Translations of
MATHEMATICAL
MONOGRAPHS

Volume 75

Orthogonal Series

B. S. Kashin
A. A. Saakyan

American Mathematical Society

Orthogonal Series

This page intentionally left blank

10.1090/mmono/075

TRANSLATIONS OF MATHEMATICAL MONOGRAPHS

VOLUME 75

Orthogonal Series

B. S. KASHIN

A. A. SAAKYAN

American Mathematical Society · Providence · Rhode Island

БОРИС СЕРГЕЕВИЧ КАШИН, АРТУР АРТУШОВИЧ СААКЯН

ОРТОГОНАЛЬНЫЕ РЯДЫ

«НАУКА», МОСКВА, 1984

Translated from the Russian by Ralph P. Boas
Translation edited by Ben Silver

2000 *Mathematics Subject Classification*. Primary 42–XX.

ABSTRACT. In this book we present the fundamental methods of the theory of orthogonal series. We study general orthonormal systems as well as specific systems (for example, the Haar and Franklin systems). We present both classical results and those obtained more recently. All propositions that fall outside the scope of university courses are provided with full proofs.
The book is intended for specialists in mathematical analysis, graduate students, and students in advanced courses in related fields.
Bibliography: 202 titles.

Library of Congress Cataloging-in-Publication Data

Kashin, B. S.
[Ortogonal'nye ri͡ady. English]
Orthogonal series/by B. S. Kashin, A. A. Saakyan; [translated from the Russian by Ralph P. Boas; translation edited by Ben Silver].
p. cm. — (Translations of mathematical monographs; v. 75)
Translation of: Ortogonal'nye ri͡ady.
Bibliography: p.
Includes index.
ISBN 0-8218-4527-6 (hard cover)
ISBN 0-8218-3789-3 (soft cover)
1. Series, Orthogonal. I. Saaki͡an, A. A. (Artur Artushovich) II. Title. III. Series.

QA404.5.K3413 1989 89-333
515′.2433—dc19 CIP

10 9 8 7 6 5 4 3 2 1 10 09 08 07 06 05

Contents

This page intentionally left blank

Preface

This book is concerned with the theory of general orthogonal series. The theory originated at the beginning of the present century as a natural generalization, based on Lebesgue integration, of the theory of trigonometric series, but has been developed most actively in the past twenty-five years. By now it has become clear that:

many propositions about properties of the trigonometric system are general in nature and remain valid for a broad class of orthonormal systems:

the study of systems of functions more general than orthonormal systems frequently reduces to the study of the latter:

in many problems, nonclassical orthonormal systems turn out to be "better" than classical systems:

the results and methods of the theory of general orthogonal systems have a variety of applications outside this theory.

These observations, which speak to the importance of the systematic study of the properties of a variety of orthonormal systems, are confirmed to a substantial extent by the contents of this book. We note in this connection that the book does not claim to be a complete treatment of the subject, and does not even touch on some important topics. We have also not attempted to present results in their most general form. Our principal aim is to present the main ideas and methods of the theory of orthogonal series. Our choice of material has been much influenced by Men′shov and Ul′yanov's seminar on the theory of real functions, which has long been active at Moscow University as a "continuous extension" of Luzin's twenty-five year seminar. We have also used the lectures on the theory of orthogonal series that the first author gave at Moscow University during 1979–1981. A significant number of the theorems proved in the book have not appeared in the monographic literature, and we hope that even specialists will find something new here. Nevertheless, we are largely oriented toward beginning mathematicians, and have therefore adhered to the rule of proving all propositions that fall outside the scope of university courses. We assume that the reader is familiar with the contents of Kolmogorov and Fomin's *Elements of the Theory of Functions and Functional Analysis*, and also with the basic theory of functions of a complex variable (for example, with the content of [107]). The essential additional material from the theory of functions and functional analysis is presented in two appendices. Information on the source of the results presented in the book is given in the Notes, which also contain commentaries and the proofs of some additional facts.

In addition, we thank our colleagues K. I. Oskolkov, A. A. Talalyan, K. Tandori, Z. Ciesielski, and P. Oswald for advice and assistance.

B. S. Kashin
A. A. Saakyan

Summary of Notation

We use, without explanation, a number of standard notations (used, in particular, in [89]), for example R^n, $L^p(0,1)$, l^p, and $C(0,1)$. In addition:

$L^0(0,1)$ and $L^0(R^1)$ are the spaces of functions that are measurable and finite almost everywhere on $[0,1]$ or R^1.

$C(-\pi,\pi)$ and $L^p(-\pi,\pi)$ are the spaces of functions of period 2π on R^1, continuous or of summable pth power on $[-\pi,\pi]$, respectively.

$\mathscr{D}_N$, $N = 1,2,\dots$, is the N-dimensional space of piecewise constant functions defined on $[0,1]$:

$$\mathscr{D}_N = \left\{ f(x)\colon f(x) = \text{const} = c_i \quad \text{if } x \in \left(\frac{i-1}{N}, \frac{i}{N}\right),\ i = 1,\dots,N; \right.$$
$$\left. f\left(\frac{i}{N}\right) = \frac{c_i + c_{i+1}}{2}, \quad \text{if } 1 \le i < N,\ f(0) = c_1,\ f(1) = c_N \right\}.$$

Similarly, for any interval $[a,b]$

$$\mathscr{D}_N(a,b) = \{f(x), x \in [a,b]\colon g(t) = f(a + t(b-a)) \in \mathscr{D}_N\}.$$

The norm of the matrix $H = \{h_{ij}\}$, $i = 1,\dots,m$, $j = 1,\dots,n$ is the number

$$\|H\| = \sup_{\{x_i\},\{y_j\}\colon\ \sum_1^m x_i^2 = \sum_1^n y_j^2 = 1} \sum_{i=1}^m \sum_{j=1}^n h_{ij} x_i y_j.$$

$m(E)$ (or $|E|$ if E is an interval (a,b)) is the Lebesgue measure of the set $E \subset R^1$.

$\operatorname{card} E$ is the number of elements of the finite set E.

If x is an element of the Banach space X and y is a bounded linear functional on X, then $\langle x,y\rangle$ or $\langle y,x\rangle$ denotes the functional y at the element x. When X is a Hilbert space, (x,y) is sometimes used instead of $\langle x,y\rangle$.

If $f(x)$ is a real or complex function, $\operatorname{supp} f(x) = \{x\colon f(x) \neq 0\}$.

If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are two sequences of positive numbers, the notation $a_n \asymp b_n$ means that there are constants C_1, $C_2 > 0$ such that $C_1 \leq a_n/b_n \leq C_2$, $n = 1, 2, \ldots$.

