Space Mappings with Bounded Distortion

YU. G. RESHETNYAK

Volume 73

TRANSLATIONS OF
MATHEMATICAL MONOGRAPHS

American Mathematical Society

Space Mappings with Bounded Distortion

This page intentionally left blank

10.1090/mmono/073

TRANSLATIONS OF MATHEMATICAL MONOGRAPHS

VOLUME 73

Space Mappings with Bounded Distortion

YU. G. RESHETNYAK

American Mathematical Society · Providence · Rhode Island

Ю. Г. РЕШЕТНЯК

ПРОСТРАНСТВЕННЫЕ ОТОБРАЖЕНИЯ С ОГРАНИЧЕННЫМ ИСКАЖЕНИЕМ

«НАУКА», МОСКВА, 1982

Translated from the Russian by H. H. McFaden
Translation edited by Ben Silver

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30-02, 30C60; Secondary 30C20, 30C35, 30C85, 46E35, 53B20, 31B15, 35J99, 53A30.

ABSTRACT. This monograph is an exposition of the main results obtained in recent years by Soviet and foreign mathematicians in the theory of mappings with bounded distortion. The book relates to an active direction in contemporary mathematics. The mathematical apparatus contained in it can be applied to a broad spectrum of problems that go beyond the context of the main topic of investigation. A number of questions in the theory of partial differential equations and the theory of functions with generalized derivatives are expounded for the first time in the world monograph literature. The book is intended for research workers, graduate students, and university students concerned with questions in analysis and function theory.

Bibliography: 183 titles

Library of Congress Cataloging-in-Publication Data

Reshetnîak, Îuriĭ Grigor′evich.
[Prostranstvennye otobrazheniîa s ogranichennym iskazheniem. English]
Space mappings with bounded distortion/Yu. G. Reshetnyak.
p. cm. – (Translations of mathematical monographs; v. 73)
Translation of: Prostranstvennye otobrazheniîa s ogranichennym iskazheniem.
Bibliography: p. 347
ISBN 0-8218-4526-8 (alk. paper)
1. Conformal mapping. I. Title. II. Series.
QA646.R3813 1989 89-72
515.9–dc 19 CIP

Contents

This page intentionally left blank

Foreword to the English Translation

This text differs from the Russian original in the following respects. First, Chapter I has been completely reworked. The author has tried to give a more complete exposition of the preliminary facts needed for reading the main text. In particular, complete proofs are presented for all the required properties of Möbius mappings. In Chapter II some errors in the original are corrected, and improvements in the text are introduced where it seemed possible by slight changes to strengthen individual results or to make the presentation more complete and clear. Moreover, §12 is added to Chapter II. It contains a survey of certain further investigations of questions close to the main topic of the book. Here the author has confined himself mainly to an account of work carried out in Novosibirsk and little known outside the USSR. The bibliography has been enlarged accordingly .

Chapter III remains almost without changes.

This page intentionally left blank

From the Author

The book before the reader is devoted to an exposition of results of investigations carried out mainly over the last 10–15 years concerning certain questions in the theory of quasiconformal mappings.

The principal objects of investigation—mappings with bounded distortion—are a kind of n-space analogue of holomorphic functions. As is known, every holomorphic function is characterized geometrically by the fact that the mapping of a planar domain it implements is conformal. In the n-space case the condition of conformality singles out a very narrow class of mappings. As Liouville showed back in 1850, already in three-dimensional Euclidean space there are no conformal mappings besides those which are compositions of finitely many inversions with respect to spheres. Such mappings are called *Möbius mappings*. They form a finite-dimensional Lie group which includes the group of motions of the space $\mathbf{R}^n$ and is only slightly broader than this group. However, if one weakens the condition of conformality, replacing it by the condition of quasiconformality, then a considerably broader class of mappings emerges.

To give the reader an idea about the subject of the book we present some explanations (the exact definitions are contained in the main text). A mapping of a domain in an n-dimensional space is called a *mapping with bounded distortion* if it satisfies definite requirements of regularity, preserves the orientation of every small domain, and (the main point) satisfies the following condition. There exists a constant q, $1 \leq q < \infty$, such that an infinitesimally small sphere is transformed by the mapping into either a point or an infinitesimally small ellipsoid for which the ratio of the largest semiaxis to the smallest does not exceed the constant q. If, moreover, the mapping is also topological (a homeomorphism), then it is said to be quasiconformal.

The regularity requirement mentioned here is that the components of the vector-valued function determining the given mapping must have first-order generalized derivatives that are locally integrable to the power n. The condition about preserving the orientation of a small domain is analytically equivalent to the Jacobian of the mapping being nonnegative. For an arbitrary mapping with bounded distortion there can exist points such that the mapping is not a homeomorphism in any neighborhood of them. These points are called *branch points* of the mapping. The dimension of the set of branch points does not exceed $n-2$. For example, the set of branch points can be a curve in three-dimensional space. For a holomorphic function of a single variable the branch points are simply the zeros of its derivative.

We note that in the two-dimensional case the study of arbitrary mappings with bounded distortion is easily reduced to the consideration of homeomorphic quasiconformal mappings and holomorphic functions of a single variable.

