operator with nonregular splitting boundary conditions*, Funktsional. Anal. i Prilozhen. **10** (1976), no. 4, 69–80; English transl. in Functional Anal. Appl. **10** (1976).

325. ——, *On the basis property for eigenfunctions of an ordinary differential operator*, Uspekhi Mat. Nauk **34** (1979), no. 5(209), 235–236; English transl. in Russian Math. Surveys **34** (1979).

326. ——, *On the basis property for the eigenvectors of quadratic operator pencils*, Mat. Zametki **30** (1981), 371–385; English transl. in Math. Notes **30** (1981).

327. ——, *On the basis property for the eigenfunctions of ordinary differential operators with integral boundary conditions*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1982**, no. 6, 12–21; English transl. in Moscow Univ. Math. Bull. **37** (1982).

328. ——, *Boundary value problems for ordinary differential equations with a parameter in the boundary conditions*, Trudy Sem. Petrovsk. Vyp. 9 (1983), 190–229; English transl. in J. Soviet Math. **33** (1986), no. 6.

329. ——, *Some questions in the theory of polynomial operator pencils*, Uspekhi Mat. Nauk **38** (1983), no. 3(231), 189–190; English transl. in Russian Math. Surveys **38** (1983).

330. ——, *Operator-differential equations on a half-line and related spectral problems for selfadjoint operator pencils*, Dokl. Akad. Nauk SSSR **276** (1984), 309–314; English transl. in Soviet Math. Dokl. **29** (1984).

331. M. A. Shubin, *Pseudodifferential operators and spectral theory*, "Nauka", Moscow, 1978; English transl., Springer-Verlag, 1986.

332. E. I. Sigal, *Partial multiplicities of a product of operator-valued functions*, Mat. Issled. **8** (1973), no. 4(30), 65–79. (Russian)

333. I. M. Spitkovskii, *Stability of the partial indices of the Riemann boundary value problem with a strictly nonsingular matrix*, Dokl. Akad. Nauk SSSR **218** (1974), 46–49; English transl. in Soviet Math. Dokl. **15** (1974).

334. Béla Sz.-Nagy and Ciprian Foias, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson, Paris, and Akad. Kiadó, Budapest, 1967; rev. English transl., North-Holland, Amsterdam, and Akad. Kiadó, Budapest, 1971.

335. Ya. D. [J. D.] Tamarkin, *Some general problems of the theory of ordinary linear differential equations and expansion of an arbitrary function in series of fundamental functions*, Dissertation, Petrograd, 1917; abridged English transl., [336].

336. ——, *Some general problems of the theory of ordinary linear differential equations and expansion of an arbitrary function in series of fundamental functions*, Math. Z. **27** (1927/28), 1–54.

337. D. I. Taragan, *Some criteria for the existence of a divisor of a holomorphic operator-valued function*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 1, 84–85; English transl. in Functional Anal. Appl. **18** (1984).

338. J. Terray and P. Lancaster, *On the numerical calculation of eigenvalues and eigenvectors of operator polynomials*, J. Math. Anal. Appl. **60** (1977), 370–378.

339. Bui An Ton, *Asymptotic distribution of eigenvalues and eigenfunctions for general linear elliptic boundary value problems*, Trans. Amer. Math. Soc. **122** (1966), 516–546.

340. V. P. Trofimov, *On the root subspaces of operators depending analytically on a parameter*, Mat. Issled. **3** (1968), no. 3(9), 117–125. (Russian)

341. Robert E. L. Turner, *Perturbation of ordinary differential operators*, J. Math. Anal. Appl. **13** (1966), 447–457.

342. ——, *Some variational principles for a nonlinear eigenvalue problem*, J. Math. Anal. Appl. **17** (1967), 151–160.

343. ——, *A class of nonlinear eigenvalue problems*, J. Functional Anal. **7** (1968), 297–322.