We use the following abbreviations:

An O.N.S. is an orthonormal system—recall that a system of functions $\{\varphi_n(x)\}_{n=1}^\infty \subset L^2(a, b)$ is *orthonormal* if

$$\int_a^b \varphi_n(x)\varphi_m(x)\,dx = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases} \quad n, m = 1, 2, \ldots ;$$

A C.O.N.S. is a complete orthonormal system;

a.e.="almost everywhere."

A phrase of the form "see Theorem 5.1" or "see §3.2" is a reference to Theorem 1 of Chapter 5 or to §2 of Chapter 3. The chapter number is omitted for references within a chapter.

Translator's note. Definitions, theorems, corollaries, and propositions are numbered consecutively (independently of each other) in each chapter. Lemmas are numbered consecutively for each theorem.

APPENDIX 1

Some Topics from the Theory of Functions of a Real Variable and Functional Analysis

§1. Equations for integrals. Moduli of continuity

1°. PROPOSITION 1. *Let* $f \in L^1(0,1)$, *and for* $t \in R^1$ *let*

$$\lambda_f(t) = m\{x \in (0,1) : |f(x)| > t\}; \tilde{\lambda}_f(t) = m\{x \in (0,1) : f(x) > t\}.$$

Then

$$\int_0^1 f(x)\,dx = -\int_{-\infty}^{\infty} t\,d\tilde{\lambda}_f(t), \tag{1}$$

and if $f \in L^p(0,1)$, $0 < p < \infty$, *then*

$$\int_0^1 |f(x)|^p\,dx = -\int_0^{\infty} t^p\,d\lambda_f(t) = p\int_0^{\infty} t^{p-1}\lambda_f(t)\,dt. \tag{2}$$

PROOF. Equation (1) follows from the definitions of the Lebesgue and Lebesgue–Stieltjes integrals:

If $\{t_i^{(n)}\}_{i=-\infty}^{\infty}$, $n = 1,2,\dots$, is a sequence of partitions of the real line: $\cdots < t_{-k}^{(n)} < \cdots < t_{-1}^{(n)} < t_0^{(n)} < t_1^{(n)} < \cdots < t_k^{(n)} < \cdots$,

$$\lim_{n\to\infty}\left[\sup_{-\infty<i<\infty}(t_{i+1}^{(n)} - t_i^{(n)})\right] = 0,$$

then the integrals $\int_0^1 f_n(x)\,dx$, where $f_n(x) = t_i^{(n)}$ if $x \in E_i^{(n)} \equiv \{x \in (0,1) : t_i^{(n)} < f(x) \le t_{i+1}^{(n)}\}$, $i = 0, \pm 1, \pm 2, \dots$, tend (as $n \to \infty$) to $\int_0^1 f(x)\,dx$. On the other hand,

$$\begin{aligned}\int_0^1 f_n(x)\,dx &= \sum_{i=-\infty}^{\infty} t_i^{(n)} m(E_i^{(n)}) \\ &= -\sum_{i=-\infty}^{\infty} t_i^{(n)}[\tilde{\lambda}_f(t_{i+1}^{(n)}) - \tilde{\lambda}_f(t_i^{(n)})] \to \int_{-\infty}^{\infty} t\,d\tilde{\lambda}_f(t) \quad \text{as } n \to \infty,\end{aligned}$$

which proves (1).

Now let $f \in L^p(0,1)$, $0 < p < \infty$. By (1), if we use the equation $\tilde{\lambda}_{|f|}(t) = 1$ for $t < 0$, we obtain

$$\int_0^1 |f(x)|^p\,dx = -\int_{-\infty}^{\infty} t\,d\tilde{\lambda}_{|f|^p}(t) = -\int_0^{\infty} t\,d\tilde{\lambda}_{|f|^p}(t). \tag{2'}$$

Since

$$\begin{aligned}\tilde{\lambda}_{|f|^p}(t) &= m\{x \in (0,1)\colon |f(x)|^p > t\}\\ &= m\{x \in (0,1)\colon |f(x)| > t^{1/p}\} = \lambda_f(t^{1/p}),\end{aligned}$$

for $t > 0$, then from (2′) (making the change of variable $t = (t')^p$) we at once obtain the first equation (2). In addition, for every $A > 0$,

$$\begin{aligned}-\int_0^A t^p\,d\lambda_f(t) &= -t^p\lambda_f(f)\Big|_0^A + \int_0^A \lambda_f(t)\,dt^p\\ &= -A^p\lambda_f(A) + p\int_0^A \lambda_f(t)\cdot t^{p-1}\,dt.\end{aligned}$$

To prove the second equation in (2), it is enough to let $A \to \infty$ in the preceding relation and use the inequality

$$A^p\lambda_f(A) \le \int_{\{x\colon |f(x)|>A\}} |f(x)|^p\,dx = o(1) \quad \text{as } A \to \infty.$$

REMARK 1. It is easy to see that the finiteness of the integral

$$\int_0^{\infty} t^{p-1}\lambda_f(t)\,dt$$

is equivalent to having $f(x)$ belong to $L^p(0,1)$.

REMARK 2. If $f(x)$ is defined on R^1 then by equation (2) for $f_k(x) \equiv f(x+k)$, $x \in (0,1)$, $k = 0, \pm 1, \ldots$, if we use the equation

$$\lambda_f(t) = m\{x \in R^1\colon |f(x)| > t\} = \sum_{k=-\infty}^{\infty} \lambda_{f_k}(t)$$

we obtain

$$\begin{aligned}\int_{R^1} |f(x)|^p\,dx &= \sum_{k=-\infty}^{\infty} \int_0^1 |f_k(x)|^p\,dx\\ &= \sum_{k=-\infty}^{\infty} p\int_0^{\infty} t^{p-1}\lambda_{f_k}(t)\,dt\\ &= p\int_0^{\infty} t^{p-1}\lambda_f(t)\,dt, \qquad 0 < p < \infty. \end{aligned} \tag{3}$$

2°. *Moduli of continuity.* Let there be given a function $f \in C(0,1)$ (or $f(x) \in L^p(0,1)$, $1 \le p < \infty$). The *modulus of continuity* and the *integral modulus of continuity* are defined by the equations

$$\omega(\delta, f) = \sup_{\substack{0 \le x < x+h \le 1 \\ 0<h\le\delta}} |f(x+h) - f(x)|, \qquad 0 < \delta < 1,$$

$$\left(\omega_p(\delta, f) = \left\{ \sup_{0<h\le\delta} \int_0^{1-h} |f(x+h) - f(x)|^p \, dx \right\}^{1/p}, \qquad 0 < \delta < 1 \right). \tag{4}$$