The theory of planar quasiconformal mappings arose at the end of the 1920's in work of Grötzsch and M. A. Lavrent'ev. This theory is now a far-advanced area of the theory of functions of a complex variable which has important applications both in function theory itself and beyond its boundaries, in particular, in applied areas.

The concept of a quasiconformal mapping in n-space introduced by Lavrent'ev in 1938 in searching for a suitable tool to construct mathematical models of certain hydrodynamics phenomena. He formulated also a number of problems whose solutions later played an essential role in the development of the theory of quasiconformal mappings in n-space. However, the beginning of intensive investigations in this area dates from 1960. Further, as in the planar case, only homeomorphic quasiconformal mappings were considered at first. The systematic study of general mappings with bounded distortion was begun in 1966.

There are two basic methods in the theory of mappings with bounded distortion. One of them goes back to the classical work of Grötzsch and is based on the use of a certain quantity characterizing a family of curves or surfaces in space and called the *modulus* of the family. This method depends on inequalities describing the behavior of the modulus of a family of curves or surfaces when the family is transformed by a given mapping, as well as on certain estimates for the moduli.

The other method consists in the use of a certain apparatus in the theory of differential equations. As is known, the real and imaginary parts of a holomorphic function of a complex variable are harmonic functions. Analogously, the components of a vector-valued function representing a mapping with bounded distortion are solutions of a certain elliptic partial differential equation. The method consists in the use of this fact (and certain generalizations of it) and properties of elliptic equations, in particular, the maximum principle for elliptic equations, Harnack's inequality, etc. Some estimates relating to the concept of the capacity of a capacitor are also used.

In this monograph the second method is used to study mappings with bounded distortion. All the needed facts about elliptic equations are given. A significant part of the book is devoted to an exposition of material that is auxiliary with respect to the main topic, though it is definitely of independent interest. In particular, a proof is given of the well-known theorem of Moser and Serrin on Harnack's inequality for elliptic equations; theorems are proved on semicontinuity and convergence with a functional for functionals of the calculus of variations; the necessary facts are given about the concept of the degree of a mapping and the metric properties of mappings connected with these facts; etc. In this connection the author hopes that the book will prove to be useful not only for specialists in the theory of mappings in n-space, but also for a broader circle of readers.

The investigation of mappings with bounded distortion is based on the concept of the generalized differential of an exterior form [146]. In particular, a detailed study is made of the properties of the generalized differential, and this, in the author's opinion, can be of interest, for example, in connection with certain recent investigations of the topology of Lipschitz manifolds by analytic means [49].

Quasiconformal mappings in n-space have been used in the theory of spaces of functions with generalized derivatives ([177], [178], [46]), as well as in investigations of compact Riemannian spaces of constant negative curvature [106]. The theory of mappings in n-space with bounded distortion is one of the areas in the general metric theory of space mappings that is being intensively developed at present. Among the investigations in this area one can cite work on the theory of quasi-isometric mappings [60], the theory of quasi-Lorentz mappings [51], a series of investigations in the theory of Kleinian groups in space [75], papers on the theory of homeomorphisms of class W_n^1 ([165], [112]), and other publications.

Many interesting questions in the theory of mappings in n-space close to the topic of the book had to be omitted for lack of space. In choosing the material the author was oriented toward results used in studying the problem of stability in Liouville's theorem on conformal mappings in space.

Yu. G. Reshetnyak

This page intentionally left blank

This page intentionally left blank

Bibliography

ROBERT A. ADAMS

1. *Sobolev spaces*, Academic Press, 1975.

LARS V. AHLFORS

2. *Quasiconformal reflections*, Acta Math. **109** (1963), 291–301.
3. *Kleinsche Gruppen in der Ebene und im Raum*, Festband 70. Geburtstag R. Nevanlinna, Springer-Verlag, 1966, pp. 7–15.
4. *Conditions for quasiconformal deformations in several variables*, Contributions to Analysis (Collection Dedicated to Lipman Bers), Academic Press, 1974, p. 19–25.
5. *Invariant operators and integral representations in hyperbolic space*, Math. Scand. **36** (1975), 27–43.
6. *Quasiconformal deformations and mappings in* $\mathbf{R}^n$, J. Analyse Math. **30** (1976), 74–97.
7. *A singular integral equation connected with quasiconformal mappings in space*, Enseign. Math. (2) **24** (1978), 225–236.
8. *A singular operator in hyperbolic space*, Complex Analysis and Its Applications (I. N. Vekua Seventieth Birthday Vol.), "Nauka", Moscow, 1978, pp. 40–44.
9. *The Hölder continuity of quasiconformal deformations*, Amer. J. Math. **101** (1979), 1–9.

LARS AHLFORS AND ARNE BEURLING

10. *Conformal invariants and function-theoretic null-sets*, Acta Math. **83** (1950), 101–129.

A. D. ALEKSANDROV

11. *The existence almost everywhere of the second differential of a convex function and some properties of convex surfaces connected with it*, Leningrad Gos. Univ. Uchen. Zap. Ser. Mat. **6** (1939), 3–35. (Russian)

P. P. BELINSKIĬ

12. *On the continuity of spatial quasiconformal mappings and Liouville's theorem*, Dokl. Akad. Nauk SSSR **147** (1962), 1003–1004; English transl. in Soviet Math. Dokl. **3** (1962).
13. *Stability in the Liouville theorem on spatial quasiconformal mappings*, Some Problems in Mathematics and Mechanics (M. A. Lavrent′ev Seventieth Birthday Vol.), "Nauka", Leningrad, 1970, pp. 88–102; English transl. in Amer. Math. Soc. Transl. (2) **104** (1976).
14. *On the order of proximity of a spatial quasiconformal mapping to a conformal mapping*, Sibirsk. Mat. Zh. **14** (1973), 475–483; English transl. in Siberian Math. J. **14** (1973).