344. A. I. Vagabov, *A theorem on multiple completeness for ordinary differential pencils*, Dokl. Akad. Nauk SSSR **275** (1984), 13–17; English transl. in Soviet Math. Dokl. **29** (1984).

345. D. G. Vasil'ev, *A two-term asymptotic expression for the spectrum of a boundary value problem under internal reflection of a general form*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 4, 1–13; English transl. in Functional Anal. Appl. **18** (1984).

346. G. V. Virabyan, *On factorization of a quadratic operator pencil*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. **9** (1974), 185–188. (Russian)

347. A. I. Virozub, *On energy completeness of the system of elementary solutions of a differential equation in Hilbert space*, Funktsional. Anal. i Prilozhen. **9** (1975), no. 1, 52–53; English transl. in Functional Anal. Appl. **9** (1975).

348. ——, *Spectral properties of holomorphic operator-valued functions and their factorization*, Author's summary of Candidate's dissertation, Khar'kov. Gos. Univ., Kharkov, 1975. (Russian)

349. A. I. Virozub and V. I. Matsaev, *A theorem on canonical factorization of operator-valued functions*, Mat. Issled. **8** (1973), no. 3(29), 145–150. (Russian)

350. ——, *On spectral properties of a certain class of selfadjoint operator-valued functions*, Funktsional. Anal. i Prilozhen. **8** (1974), no. 1, 1–10; English transl. in Functional Anal. Appl. **8** (1974).

351. V. N. Viziteĭ and A. S. Markus, *On convergence of multiple expansions in the eigenvectors and associated vectors of an operator pencil*, Mat. Sb. **66 (108)** (1965), 287–320; English transl. in Amer. Math. Soc. Transl. (2) **87** (1970).

352. V. V. Vlasov, *Multiple minimality of part of the system of root vectors of a Keldysh pencil*, Dokl. Akad. Nauk SSSR **263** (1982), 1289–1293; English transl. in Soviet Math. Dokl. **25** (1982).

353. ——, *On the minimality of derived chains*, Uspekhi Mat. Nauk **37** (1982), no. 5(227), 171–172; English transl. in Russian Math. Surveys **37** (1982).

354. G. A. Voropaeva and V. P. Maslov, *Multiple completeness in the Keldysh sense, and uniqueness of a solution of the corresponding Cauchy problem*, Funktsional. Anal. i Prilozhen. **4** (1970), no. 2, 10–17; English transl. in Functional Anal. Appl. **4** (1970).

355. I. I. Vorovich and V. E. Koval'chuk, *On the basis properties of a system of homogeneous equations*, Prikl. Mat. Mekh. **31** (1967), 861–869; English transl. in J. Appl. Math. Mech. **31** (1967).

356. Johann Walter, *Regular eigenvalue problems with eigenvalue parameter in the boundary condition*, Math. Z. **133** (1973), 301–312.

357. Michiaki Watanabe, *On the asymptotic distribution of eigenvalues of nonsymmetric operators associated with strongly elliptic sesquilinear forms*, Sci. Rep. Niigata Univ. Ser. A No. 11 (1974), 69–84.

358. H. F. Weinberger, *On a nonlinear eigenvalue problem*, J. Math. Anal. Appl. **21** (1968), 506–509.

359. Bodo Werner, *Das Spektrum von Operatorscharen mit verallgemeinerten Rayleighquotienten*, Arch. Rational Mech. Anal. **42** (1971), 223–238.

360. S. Ya. Yakubov, *On two-fold completeness of the eigenelements and associated elements of a quadratic operator pencil*, Funktsional. Anal. i Prilozhen. **7** (1973), no. 1, 92–94; English transl. in Functional Anal. Appl. **7** (1973).

361. ——, *Quadratic operator pencils*, Dokl. Akad. Nauk SSSR **269** (1983), 39–42; English transl. in Soviet Math. Dokl. **27** (1983).