The modulus of continuity of second order of a function $f \in C(0,1)$, and the integral modulus of continuity of second order of a function $f \in L^p(0,1)$, $1 \le p < \infty$, are defined by

$$\omega^{(2)}(\delta, f) = \sup_{\substack{0 \le x-h < x+h \le 1 \\ 0<h\le\delta}} |f(x+h) + f(x-h) - 2f(x)|, \tag{5}$$

$0 < \delta \le \frac{1}{2}$, and

$$\omega_p^{(2)}(\delta, f) = \left\{ \sup_{0<h<\delta} \int_h^{1-h} |f(x+h) + f(x-h) - 2f(x)|^p \, dx \right\}^{1/p}. \tag{5$'$}$$

For periodic functions the moduli of continuity are defined somewhat differently:

If $f(x)$ is defined on R^1 and has period $T > 0$, then

a)

$$\omega(\delta, f) = \sup_{\substack{0<h\le\delta \\ x\in[0,T]}} |f(x+h) - f(x)|,$$

$$\omega^{(2)}(\delta, f) = \sup_{\substack{0<h\le\delta \\ x\in[0,T]}} |f(x+h) + f(x-h) - 2f(x)|,$$

$$\delta > 0, \ f \in C(0,T); \tag{6}$$

b)

$$\omega_p(\delta, f) = \left\{ \sup_{0<h\le\delta} \int_0^T |f(x+_h) - f(x)|^p \, dx \right\}^{1/p},$$

$$\omega_p^{(2)}(\delta, f) = \left\{ \sup_{0<h\le\delta} \int_0^T |f(x+h) + f(x-h) - 2f(x)|^p \, dx \right\}^{1/p},$$

$$\delta > 0, \ f \in L^p(0,T). \tag{6$'$}$$

We record the fundamental properties of moduli of continuity:

$$\begin{aligned}
\text{(I)}\quad \lim_{\delta\to 0}\omega(\delta,f) &= \lim_{\delta\to 0}\omega^{(2)}(\delta,f)\\
&= \lim_{\delta\to 0}\omega_p(\delta,f) = \lim_{\delta\to 0}\omega_p^{(2)}(\delta,f) = 0,
\end{aligned}$$

$$\begin{aligned}
\text{(II)}\qquad \omega(2\delta,f) &\le 2\omega(\delta,f),\\
\omega^{(2)}(2\delta,f) &\le 4\omega^{(2)}(\delta,f),\\
\omega_p(2\delta,f) &\le 2\omega_p(\delta,f),\\
\omega_p^{(2)}(2\delta,f) &\le 4\omega_p^{(2)}(\delta,f),\\
\text{(III)}\qquad \omega^{(2)}(\delta,f) &\le 2\omega(\delta,f),\\
\omega_p^{(2)}(\delta,f) &\le 2\omega_p(\delta,f).
\end{aligned} \tag{7}$$

Relations (II) and (III) in (7) follow immediately from the definitions. When $f \in C(0,1)$, equations (I) follow at once from the uniform continuity of f on $[0,1]$. If $f \in L^p(0,1)$, $1 \le p < \infty$, then if we find, for a given $\varepsilon > 0$, a function $g \in C(0,1)$ such that $\|g - f\|_p \le \varepsilon/3$, we will have, for sufficiently small $h > 0$,

$$\begin{aligned}
\|f(x-h) - f(x)\|_{L^p(0,1-h)} \le \|f(x+h) - g(x+h)\|_{L^p(0,1-h)}&\\
+ \|g(x+h) - g(x)\|_{L^p(0,1-h)}&\\
+ \|g(x) - f(x)\|_{L^p(0,1-h)} \le \varepsilon,&
\end{aligned}$$

which proves that $\lim_{\delta\to 0}\omega_p(\delta,f) = 0$.

PROPOSITION 2. *Let $f(x)$, $x \in R^1$, have period T and bounded variation on $[0,T]$. Then*

$$\omega_1(\delta,f) \le 4\delta V(f), \qquad 0 < \delta < T,$$

where $V(f) \equiv V_0^T(f)$ is the total variation of $f(x)$ on $[0,T]$.

PROOF. Let $f_1(x) = V_0^x(f)$, $f_2(x) = f_1(x) - f(x)$, $x \in [0,T]$. Since $f_1(x)$ and $f_2(x)$ are nondecreasing functions, we obtain for $h \in (0,T)$, $i = 1,2$,

$$\begin{aligned}
\int_0^{T-h} |f_i(x+h) - f_i(x)|\,dx &= \int_0^{T-h} f_i(x+h)\,dx - \int_0^{T-h} f_i(x)\,dx\\
&= \int_h^T f_i(x)\,dx - \int_0^{T-h} f_i(x)\,dx\\
&= \int_{T-h}^T f_i(x)\,dx - \int_0^h f_i(x)\,dx\\
&\le [f_i(T) - f_i(0)]h.
\end{aligned}$$

Consequently for $h \in (0, T)$

$$\begin{aligned}\int_0^T |f(x+h) - f(x)|\,dx &= \int_0^{T-H} + \int_{T-h}^T \\ &\le [f_1(T) - f_1(0) + f_2(T) - f_2(0)] \cdot h \\ &\quad + h \sup_{x,x' \in [0,T]} |f(x) - f(x')| \\ &\le [2\{f_1(T) - f_1(0)\} + f(0) - f(T)]h + h \cdot V(f) \\ &\le 4hV(f),\end{aligned}$$

as was to be proved.

If $\omega(\delta)$ is convex upward and $\omega(0) = 0$, we set

$$H_\omega = \{f \in C(0,1)\colon \omega(\delta, f) = O(\omega(\delta)) \text{ as } \delta \to 0\}. \tag{8}$$

The class H_{δ^α}, $0 < \alpha \le 1$ (Lipschitz class of order α), is denoted by $\operatorname{Lip}\alpha$ in the text.

3°. In studying properties of the partial sums of Fourier series in orthonormal systems, there arise the operators

$$S(f) = S(f, x) = \int_0^1 f(t) K(x,t)\,dt.$$

It is clear that if the kernel $K(x,t) \in C([0,1] \times [0,1])$, then $S(f)$ acts from $C(0,1)$ to $C(0,1)$. Moreover, it is easy to see that

$$\|S\|_{C \to C} \equiv \sup_{\|f\|_{C(0,1)} \le 1} \|S(f)\|_{C(0,1)}\|_{C(0,1)} = \max_{x \in [0,1]} \int_0^1 |K(x,t)|\,dt. \tag{9}$$

It is also clear that when $K(x,t) \in L^\infty([0,1] \times [0,1])$, the operator $S(f)$ acts from $L^1(0,1)$ to $L^\infty(0,1)$. Let us verify that in this case

$$\|S\|_{L^1 \to L^1} \equiv \sup_{\|f\|_1 \le 1} \|S(f)\|_1 = \left\| \int_0^1 |K(x,t)|\,dx \right\|_\infty . \tag{10}$$