P. P. BELINSKIĬ AND M. A. LAVRENTIEV [LAVRENT′EV]

15. *On locally quasiconformal mappings in n-space* ($n = 3$), Contribution to Analysis (Collection Dedicated to Lipman Bers), Academic Press, 1974, pp. 27–30.

O. V. BESOV

16. *Integral representations of functions in a domain with the flexible horn condition, and imbedding theorems*, Dokl. Akad. Nauk SSSR **273** (1983), 1294–1297; English transl. in Soviet Math. Dokl. **28** (1983).
17. *Estimates of the L_p-moduli of continuity and imbedding theorems for domains with the flexible horn condition*, Dokl. Akad. Nauk SSSR **275** (1984), 1036–1041; English transl. in Soviet Math. Dokl. **29** (1984).

O. V. BESOV, V. P. IL′IN, AND S. M. NIKOL′SKIĬ

18. *Integral representations of functions and imbedding theorems*, "Nauka", Moscow, 1975; English transl., Vols. I, II, Wiley, 1979.

B. V. BOYARSKIĬ [BOGDAN BOJARSKI]

19. *Homeomorphic solutions of Beltrami systems*, Dokl. Akad. Nauk SSSR. **102** (1955), 661–664. (Russian)
20. *Generalized solutions of a system of first-order differential equations of elliptic type with discontinuous coefficients*, Mat. Sb. **43** (85) (1957), 451–503. (Russian)

EUGENIO CALABI AND PHILIP HARTMAN

21. *On the smoothness of isometries*, Duke Math. J. **37** (1970), 741–750.

ALBERTO P. CALDERÓN

22. *On the differentiability of absolutely continuous functions*, Riv. Mat. Univ. Parma **1951**, 203–213.

A. P. CALDERÓN AND A. ZYGMUND

23. *Singular operators and differential equations*, Amer. J. Math. **79** (1957), 901–921.

E. DAVID CALLENDER

24. *Hölder continuity of n-dimensional quasi-conformal mappings*, Pacific J. Math. **10** (1960), 499–515.

PETRU CARAMAN

25. *n-dimensional quasiconformal mappings*, Ed. Acad. Rep. Soc. România, Bucharest, 1968; rev. English transl., Ed. Acad. Române, Bucharest, Abacus Press, Tunbridge Wells, and Haessner, Newfoundland, N. J., 1974.

A. V. CHERNAVSKIĬ

26. *Finite-to-one open mappings of manifolds*, Mat. Sb. **65 (107)** (1964), 357–369; English transl. in Amer. Math. Soc. Transl. (2) **100** (1972).
27. Addendum to [26], Mat. Sb. **66** (**108**) (1964), 471–472; English transl. in Amer. Math. Soc. Transl. (2) **100** (1972).

N. S. DAIRBEKOV

28. *On the stability of classes of conformal mappings on the plane and in space*, Sibirsk. Mat. Zh. **27** (1986), no. 5, 188–191. (Russian)

N. S. DAIRBEKOV AND A. P. KOPYLOV

29. *On stability in the C-norm of classes of solutions of systems of linear partial differential equations*, Ninth School Theory of Operators in Function Spaces (September 1984), Abstracts of Reports, Ternopol′, 1984, p. 137. (Russian)
30. *ξ-stability of classes of mappings, and systems of linear partial differential equations*, Sibirsk. Mat. Zh. **26** (1985), no. 2, 73–90; English transl. in Siberian Math. J. **26** (1985).

ENNIO DE GIORGI

31. *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43.

K. O. FRIEDRICHS

32. *On the boundary-value problems of the theory of elasticity and Korn's inequality*, Ann. of Math. (2) **48** (1947), 441–471.

F. W. GEHRING

33. *The definitions and exceptional sets for quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 281 (1960).

34. *Rings and quasiconformal mappings in space*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 98–105.

35. *Symmetrization of rings in space*, Trans. Amer. Math. Soc. **101** (1961), 499–519.

36. *A remark on the moduli of rings*, Comment. Math. Helv. **36** (1961), 42–46.

37. *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1963), 353–393.

38. *The L^p-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. **130** (1973), 265–277.

39. *Injectivity of local quasi-isometries*, Comment. Math. Helv. **57** (1982), 202–220.

40. *Characteristic properties of quasidisks*, Sém. Math. Sup., vol. 84, Presses Univ. Montréal, Montréal, 1982.

F. W. GEHRING AND F. HUCKEMANN

41. *Quasiconformal mappings of a cylinder*, Proc. Romanian-Finnish Sem. Teichmüller Spaces and Quasiconformal Mappings (Braşov, 1969), Publ. House Acad. Socialist Republic Romania, Bucharest, 1971, pp. 171–186.