362. S. Ya. Yakubov and K. S. Mamedov, *On mutliple completeness of the system of eigenelements and associated elements of a polynomial operator pencil, and on multiple expansions in this system*, Funktsional. Anal. i Prilozhen. **9** (1975), no. 1, 91–93; English transl. in Functional Anal. Appl. **9** (1975).

363. V. S. Zayachkovskiĭ and A. A. Pankov, *On stability of factorizations of polynomial operator pencils*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 2, 73–74; English transl. in Functional Anal. Appl. **17** (1983).

364. A. S. Zil'bergleĭt and Yu. I. Kopilevich, *Spectral theory of regular waveguides*, Izdat. Ioffe Fiz.-Tekhn. Inst. Akad. Nauk SSSR, Leningrad, 1983. (Russian)

365. János Bognár, *Indefinite inner product spaces*, Springer-Verlag, 1974.

366. I. Gohberg, P. Lancaster, and L. Rodman, *Matrices and indefinite scalar products*, Birkhäuser, 1983.

367. ——, *Invariant subspaces of matrices with applications*, Wiley, 1986.

368. Ya. B. Lopatinskiĭ, *The factorization of a polynomial matrix*, L'vov. Politekhn. Inst. Nauchn. Zap. Ser. Fiz.-Mat. Vyp. 38 (1956), 3–7 (1957). (Russian)

369. A. S. Markus and V. I. Matsaev, *On the basis property for a certain part of the eigenvectors and associated vectors of a selfadjoint operator pencil*, Mat. Sb. **133(175)** (1987), 293–313; English transl. in Math. USSR Sb. **61** (1988).

370. George D. Birkhoff, *On the asymptotic character of the solutions of certain linear differential equations containing a parameter*, Trans. Amer. Math. Soc. **9** (1908), 219–231.

Subject Index

List of Notation

The symbols **C**, **R**, and **N** have the standard meaning.

$\sigma = \operatorname{Re}\lambda,\ \tau = \operatorname{Im}\lambda,\ \varphi = \arg\lambda,\ \Omega(\theta) = \{\lambda\colon |\varphi| < \theta\}$

$D_r = \{\lambda\colon |\lambda| < r\},\ \Gamma_0 = \{t\colon |t| = 1\}$

$C(\Gamma, L(\mathfrak{H})),\ C_+(\Gamma, L(\mathfrak{H})),\ C_-^0(\Gamma, L(\mathfrak{H}))$ §22.4

$\mathfrak{D}(A)$ §2.1

$\det(I + K)$ §2.5

$\det_k(I - T)$ §18.2

$\mathfrak{H}$ §2.*I*

H^2, H^∞ §34.4

$H^2(\mathfrak{H}), H^\infty(\Gamma_0, L(\mathfrak{H}))$ §34.2

$H^\infty(\mathbf{R}, L(\mathfrak{H}))$ §34.6

I §2.1

$\operatorname{Im} A$ §2.1

$\operatorname{ind} f(t),\ \operatorname{ind}_\Gamma f(t)$ §25.1

$\operatorname{Ker} A$ §2.1

$L(\mathfrak{H})$ §2.1

$L^2(\mathfrak{H}),\ L_+^2(\mathfrak{H}),\ L_-^2(\mathfrak{H})$ §24.1

$L^\infty(\Gamma_0, L(\mathfrak{H}))$ §34.2

$L^\infty(\mathbf{R}, L(\mathfrak{H}))$ §34.4

$m(g_0), m(\lambda_0, A(\lambda))$ §11.2

$\mathfrak{M}(Q)$ §34.2

$N(U, A),\ N(r, A),\ N_+(r, A),\ N_+(\rho, r, A)$ §6.4

$N(r, \theta, A),\ N(\rho, r, \theta, A)$ §8.2

$N_k(r, A(\lambda))$ §15.2

$n(r, H)$ §6.6

$p_k(f)$ §31.1

$R(A)$ §26.3

$R_\lambda(A)$ §2.1

$r(B)$ §24.1

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