Set $I(t) = \int_0^t |K(x,t)|\,dx$, where $t \in [0,1]$. If $\|f\|_1 \le 1$, then

$$\|S(f)\|_1 \le \int_0^1 |f(t)| \int_0^1 |K(x,t)|\,dx\,dt \le \|I(t)\|_\infty,$$

i.e.,

$$\|S\|_{L^1 \to L^1} \le \left\| \int_0^1 |K(x,t)|\,dx \right\|_\infty . \tag{11}$$

To prove an inequality in the opposite direction, we consider the set E of the points $y \in (0,1)$ for which

$$\lim_{n\to\infty} 2n \int_{y-1/n}^{y+1/n} K(x,t)\,dt = K(x,y) \quad \text{for almost all } x \in (0,1). \tag{12}$$

By Lebesgue's theorem on the differentiation of absolutely continuous functions and Fubini's theorem,

$$m(E) = 1. \tag{13}$$

For a given $y \in E$ we set

$$f_n(t) = \begin{cases} 2n & \text{if } t \in (y-1/n, y+1/n), \\ 0 & \text{if } t \in (0,1)\setminus(y-1/n, y+1/n), \end{cases} \qquad n = 1,2,\dots.$$

Then $\|f_n(t)\|_1 = 1$, $n = 1,2,\dots$, and by applying Fatou's theorem we find that for every $y \in E$

$$\begin{aligned} I(y) = \int_0^1 |K(x,y)|\,dy &\le \overline{\lim_{n\to\infty}} \int_0^1 \left| 2n \int_{y-1/n}^{y+1/n} K(x,t)\,dt \right| dx \\ &= \overline{\lim_{n\to\infty}} \int_0^1 \left| \int_0^1 f_n(t) K(x,t)\,dt \right| dx = \overline{\lim_{n\to\infty}} \|S(f_n)\|_1 \le \|S\|_{L^1\to L^1}. \end{aligned}$$

Therefore (see (13)) $\|I(y)\|_\infty < \|S\|_{L^1\to L^1}$ and (also see (11)) we obtain inequality (10), as required.

§2. The maximal function and interpolation theorems

1°. *Marcinkiewicz's interpolation theorem.*

DEFINITION 1. An operator T from $L^p(R^1)$ $(1 \le p < \infty)$ to $L^0(R^1)$ is an *operator of weak type* (p,p) if

$$m\{x \in R^1 : |T(f,x)| > y\} \le \frac{M}{y^p} \|f\|^p_{L^p(R^1)}, \qquad f \in L^p(R^1), \tag{14}$$

for every $y > 0$, and an *operator of type* (p,p) if

$$\|T(f)\|_{L^p(R^1)} \le M \|f\|_{L^p(R^1)}, \qquad f \in L^p(R^1) \tag{15}$$

(the constants M in (14) and (15) are independent of $f(x)$). If $p = \infty$, we say that T is of *weak type* (p,p) if it is of type (p,p).

It follows immediately from Chebyshev's inequality that an operator of type (p,p) is also of weak type (p,p). We have the following theorem.

THEOREM 1. *If a linear operator T has weak type (p_1, p_1) and also weak type (p_2, p_2) $(1 \leq p_1 \leq p_2 \leq \infty)$, then t has type (p, p) for every p on the interval (p_1, p_2).*

REMARK. We assume that T can be extended to a linear operator on the space

$$L^{p_1}(R^1) \oplus L^{p_2}(R^1) = \{f(x)\colon f(x) = f_1(x) + f_2(x), \\ f_1(x) \in L^{p_1}(R^1), f_2(x) \in L^{p_2}(R^1)\}, \tag{16}$$

i.e., (see (16)) $T(f, x) = T(f_1, x) + T(f_2, x)$ for every function $f \in L^{p_1}(R^1) \oplus L^{p_2}(R^1)$. It is easy to see that this definition of $T(f, x)$ is independent of the choice of $f_1(x)$ and $f_2(x)$ in (16). Since $L^p(R_1) \subset L^{p_1}(R^1) \oplus L^{p_2}(R^1)$, $p_1 < p < p_2$, it follows that T is also defined on $L^p(R^1)$.

PROOF. We carry out the proof of Theorem 1 on the assumption that $p_2 < \infty$ (it is only this case that we actually use in the text). If $p_2 = \infty$ the proof requires a few changes (in fact, the necessary discussion is given below in the proof of Theorem 2).

We consider a function $f \in L^p(0, 1)$ and a number $t \geq 0$, and estimate the function

$$\lambda(t) = m\{x \in R^1\colon |T(f, x)| > t\}.$$

For $x \in R^1$, set

$$h(x) = \begin{cases} f(x) & \text{if } |f(x)| > t, \\ 0 & \text{if } |f(x)| \leq t, \end{cases} \qquad g(x) = f(x) - h(x),$$

Then $h \in L^{p_1}(R^1)$, $g \in L^{p_2}(R^1)$, and $T(f, x) = T(h, x) + T(g, x)$. Consequently

$$\lambda(t) \leq m\{x \in R^1\colon |T(h, x)| > t/2\} + m\{x \in R^1\colon |T(g, x)| > t/2\}.$$

Hence, using the inequalities for weak types (p_1, p_1) and (p_2, p_2) (see (14)), we find that when $t > 0$

$$\begin{aligned} \lambda(t) &\leq \frac{2^{p_1} M_1}{t^{p_1}} \int_{R^1} |h(x)|^{p_1}\, dx + \frac{2^{p_2} M_2}{t^{p_2}} \int_{R^1} |g(x)|^{p_2}\, dx \\ &= \frac{2^{p_1} M_1}{t^{p_1}} \int_{\{x \in R^1\colon\, |f(x)| > t\}} |f(x)|^{p_1}\, dx \\ &\quad + \frac{2^{p_2} M_2}{t^{p_2}} \int_{\{x \in R^1\colon\, |f(x)| \leq t\}} |f(x)|^{p_2}\, dx. \end{aligned}$$

From the preceding inequality and (3) we obtain

$$\begin{aligned}
\|T(f)\|^p_{L^p(R^1)} &= p\int_0^\infty t^{p-1}\lambda(t)\,dt \\
&\le p2^{p_1}M_1\int_0^\infty \frac{t^{p-1}}{t^{p_1}}\int_{\{x\in R^1:\ |f(x)|>t\}}|f(x)|^{p_1}\,dx\,dt \\
&\quad + p2^{p_2}M_2\int_0^\infty \frac{t^{p-1}}{t^{p_2}}\int_{\{x\in R^1:\ |f(x)|>t\}}|f(x)|^{p_2}\,dx\,dt \\
&= p2^{p_1}M_1\int_{R^1}|f(x)|^{p_1}\int_0^{|f(x)|}t^{p-p_1-1}\,dt\,dx \\
&\quad + p2^{p_2}M_2\int_{R^1}|f(x)|^{p_2}\int_{|f(x)|}^\infty t^{p-p_2-1}\,dt\,dx \\
&= \frac{p\cdot 2^{p_1}M_1}{p-p_1}\int_{R^1}|f(x)|^p\,dx + \frac{p\cdot 2^{p_2}M_2}{|p_2|-|p|}\int_{R^1}f(x)|^p\,dx \\
&= M\cdot\|f\|^p_{L^p(R^1)},
\end{aligned}$$

i.e., T has type (p,p). This completes the proof of Theorem 1.