F. W. GEHRING AND J. VÄISÄLÄ

42. *The coefficients of quasiconformality of domains in space*, Acta Math. **114** (1965), 1–70.

V. M. GOL′DSHTEĬN

43. *A homotopy property of mappings with bounded distortion*, Sibirsk. Mat. Zh. **11** (1970), 999–1008; English transl. in Siberian Math. J. **11** (1970).

44. *On the behavior of mappings with bounded distortion when the distortion coefficient is close to* 1, Sibirsk. Mat. Zh. **12** (1971), 1250–1258; English transl. in Siberian Math. J. **12** (1971).

45. *Extension of functions in the classes $B^l_{p,q}$ across quasiconformal boundaries*, Theory of Cubature Formulas and Applications of Functional Analysis to Problems of Mathematical Physics, (Proc. Sem. S. L. Sobolev, 1979, vol. 1), Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1979, pp. 12–32.

VLADIMIR MIKAILOVITCH GOL′DŠTEĬN [V. M. GOL′DSHTEĬN] AND SERGEI KONSTANTINOVITCH VODOP′JANOV [S. K. VODOP′YANOV]

46. *Prolongement des fonction de classe $\mathscr{L}^1_p$ et applications quasi conforme*, C. R. Acad. Sci. Paris Sér. A–B **290** (1980), A453–A456.

V. M. GOL'DSHTEĬN

47. *Extension of functions with first generalized derivatives from plane domains*, Dokl. Akad. Nauk SSSR **257** (1981), 268–271; English transl. in Soviet Math. Dokl. **23** (1981).

V. M. GOL'DSHTEĬN AND YU. G. RESHETNYAK

48. *Introduction to the theory of functions with generalized derivatives and quasiconformal mappings*, "Nauka", Novosibirsk, 1983; English transl., *Foundations of the theory of functions with generalized derivatives and space quasiconformal mappings*, Reidel (to appear).

V. M. GOL'DSHTEĬN, V. I. KUZ'MINOV, AND I. A. SHVEDOV

49. *Classes of differential forms analogous to classes of Sobolev functions*, Preprint, Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1981. (Russian) R. Zh. Mat. **1982**, 2A655.

L. G. GUROV

50. *On the stability of Lorentz mappings*, Dokl. Akad. Nauk SSSR **213** (1973), 267–269; English transl. in Soviet Math. Dokl. **14** (1973).
51. *Stability estimates for Lorentz mappings*, Sibirsk. Mat. Zh. **15** (1974), 498–515; English transl. in Siberian Math. J. **15** (1974).
52. *On the stability of Lorentz transformations. Estimates for the derivatives*, Dokl. Akad. Nauk SSSR **220** (1975), 273–276; English transl. in Soviet Math. Dokl. **16** (1975).
53. *On the stability of Lorentz transformations in space*, Sibirsk. Mat. Zh. **21** (1980), no. 2, 51–60; English transl. in Siberian Math. J. **21** (1980).

L. G. GUROV AND YU. G. RESHETNYAK

54. *On an analogue of the concept of a function with bounded mean oscillation*, Sibirsk. Mat. Zh. **17** (1976), 540–546; English transl. in Siberian Math. J. **17** (1976).

SZE-TSEN HU

55. *Homotopy theory*, Academic Press, 1959.

V. P. IL'IN

56. *On an imbedding theorem for a limiting exponent*, Dokl. Akad. Nauk SSSR **96** (1954), 905–908. (Russian)

V. V. IVANOV

57. *Smooth isomorphisms not having quasiconformal fractional powers*, Sibirsk. Mat. Zh. **27** (1986), no. 3, 103–111; English transl. in Siberian Math. J. **27** (1986).

TADEUSZ IWANIEC

58. *On L^p-integrability in PDEs and quasiregular mappings for large exponents*, Ann. Acad. Sci. Fenn. Ser. A I Math. **7** (1982), 301–322.
59. *Regularity theorems for solutions of partial differential equations for quasiconformal mappings in several dimensions*, Dissertation, Polish Acad. of Sci., Warsaw, 1982.

FRITZ JOHN

60. *Rotation and strain*, Comm. Pure Appl. Math. **14** (1961), 391–413.
61. *On quasi-isometric mappings*. I, Comm. Pure Appl. Math. **21** (1968), 77–110.
62. *On quasi-isometric mappings*. II, Comm. Pure Appl. Math. **22** (1969), 265–278.

F. JOHN AND L. NIRENBERG

63. *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

PETER W. JONES

64. *Quasiconformal mappings and extendibility of functions in Sobolev spaces*, Acta Math. **147** (1981), 71–88.

A. P. KOPYLOV

65. *On the behavior of quasiconformal space mappings close to conformal mappings on hyperplanes*, Dokl. Akad. Nauk SSSR **209** (1973), 1278–1280; English transl. in Soviet Math. Dokl. **14** (1973).
66. *On approximation of quasiconformal space mappings close to conformal mappings by smooth quasiconformal mappings*, Sibirsk. Mat. Zh. **13** (1972), 94–106; English transl. in Siberian Math. J. **13** (1972).
67. *Integral averagings and quasiconformal mappings*, Dokl. Akad. Nauk SSSR **231** (1976), 289–291; English transl. in Soviet Math. Dokl. **17** (1976).
68. *On the removability of a ball for space mappings close to conformal mappings*, Dokl. Akad. Nauk SSSR **234** (1977), 525–527; English transl. in Soviet Math. Dokl. **18** (1977).
69. *Stability of classes of multidimensional holomorphic mappings*. I: *The concept of stability. Liouville's theorem*, Sibirsk. Mat. Zh. **23** (1982), no. 2, 83–111; English transl. in Siberian Math. J. **23** (1982).
70. *Stability of classes of multidimensional holomorphic mappings*. II: *Stability of classes of holomorphic mappings*, Sibirsk. Mat. Zh. **23** (1982), no. 4, 65–89; English transl. in Siberian Math. J. **23** (1982).