2°. *The maximal function.*

DEFINITION 2. Let $f(x)$, $x\in R^1$, be summable on every interval $(-A,A)$, $A>0$. The *maximal function* of $f(x)$ is the function

$$M(f,x)=\sup_{I\ni x}|I|^{-1}\int_I|f(t)|\,dt,\qquad x\in R^1,$$

where the supremum is taken over all intervals I that contain x.

THEOREM 2. a) *If $f(x)\in L^1(R^1)$, then for every $t>0$*

$$m\{x\in R^1\colon M(f,x)>t\}\le\frac{5}{t}\int_{R^1}|f(x)|\,dx.$$

b) *If $f(x)\in L^p(R^1)$, $1<p\le\infty$, then*

$$\|M(f)\|_{L^p(R^1)}\le C_p\|f\|_{L^p(R^1)},$$

where C_p is a constant that depends only on p.

LEMMA 1. *Let $E\subset R^1$ be a measurable set and $\{I_\alpha\}$ a family of intervals of bounded length which cover E: $E\subset\bigcup_\alpha I_\alpha$. Then we can extract from this family a sequence I_1, $I_2,\dots$ of pairwise disjoint intervals (finite or infinite) such that*

$$\sum_k|I_k|\ge\frac{1}{5}m(E). \tag{17}$$

PROOF OF LEMMA 1. We will construct the intervals I_1, $I_2,\dots,$ inductively. At the first step we select from $\{I_\alpha\}$ an interval I_1 for which

$|I_1| \geq \frac{1}{2}\sup_\alpha |I_\alpha|$. Now suppose that $I_1, \ldots, I_k$ have already been constructed; we find an interval I_{k+1}, disjoint from $I_1, \ldots, I_k$, with

$$|I_{k+1}| \geq \tfrac{1}{2}\sup\{|I_\alpha| \colon I_\alpha \cap I_j = \varnothing,\ j = 1, \ldots, k\}.$$

If there is no such interval, the construction stops.

We have now defined the sequence $I_1, I_2, \ldots$. Let us prove (17). We need to consider only the case when $\sum_k |I_k| < \infty$. Let I_k^* be an interval with the same midpoint as I_k but five times as long: $|I_k^*| = 5|I_k|$. Let us show that

$$E \subset \bigcup_k I_k^*. \tag{18}$$

It is enough to verify that every I in $\{I_\alpha\}$ is contained in $\bigcup_k I_k^*$.

Since $\sum_k |I_k| < \infty$, we have $\lim_{k\to\infty} |I_k| = 0$ and, for sufficiently large k,

$$|I_{k+1}| > |I|/2. \tag{19}$$

Let k_0 $(1 \leq k_0 < \infty)$ be the smallest k satisfying (19). By construction, I intersects one of the intervals $I_1, \ldots, I_{k_0}$, i.e., for some j we have $1 \leq j \leq k_0$, $I \cap I_j \neq \varnothing$, and $\frac{1}{2}|I| \leq |I_j|$. But then $I \subset I_j^*$ and we have obtained (18). Finally, from (18) we have

$$m(E) \leq \sum_k |I_k^*| = 5\sum_k |I_k|.$$

Proof of Theorem 2. Let $f(x) \in L^1(R^1)$. Choose a number $t > 0$ and set

$$E_t = \{x \in R^1 \colon M(f, x) > t\}.$$

By the definition of the maximal function, for every $x \in E_t$ there is an interval I_x such that

$$I_x \ni x \quad \text{and} \quad \int_{I_x} |f(u)|\, du > t|I_x|. \tag{20}$$

The family $\{I_x\}_{x\in E_t}$ of bounded intervals (see (20)) covers E_t. By Lemma 1 there is a sequence $\{I_k\}$ of disjoint intervals with

$$\sum_k |I_k| \geq \frac{1}{5} m(E_t). \tag{21}$$

Since (see (20))

$$\sum_k |I_k| < \frac{1}{t}\sum_k \int_{I_k} |f(u)|\, du \leq \frac{1}{t}\int_{R^1} |f(u)|\, du,$$

it follows from (21) that $m(E_t)$ is finite and

$$m(E_t) < \frac{5}{t}\int_{R^1} |f(u)|\,du.$$

Now we can prove Proposition b). For $p = \infty$ the inequality b) (with $C_\infty = 1$) is evident. If $1 < p < \infty$, then with a given $t > 0$ we obtain

$$h(x) = \begin{cases} f(x) & \text{if } |f(x)| > \frac{t}{2}, \\ 0 & \text{if } |f(x)| \le \frac{t}{2}, \end{cases} \quad x \in R^1.$$

Then $|f(x)| \le |h(x)| + t/2$ and $M(f,x) \le M(h,x) + t/2$, $x \in R^1$. Consequently

$$E_t \equiv \{x \in R^1 : M(f,x) > t\} \subset \{x \in R^1 : M(h,x) > t/2\}$$

and (see a))

$$m(E_t) \le \frac{10}{t}\|h\|_{L^1(R^1)} = \frac{10}{t}\int_{\{x\in R^1 :\ |f(x)|>t/2\}} |f(x)|\,dx.$$

Finally, we use (3):

$$\begin{aligned}\int_{R^1} M^p(f,x)\,dx &= p\int_0^\infty t^{p-1} m(E_t)\,dt \\ &\le 10p\int_0^\infty t^{p-1}\cdot\frac{1}{t}\int_{\{x\in R^1 :\ |f(x)|>t/2\}} |f(x)|\,dx \\ &= 10p\int_{R^1} |f(x)| \int_0^{2|f(x)|} t^{p-2}\,dt\,dx = C_p\int_{R^1} |f(x)|^p\,dx.\end{aligned}$$

REMARK. Lebesgue's theorem on the differentiability a.e. of an indefinite integral is easily deduced from part a) of Theorem 2.