71. *Stability of classes of multidimensional holomorphic mappings. III: Properties of mappings close to holomorphic mappings*, Sibirsk. Mat. Zh. **24** (1983), no. 3, 70–91; English transl. in Siberian Math. J. **24** (1983).

72. *On boundary values of mappings of the half-space that are close to being conformal*, Sibirsk. Mat. Zh. **24** (1983), no. 5, 76–93; English transl. in Siberian Math. J. **24** (1983).

M. Kreĭnes

73. *Sur une classe de fonctions de plusieurs variables*, Mat. Sb. **9** (**51**) (1941), 713–720.

S. L. Krushkal′

74. *On the absolute continuity and differentiability of certain classes of mappings of multidimensional domains*, Sibirsk. Mat. Zh. **6** (1965), 692–696. (Russian)

S. L. Krushkal′, B. N. Apanasov, and N. A. Gusevskiĭ

75. *Kleinian groups and uniformization in examples and problems*, "Nauka", Novosibirsk, 1981; English transl., Amer. Math. Soc., Providence, R. I., 1986.

K. Kuratowski

76. *Topology*. Vol. 2, 4th ed., PWN, Warsaw, and Academic Press, New York, 1968.

O. A. Ladyzhenskaya and N. N. Ural′tseva

77. *Linear and quasilinear elliptic equations*, 2nd ed., "Nauka", Moscow, 1973; English transl. of 1st ed., Academic Press, 1968.

O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural′tseva

78. *Linear and quasilinear equations of parabolic type*, "Nauka", Moscow, 1967; English transl., Amer. Math. Soc., Providence, R. I., 1968.

M. Lavrentieff [M. A. Lavrent′ev]

79. *Sur une classe de représentations continues*, Mat. Sb. **42** (1935), 407–424.

80. *Sur un critère différentiel des transformations homéomorphes des domaines à trois dimensions*, C. R. (Dokl.) Acad. Sci. URSS **20** (1938), 241–242.

81. *Stability in Liouville's theorem*, Dokl. Akad. Nauk SSSR **95** (1954), 925–926. (Russian)

82. *On the theory of quasi-conformal mappings of three-dimensional domains*, J. Analyse Math. **19** (1967), 217–225.

M. A. Lavrent′ev and P. P. Belinskiĭ

83. *Some problems of geometric function theory*, Trudy Mat. Inst. Steklov. **128** (1972), 34–40; English transl. in Proc. Steklov Inst. Mat. **128** (1972).

Olli Lehto

84. *Remarks on the integrability of the derivatives of quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 371 (1965).
85. *Univalent functions and Teichmüller spaces*, Springer-Verlag, 1987.

Jacqueline Lelong-Ferrand

86. *Interprétation fonctionnelle des homéomorphismes quasi-conformes*, C. R. Acad. Sci. Paris Sér. A–B **272** (1971), A1390–A1392.
87. *Geometrical interpretations of scalar curvature and regularity of conformal homeomorphisms*, Differential Geometry and Relativity (A. Lichnerowicz Sixtieth Birthday Vol.), Reidel, 1976, pp. 91–105.

Lawrence G. Lewis

88. *Quasiconformal mappings and Royden algebras in space*, Trans. Amer. Math. Soc. **158** (1971), 481–492.

Charles Loewner

89. *On the conformal capacity in space*, J. Math. and Mech. **8** (1959), 411–414.

A. Markouchevitch [A. I. Markushevich]

90. *Sur certaines classes de transformations continues*, C. R. (Dokl.) Acad. Sci. URSS **28** (1940), 301–304.

O. Martio

91. *A capacity inequality for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 474 (1970).
92. *Definitions for uniform domains*, Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980), 197–205.

O. Martio and J. Sarvas

93. *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1979), 383–401.

O. Martio, S. Rickman, and J. Väisälä

94. *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 448 (1969).

95. *Distortion and singularities for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 465 (1970).

96. *Topological and metric properties of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 488 (1971).

D. MENCHOFF [D. E. MEN'SHOV]

97. *Sur une généralisation d'un théorème de M. H. Bohr*, Mat. Sb. **2** (**44**) (1937), 339–356.

NORMAN G. MEYERS AND ALAN ELCRAT

98. *Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions*, Duke Math. J. **42** (1975), 121–136.

S. G. MIKHLIN

99. *Multidimensional singular integrals and integral equations*, Fizmatgiz, Moscow, 1962; English transl., Pergamon Press, 1965.

V. M. MIKLYUKOV

100. *On removable singularities of quasiconformal mappings in space*, Dokl. Akad. Nauk SSSR **188** (1969), 525–527; English transl. in Soviet Math. Dokl. **10** (1969).