§3. Some topics from functional analysis

1°. DEFINITION 3. Let $\Phi = \{\varphi_n(x)\}_{n-1}^N \subset L^2(E)$ be a system of functions defined on a set E ($E \subset R^2, 1 \le N \le \infty$). The *Gram matrix* of Φ is the matrix

$$G = G_\Phi = \{u_{n,j}\}_{n,j=1}^N, \quad \text{where } u_{n,j} = \int_E \varphi_n(x)\varphi_j(x)\,dx.$$

It is not difficult to verify (this is done for all Euclidean spaces in courses on linear algebra) the following proposition.

PROPOSITION 3. *A system $\Phi = \{\varphi_n\}_{n=1}^N \subset L^2(E)$, $1 \le N < \infty$, is linearly independent if and only if the determinant* $\det G_\Phi \neq 0$.

PROPOSITION 4. *Let there be given in $L^2(E)$ two systems $\Phi = \{\varphi_n(x)\}_{n=1}^N$ and $\Psi = \{\psi_n(x)\}_{n=1}^\infty$ of functions, $1 \le N < \infty$, and let L_Φ, L_Ψ be the* *subspaces spanned by the functions* $\{\varphi_n\}_{n=1}^N$ *and* $\{\psi_n\}_{n=1}^N$, *respectively. Then if* Φ *and* Ψ *have equal Gram matrices* ($G_\Phi = G_\Psi$), *there is an isometry* $T\colon L_\Phi \to L_\Psi$ (*i.e.*, $\|T(f)\|_{L^2(E)} = \|f\|_{L^2(E)}$ *if* $f \in L_\Phi$) *such that*

$$T(\varphi_n) = \psi_n, \qquad n = 1, \ldots, N. \tag{22}$$

2°. *Minimax theorem.*

THEOREM (VON NEUMANN). *Let* $A = \{a_{i,j}\}$ ($a_{i,j} \geq 0$; $1 \leq i \leq m$, $1 \leq j \leq n$) *be a matrix with nonnegative elements and*

$$\sigma_m = \{x \in R^m\colon x = (x_1, \ldots, x_m), x_i \geq 0, \sum_{i=1}^m x_i = 1\}.$$

Then the quadratic form

$$F(x,y) = \sum_{i=1}^m \sum_{j=1}^n a_{i,j} x_i y_j \qquad (x = \{x_i\}, y = \{y_j\})$$

satisfies

$$\min_{y\in\sigma_n} \max_{x\in\sigma_m} F(x,y) = \max_{x\in\sigma_m} \min_{y\in\sigma_n} F(x,y). \tag{23}$$

PROOF. Let γ and ρ denote the left-hand and right-hand sides of (23). The inequality $\gamma \geq \rho$ can be verified very simply. In fact, if $x_0 \in \sigma_m$ is a point such that $\min_{y\in\sigma_n} F(x_0, y) = \rho$, then by using the inequality $\max_{x\in\sigma_m} F(x,y) \geq F(x_0, y)$, $y \in \sigma_n$, we obtain

$$\gamma = \min_{\nu\in\sigma_n} \max_{x\in\sigma_m} F(x,y) \geq \min_{y\in\sigma_n} F(x_0, y) = \rho.$$

To prove the converse inequality $\gamma \leq \rho$ we consider the closed sets

$$A(x, \varepsilon) = \{y \in \sigma_n\colon F(x,y) \leq \rho + \varepsilon\}, \qquad x \in \sigma_m, \quad \varepsilon > 0.$$

It is enough to show that for each $\varepsilon > 0$

$$\bigcap_{\nu=1}^s A(x^\nu, \varepsilon) \neq \varnothing, \qquad s = 1, 2, \ldots, \tag{24}$$

where $x^\nu = \{x_1^\nu, \ldots, x_m^\nu\}$, $\nu = 1, 2, \ldots$, is a sequence which is everywhere dense in σ_m. In fact, if follows from (24) and the fact that $A(x, \varepsilon)$ is closed that there is a point $y_0 \in \sigma_n$ such that $y_0 \in A(x^\nu, \varepsilon)$ for $\nu = 1, 2, \ldots$, i.e.,

$$\max_{1\leq\nu<\infty} F(x^\nu, y_0) \leq \rho + \varepsilon. \tag{25}$$

Since $F(x,y)$ is continuous, it follows from (25) that $\max_{x\in\sigma_m} F(x, y_0) \leq \rho + \varepsilon$, whence, since $\varepsilon > 0$ is arbitrary, we obtain the inequality $\gamma \leq \rho$.

Let the numbers $s = 1, 2, \ldots$ and $\varepsilon > 0$ be given, and let

$$P = \{(\xi_1, \ldots, \xi_s) \in R^s : \xi_\nu \leq \rho + \varepsilon, \nu = 1, \ldots, s\}.$$

In addition, let $Q \subset R^s$ be the set of vectors of the form

$$(F(x^1, y), F(x^2, y), \ldots, F(x^s, y)), \qquad y \in \sigma_n.$$

It is clear that Q is convex, closed and bounded. Let us show that

$$P \cap Q \neq \varnothing. \tag{26}$$

Suppose, on the contrary, that $P \cap Q = \varnothing$. Then by a corollary of the Hahn–Banach theorem the convex sets P and Q are separated by a linear functional: There are a vector $(\alpha_1, \ldots, \alpha_s) \in R^s$ and a number $\alpha \in R^1$ such that

$$\begin{aligned} &1) \sum_{\nu=1}^{s} |\alpha_\nu| = 1; \\ &2) \sum_{\nu=1}^{s} \alpha_\nu \xi_\nu \leq \alpha \quad \text{if } (\xi_1, \ldots, \xi_s) \in P; \\ &3) \sum_{\nu=1}^{s} \alpha_\nu \xi_\nu \geq \alpha \quad \text{if } (\xi_1, \ldots, \xi_s) \in Q. \end{aligned} \tag{27}$$

Since the vectors $(0, \ldots, 0, \xi_\nu, 0, \ldots, 0) \in P$ if $\xi_\nu \leq \rho$ and $1 \leq \nu \leq s$, it follows from (27), 2) that $\alpha_\nu \geq 0$, $\nu = 1, \ldots, s$. Consider the vector

$$x^0 = (x_1^0, \ldots, x_m^0) = \sum_{\nu=1}^{s} \alpha_\nu x^\nu$$

(i.e., $x_i^0 = \sum_{\nu=1}^{s} \alpha_\nu x_i^\nu$). Since $\alpha_\nu \geq 0$ and $x^\nu \in \sigma_m$, $1 \leq \nu \leq s$, then $x_0^i \geq 0$ and (see 1))

$$\sum_{i=1}^{m} x_i^0 = \sum_{\nu=1}^{s} \alpha_\nu \sum_{i=1}^{m} x_i^\nu = \sum_{\nu=1}^{s} \alpha_\nu = 1,$$

i.e., $x^0 \in \sigma_m$.