101. *On the asymptotic properties of subsolutions of quasilinear elliptic equations and mappings with bounded distortion*, Mat. Sb. **111 (153)** (1980), 42–66; English transl. in Math. USSR Sb. **39** (1981).

CHARLES B. MORREY, JR.

102. *Multiple integral problems in the calculus of variations and related topics*, Univ. California Publ. Math. (N.S.) **1** (1943), 1–130.

JÜRGEN MOSER

103. *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **13** (1960), 457–468.

104. *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591.

P. P. MOSOLOV AND V. P. MYASNIKOV

105. *A proof of Korn's inequality*, Dokl. Akad. Nauk SSSR **201** (1971), 36–39; English transl. in Soviet Math. Dokl. **12** (1971).

G. D. MOSTOW

106. *Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms*, Inst. Hautes Études Sci. Publ. Math. No. 34 (1968), 53–104.

MITSURU NAKAI

107. *Algebraic criterion on quasiconformal equivalence of Riemann surfaces*, Nagoya Math. J. **16** (1960), 157–184.

JINDŘICH NEČAS

108. *Sur les normes équivalentes dans $W_p^{(k)}(\Omega)$ et sur la coercitivité des formes formellement positives*, Équations aux Dérivées Partielles (1965), Sém. Math. Sup., vol. 19, Presses Univ. Montréal, Montréal, 1966, pp. 102–128.

ROLF NEVANLINNA

109. *Eindeutige analytische Funktionen*, Springer-Verlag, 1936.

S. M. NIKOL'SKIĬ

110. *Properties of certain classes of functions of several variables on differentiable manifolds*, Mat. Sb. **33 (75)** (1953), 261–326; English transl. in Amer. Math. Soc. Transl. (2) **80** (1969).

LOUIS NIRENBERG

111. *On nonlinear elliptic partial differential equations and Hölder continuity*, Comm. Pure Appl. **6** (1953), 103–156, 395.

I. S. OVCHINNIKOV AND G. D. SUVOROV

112. *Transformations of the Dirichlet integral and space mappings*, Sibirsk. Mat. Zh. **6** (1965), 1292–1314. (Russian)

E. A. POLETSKIĬ

113. *The modulus method for nonhomeomorphic quasiconformal mappings*, Mat. Sb. **83 (125)** (1970), 261–272; English transl. in Math. USSR Sb. **12** (1970).

114. *On the removal of singularities of quasiconformal mappings*, Mat. Sb. **92 (134)** (1973), 242–256; English transl. in Math. USSR Sb. **21** (1973).

G. PÓLYA AND G. SZEGÖ

115. *Isoperimetric inequalities in mathematical physics*, Princeton Univ. Press, Princeton, N. J., 1951.

T. RADO AND P. V. REICHELDERFER

116. *Continuous transformations in analysis. With an introduction to algebraic topology*, Springer-Verlag, 1955.

P. K. RASHEVSKIĬ

117. *Riemannian geometry and tensor analysis*, 3rd ed., "Nauka", Moscow, 1967; German transl. of 1st ed., VEB Deutscher Verlag Wiss., Berlin, 1959.

H. M. REIMANN

118. *Ordinary differential equations and quasiconformal mappings*, Invent. Math. **33** (1976), 247–270.

YU. G. RESHETNYAK

119. *On a sufficient condition for Hölder continuity of a mapping*, Dokl. Akad. Nauk SSSR **130** (1960), 507–509; English transl. in Soviet Math. Dokl. **1** (1960).
120. *On conformal mappings of space*, Dokl. Akad. Nauk SSSR **130** (1960), 1196–1198; English transl. in Soviet Math. Dokl. **1** (1960).
121. *Stability in the Liouville theorem on conformal mappings*, Some Problems in Mathematics and Mechanics (M. A. Lavrent'ev, Sixtieth Birthday Vol.), Izdat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1961, pp. 219–223. (Russian)
122. *On stability in Liouville's theorem on conformal mappings of space*, Dokl. Akad. Nauk SSSR **152** (1963), 286–287; English transl. in Soviet Math. Dokl. **4** (1963).
123. *Some geometric properties of functions and mappings with generalized derivatives*, Sibirsk. Mat. Zh. **7** (1966), 886–919; English transl. in Siberian Math. J. **7** (1966).
124. *Estimates of the modulus of continuity for certain mappings*, Sibirsk. Mat. Zh. **7** (1966), 1106–1114; English transl. in Siberian Math. J. **7** (1966).
125. *Space mappings with bounded distortion*, Sibirsk. Mat. Zh. **8** (1967), 629–658; English transl. in Siberian Math. J. **8** (1967).
126. *Liouville's conformal mapping theorem under minimal regularity assumptions*, Sibirsk. Mat. Zh. **8** (1967), 835–840; English transl. in Siberian Math. J. **8** (1967).
127. *General theorems on semicontinuity and convergence with functionals*, Sibirsk. Mat. Zh. **8** (1967), 1051–1069; English transl. in Siberian Math. J. **8** (1967).
128. *On the set of singular points of solutions of certain nonlinear elliptic equations*, Sibirsk. Mat. Zh. **9** (1968), 354–367; English transl. in Siberian Math. J. **9** (1968).
129. *On a condition for boundedness of the index for mappings with bounded distortion*, Sibirsk. Mat. Zh. **9** (1968), 368–374; English transl. in Siberian Math. J. **9** (1968).
130. *Mappings with bounded distortion as extremals of integrals of Dirichlet type*, Sibirsk. Mat. Zh. **9** (1968), 652–666; English transl. in Siberian Math. J. **9** (1968).

131. *Generalized derivatives and differentiability almost everywhere*, Mat. Sb. **75 (117)** (1968), 323–334; English transl. in Math. USSR Sb. **4** (1968).
132. *On the concept of capacity in the theory of functions with generalized derivatives*, Sibirsk. Mat. Zh. **10** (1969), 1109–1138; English transl. in Siberian Math. J. **10** (1969).
133. *On extremal properties of mappings with bounded distortion*, Sibirsk. Mat. Zh. **10** (1969), 1300–1310; English transl. in Siberian Math. J. **10** (1969).
134. *The local structure of mappings with bounded distortion*, Sibirsk. Mat. Zh. **10** (1969), 1311–1333; English transl. in Siberian Math. J. **10** (1969).
135. *Estimates for certain differential operators with finite-dimensional kernels*, Sibirsk. Mat. Zh. **11** (1970), 414–428; English transl. in Siberian Math. J. **11** (1970).
136. *On a stability estimate in Liouville's theorem on conformal mappings of multidimensional spaces*, Sibirsk. Mat. Zh. **11** (1970), 1112–1139; English transl. in Siberian Math. J. **11** (1970).
137. *On the set of branch points of mappings with bounded distortion*, Sibirsk. Mat. Zh. **11** (1970), 1333–1339; English transl. in Siberian Math. J. **11** (1970).
138. *Certain integral representations of differentiable functions*, Sibirsk. Mat. Zh. **12** (1971), 420–432; English transl. in Siberian Math. J. **12** (1971).
139. *Lectures on mathematical analysis. Mathematical analysis on manifolds*, Novosibirsk. Gos. Univ., Novosibirsk, 1972. (Russian)
140. *Stability in Liouville's theorem on conformal mappings of a space for domains with nonsmooth boundary*, Sibirsk. Mat. Zh. **17** (1976), 361–369; English transl. in Siberian Math. J. **17** (1976).
141. *Stability estimates in Liouville's theorem and L_p-integrability of the derivatives of quasiconformal mappings*, Sibirsk. Mat. Zh. **17** (1976), 868–896; English transl. in Siberian Math. J. **17** (1976).
142. *Stability estimates in the class W_p^1 for Liouville's theorem on conformal mappings for a closed domain*, Sibirsk. Mat. Zh. **17** (1976), 1382–1394; English transl. in Siberian Math. J. **17** (1976).
143. *Differentiability properties of quasiconformal mappings and conformal mappings of Riemannian spaces*, Sibirsk. Mat. Zh. **19** (1978), 1166–1183; English transl. in Siberian Math. J. **19** (1978).
144. *Stability theorems in geometry and analysis*, "Nauka", Novosibirsk, 1982. (Russian)

Yu. G. Reshetnyak and B. V. Shabat

145. *On quasiconformal mappings in space*, Proc. Fourth All-Union Math. Congr. (Leningrad, 1961), Vol. II, "Nauka", Leningrad, 1964, pp. 672–680. (Russian)

Georges de Rham

146. *Variétés différentiables. Formes, courants, formes harmoniques*, Actualités Sci. Indust., no. 1222, Hermann, Paris, 1955.

Seppo Rickman

147. *On the number of omitted values of entire quasiregular mappings*, J. Analyse Math. **37** (1980), 100–117.

Frédéric Riesz and Béla Sz. Nagy

148. *Leçons d'analyse fonctionnelle*, 2nd ed., Akad. Kiadó, Budapest, 1953; English transl., Ungar, New York, 1955.

Stanisław Saks

149. *Theory of the integral*, 2nd rev. ed., PWN, Warsaw, 1937; reprint, Dover, New York, 1964.

V. I. Semenov

150. *On one-parameter groups of quasiconformal homeomorphisms in a Euclidean space*, Sibirsk. Mat. Zh. **17** (1976), 177–193; English transl. in Siberian Math. J. **17** (1976).
151. *Semigroups of quasi-Lorentz mappings*, Sibirsk. Mat. Zh. **18** (1977), 713; English transl. in Siberian Math. J. **18** (1977).
152. *Uniform estimates of stable isometries*, Sibirsk. Mat. Zh. **27** (1986), no. 3, 193–199; English transl. in Siberian Math. J. **27** (1986).
153. *On uniform stability estimates for quasiconformal and quasi-isometric mappings in space*, Dokl. Akad. Nauk SSSR **286** (1986), 295–297; English transl. in Soviet Math. Dokl. **33** (1986).
154. *Stability estimates for quasiconformal space mappings of a star-shaped domain*, Sibirsk. Mat. Zh. **28** (1987), no. 6, 102–118; English transl. in Siberian Math. J. **28** (1987).

James Serrin

155. *A Harnack inequality for nonlinear equations*, Bull. Amer. Math. Soc. **69** (1963), 481–486.
156. *Local behavior of solutions of quasi-linear equations*, Acta. Math. **111** (1964), 247–302.