On the other hand, since the s-component vector $(\rho + \varepsilon, \ldots, \rho + \varepsilon) \in P$, we can deduce from 2) and 3) that for every y

$$F(x^0, y) = \sum_{\nu=1}^{s} \alpha_\nu F(x^\nu, y) \geq \alpha \geq \sum_{\nu=1}^{s} \alpha_\nu (\rho + \varepsilon) > \rho,$$

which contradicts the definition of ρ. Consequently (26) is established.

We now take an arbitrary vector $(\xi_1^0, \ldots, \xi_s^0) \in P \cap Q$. Then $\xi_\nu^0 \leq \rho \uparrow \varepsilon$ and $\xi_\nu^0 = F(x^\nu, y^0)$, $\nu = 1, \ldots, s$, $y^0 \in \sigma_n$ (see the definition of P and Q), i.e., $y^0 \in A(x^\nu, \varepsilon)$, $\nu = 1, \ldots, s$. This establishes (24) and hence the theorem is proved.

3°. *Proof of Theorem* 1.6 (*see* §1.4).

Necessity. Let $\{x_n\}_{n=1}^\infty$ be a basis in the Banach space X, i.e., there exists, for every $x \in X$, a unique series

$$\sum_{n=1}^\infty a_n(x)x_n = x,$$

that converges to x in the norm of X.

It is clear that $\{x_n\}$ is dense in X. Consider the set Y of sequences of numbers $A = \{a_n\}_{n=1}^\infty$ for which the series $\sum_{n=1}^\infty a_n x_n$ converges in the norm of X. For $A = \{a_n\} \in Y$, set

$$\|A\|_Y = \sup_{1 \le N < \infty} \left\| \sum_{n=1}^N a_n x_n \right\|_X . \tag{28}$$

It is easily seen that (28) defines a norm on the linear space Y. Let us show that Y is a Banach space under the norm (28).

Let $A^{(k)} = \{a_n^{(k)})_{n=1}^\infty \in Y$, $k = 1, 2, \dots$, and

$$\|A^{(k)} - A^{(k+p)}\|_Y \to 0 \quad \text{if } k \to \infty,\ p = 1, 2, \dots .$$

Then by (28), for every $\varepsilon > 0$ there is a number k_0 such that, for N, R, $p = 1, 2, \dots$,

$$\left\| \sum_{n=N}^{N+R} a_n^{(k)} x_n - \sum_{n=N}^{N+R} a_n^{(k+p)} x_n \right\|_X \le \varepsilon, \quad \text{if } k \ge k_0. \tag{29}$$

It follows, in particular, from (29) that the limit $\lim_{k\to\infty} a_n^{(k)} = a_n$ exists for $n = 1, 2, \dots$. If we fix k, N, and R in (29) and let p tend to infinity, we find

$$\left\| \sum_{n=N}^{N+R} a_n^{(k)} x_n - \sum_{n=N}^{N+R} a_n x_n \right\|_X \le \varepsilon, \qquad k \ge k_0,\ N, R = 1, 2, \dots . \tag{30}$$

Since $\{a_n^{k_0}\} \in Y$, we have, for sufficiently large $N_0 = N_0(\varepsilon)$,

$$\left\| \sum_{n=N}^{N+R} a_n^{(k_0)} x_n \right\|_X \le \varepsilon, \quad \text{if } N \ge N_0,\ R = 1, 2, \dots .$$

It follows from this and from (30) that when $N > N_0$

$$\left\| \sum_{n=N}^{N+R} a_n x_n \right\|_X \le 2\varepsilon, \qquad R = 1, 2, \dots,$$

i.e., the series $\sum_{n=1}^\infty a_n x_n$ converges in X, and that $A = \{a_n\} \in Y$. Inequality (30) shows that $\lim \|A^{(k)} - A\|_Y = 0$. Therefore Y is a Banach space.

Now we consider the linear operator $T\colon Y \to X$ which assigns to each element $A = \{a_n\} \in Y$ the sum (in X) $\sum_{n=1}^{\infty} a_n x_n$; that is,

$$T(A) \equiv T(\{a_n\}) = \sum_{n=1}^{\infty} a_n x_n.$$

From the fact that $\{x_n\}_{n=1}^{\infty}$ is a basis in X and from the definition of the norm in Y (see (28)) it follows immediately that T is a bounded linear operator which provides a one-to-one mapping of Y onto X. By Banach's theorem on inverse operators, T^{-1} is also bounded, i.e., there is a constant $M > 0$ such that

$$\sup_{1 \le N < \infty} \left\| \sum_{n=1}^{N} a_n(x) \cdot x_n \right\|_X \le M \|x\|_X, \qquad x \in X. \tag{31}$$

Hence we have shown that the basis $\{x_n\}_{n=1}^{\infty}$ satisfies condition c) (see the statement of Theorem 1.6).

In addition, it follows from (31) that the linear functionals $a_n(x)$, $x \in X$, $n = 1, 2, \ldots$, are bounded, and therefore (see Theorem 1.2) that $\{x_n\}_{n=1}^{\infty}$ is minimal.

Sufficiency. Let $\{x_n\}_{n=1}^{\infty}$ be complete and minimal in X, and let $\{y_n\}_{n=1}^{\infty} \subset X^*$ be the system dual to $\{x_n\}$ (the existence and uniqueness of $\{y_n\}$ was established in Chapter 1). In addition, let there be a number M such that for every $x \in X$

$$\left\| \sum_{n=1}^{N} \langle y_n, x \rangle x_n \right\|_X \le M \|x\|_X, \qquad n = 1, 2, \ldots \tag{32}$$

Let us show that $\{x_n\}$ is a basis in X. Choose an element x in X and an $\varepsilon > 0$. Since $\{x_n\}$ is complete, there is a polynomial $P = \sum_{n=1}^{R} a_n x_n$ in this system for which

$$\|P - x\|_X \le \varepsilon. \tag{33}$$

Since $\sum_{n=1}^{N} \langle y_n, P \rangle x_n = P$ if $N \ge R$, we find from (32) and (33) that

$$\begin{aligned}
\left\| \left\{ \sum_{n=1}^{N} \langle y_n, x \rangle x_n \right\} - x \right\|_X &\le \left\| \left\{ \sum_{n=1}^{N} \langle y_n, x \rangle x \right\} - P \right\|_X + \|P - x\|_X \\
&\le \left\| \sum_{n=1}^{N} \langle y_n, x - P \rangle x_n \right\|_X + \varepsilon \\
&\le M \|x - P\|_X + \varepsilon \le (M+1)\varepsilon, \quad \text{if } N \ge R,
\end{aligned}$$

i.e., the series $\sum_{n=1}^{\infty}\langle y_n, x\rangle x_n$ converges to x in X. The uniqueness of this series follows from the biorthonormality of the system $\{x_n, y_n\}$: if $\sum_{n=1}^{\infty} a_n x_n$ converges to x, then for $n = 1, 2, \ldots,$

$$\langle y_n, x\rangle = \left\langle y_n, \sum_{k=1}^{\infty} a_k x_k \right\rangle = \sum_{k=1}^{\infty} a_k \langle y_n, x_k\rangle = a_n.$$

This completes the proof of the theorem.