B. V. Shabat

157. *The modulus method in space*, Dokl. Akad. Nauk SSSR **130** (1960), 1210–1213; English transl. in Soviet Math. Dokl. **1** (1960).

158. *On the theory of quasiconformal mappings in space*, Dokl. Akad. Nauk SSSR **132** (1960), 1045–1048; English transl. in Soviet Math. Dokl. **1** (1960).

S. Z. SHEFEL′

159. *Conformal correspondence of metrics, and the smoothness of isometric immersions*, Sibirsk. Mat. Zh. **20** (1979), 397–401; English transl. in Siberian Math. J. **20** (1979).

160. *Smoothness of a conformal mapping of Riemannian spaces*, Sibirsk. Mat. Zh. **23** (1982), no. 1, 153–159; English transl. in Siberian Math. J. **23** (1982).

V. I. SMIRNOV

161. *A course in higher mathematics*, Vol. 5, rev. ed., Fizmatgiz, Moscow, 1959; English transl., Pergamon Press, Oxford, and Addison-Wesley, Reading, Mass., 1964.

S. L. SOBOLEV

162. *Applications of functional analysis in mathematical physics*, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; English transl., Amer. Math. Soc., Providence, R. I., 1963.

W. STEPANOFF [V. V. STEPANOV]

163. *Sur les conditions de l'existence de la différentielle totale*, Mat. Sb. **32** (1924/25), 511–527.

SHLOMO STERNBERG

164. *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

G. D. SUVOROV

165. *Extensions of topological structures and metric properties of mappings*, Contributions to Extension Theory of Topological Structures (Proc. Sympos., Berlin, 1967; J. Flachsmeyer et al., editors), VEB Deutscher Verlag Wiss., Berlin, 1969, pp. 257–273; English transl. in Amer. Math. Soc. Transl. (2) **134** (1987).

A. V. SYCHEV

166. *Moduli and n-dimensional quasiconformal mappings*, "Nauka", Novosibirsk, 1983. (Russian)

HANS TRIEBEL

167. *Interpolation theory, function spaces, differential operators*, VEB Deutscher Verlag Wiss., Berlin, 1977, and North-Holland, Amsterdam, 1978.

D. A. TROTSENKO

168. *Properties of domains with nonsmooth boundary*, Sibirsk. Mat. Zh. **22** (1981), no. 4, 221–224. (Russian)
169. *Extension from a domain and approximation of spatial quasiconformal mappings with small distortion coefficient*, Dokl. Akad. Nauk SSSR **270** (1983), 1331–1333; English transl. in Soviet Math. Dokl. **27** (1983).
170. *Extension of spatial quasiconformal mappings that are close to conformal mappings*, Sibirsk. Mat. Zh. **28** (1987), no. 6, 126–133; English transl. in Siberian Math. J. **28** (1987).

JUSSI VÄISÄLÄ

171. *On quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. Ser. A I No. 298 (1961).
172. *Discrete open mappings on manifolds*, Ann. Acad. Sci. Fenn. Ser. A I No. 392 (1966).
173. *Removable sets for quasiconformal mappings*, J. Math. and Mech. **19** (1969/70), 49–51.
174. *Modulus and capacity inequalities for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 509 (1972).
175. *A survey of quasiregular maps in R^n*, Proc. Internat. Congr. Math. (Helsinki 1978), vol. 2, Acad. Sci. Fenn., Helsinki, 1980, pp. 685–691.

M. YU. VASIL'CHIK

176. *On a lower estimate of the distortion coefficient for infinitely close domains*, Sibirsk. Mat. Zh. **19** (1978), 547–554; English transl. in Siberian Math. J. **19** (1978).

S. K. VODOP'YANOV AND V. M. GOL'DSHTEĬN

177. *Lattice isomorphisms of the spaces W_n^1 and quasiconformal mappings*, Sibirsk. Mat. Zh. **16** (1975), 224–246; English transl. in Siberian Math. J. **16** (1975).
178. *Quasiconformal mappings and spaces of functions with first generalized derivatives*, Sibirsk. Mat. Zh. **17** (1976), 515–531; English transl. in Siberian Math. J. **17** (1976).
179. *Sobolev spaces and special classes of mappings*, Novosibirsk. Gos. Univ., Novosibirsk, 1981. (Russian)

S. K. VODOP'YANOV, V. M. GOL'DSHTEĬN, AND T. G. LATFULLIN

180. *A criterion for extendibility of L_2^1-functions from unbounded plane domains*, Sibirsk. Mat. Zh. **20** (1979) 416–420; English transl. in Siberian Math. J. **20** (1979).

WILLIAM P. ZIEMER

181. *Extremal length and p-capacity*, Michigan Math. J. **16** (1969), 43–51.

V. A. ZORICH

182. *A theorem of M. A. Lavrent'ev on quasiconformal space maps*, Mat. Sb. **74** (**116**) (1967), 417–433; English transl. in Math. USSR Sb. **3** (1967).

A. ZYGMUND

183. *Trigonometric series*, 2nd rev. ed., Vol. I, Cambridge Univ. Press, 1959.

ABCDEFGHIJ - 89

ISBN 978-0-8218-4526-4
9 780821 845264
MMONO/73