This page intentionally left blank

APPENDIX 2

Some Topics from the Theory of Functions of a Complex Variable

Here we prove some results from the theory of functions of a complex variable which are used in the text (in §§5.2, 5.3, and 6.5), but which fall outside the limits of the usual university curriculum.

§1. The Poisson integral

If $f(x)$ and $g(x)$, $x \in R^1$, are summable on $[-\pi, \pi]$, periodic with period 2π, and complex-valued, we denote by $f * g(x)$ the convolution

$$f * g(x) = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x+t) g(t)\, dt.$$

It follows easily from Fubini's theorem that a convolution of summable functions is also summable on $[-\pi, \pi]$ and

$$c_n(f * g) = c_n(f) \cdot c_{-n}(g), \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{1}$$

where $\{c_n(f)\}$ are the Fourier coefficients of $f(x)$:

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt, \qquad n = 0, \pm 1, \pm 2, \ldots .$$

Let $f \in L^1(-\pi, \pi)$. For $0 \le r < 1$ we consider the function

$$f_r(x) = \sum_{n=-\infty}^{\infty} c_n(f) r^{|n|} e^{inx}, \quad x \in [-\pi, \pi], \tag{2}$$

where the series on the right of (2) converges uniformly in x for every given r, $0 \le r < 1$. The Fourier coefficients of $f_r(x)$ are $c_n(f_r) = c_n(f) \cdot r^{|n|}$, $n = 0, \pm 1, \pm 2, \ldots$; and this means, because of (1), that $f_r(x)$ can be represented as a convolution:

$$f_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) P_r(t)\, dt, \tag{3}$$

where

$$P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int}, \qquad t \in [-\pi, \pi]. \tag{4}$$

The function $P_r(t)$ of two variables, $0 \le r < 1$, $t \in [-\pi, \pi]$, is the *Poisson kernel*, and the integral (3) is the *Poisson integral*. It is easily verified that

$$P_r(t) = \operatorname{Re}\left[1 + 2\sum_{n=1}^{\infty} r^n e^{int}\right] = \operatorname{Re}\frac{1 + re^{it}}{1 - re^{it}}.$$

Consequently

$$P_r(t) = \frac{1 - r^2}{1 + r^2 - 2r\cos t}, \qquad 0 \le r < 1,\ t \in [-\pi, \pi]. \tag{5}$$

If $f \in L^1(-\pi, \pi)$ is a real function, then since $c_{-n}(f) = \overline{c_n(f)}$, $n = 0, \pm 1, \pm 2, \dots$, we find from (2) that

$$\begin{aligned} f_r(x) &= c_0(f) + \sum_{n=1}^{\infty} c_n(f) r^n e^{inx} + \sum_{n=1}^{\infty} \overline{c_n(f)} r^n e^{-inx} \\ &= \tfrac{1}{2}[F(re^{ix}) + \overline{F(re^{ix})}] = \operatorname{Re} F(re^{ix}), \end{aligned} \tag{6}$$

where

$$F(z) = c_0(f) + 2\sum_{n=1}^{\infty} c_n(f) r^n e^{inx} \qquad (z = re^{ix}) \tag{7}$$

is analytic in the unit disk. Equation (6) shows that for every real $f \in L^1(-\pi, \pi)$ the Poisson integral (3) defines a harmonic function in the unit disk,

$$u(z) = f_r(e^{ix}), \qquad z = re^{ix}, \quad 0 \le r < 1, \quad x \in [-\pi, \pi].$$

Moreover, the harmonic conjugate $v(z)$ of $u(z)$, with $v(0) = 0$, is given by the formula

$$v(z) = \operatorname{Im} F(z) = \sum_{n=-\infty}^{\infty} [-i(\operatorname{sgn} n) r^{|n|} c_n(f)] e^{inx}, \tag{8}$$

where, as usual,

$$\operatorname{sgn}\alpha = \begin{cases} \dfrac{\alpha}{|\alpha|} & \text{if } \alpha \ne 0, \\ & \qquad\qquad \alpha \in R^1 \\ 0 & \text{if } \alpha = 0, \end{cases} \tag{9}$$

PROPOSITION 1. *Let $u(z)$ be harmonic (or analytic) in the disk $|z| < 1+\varepsilon$ ($\varepsilon > 0$), and let $f(x) = u(e^{ix})$, $x \in [-\pi, \pi]$. Then*

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x + t) P_r(t)\, dt \qquad (z = re^{ix},\ |z| < 1). \tag{10}$$

Since the Poisson kernel is a real function, it is enough to verify (10) in the case when $u(z)$ is an analytic function:

$$u(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad |z| < 1 + \varepsilon.$$

But then

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{ix}) e^{-inx}\, dx = \begin{cases} c_n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0, \end{cases}$$

and (10) follows at once from (2) and (3).

Before we turn to the investigation of the behavior of $f_r(x)$ as $r \to 1$, we note some properties of the Poisson kernel:

$$\begin{aligned} &\text{a)} \quad P_r(t) \geq 0, \qquad \leq r < 1, \quad t \in [-\pi, \pi]. \\ &\text{b)} \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t)\, dt = 1, \qquad 0 \leq r < 1. \\ &\text{c)} \quad \text{for every } \delta > 0 \\ &\mu_\delta(r) = \sup_{\delta \leq |t| \leq \pi} P_r(t) = o(1) \quad \text{as } r \to 1. \end{aligned} \tag{11}$$

Relations a) and c) follow at once from (5); and to prove (b) it is enough to set $f(x) \equiv 1$ in (2) and (3).

THEOREM 1. *For every (complex-valued) function $f \in L^p(-\pi, \pi)$, $1 \leq p < \infty$, we have the equation*

$$\lim_{r \to 1} \|f - f_r\|_{L^p(-\pi,\pi)} = 0$$

[*see* (3)]; *if also $f(x)$ is continuous on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$, then*

$$\lim_{r \to 1} \|f - f_r\|_{C(-\pi,\pi)} = 0.$$

PROOF. By (3) and property b) of the Poisson kernel,

$$f_r(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x+t) - f(x)] P_r(t)\, dt. \tag{12}$$

For every $g \in L^q(-\pi, \pi)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\|g\|_{L^q(-\pi,\pi)} = 1$, we can use Hölder's inequality and the positivity of the Poisson kernel (see a)) to obtain

$$\begin{aligned} &\int_{-\pi}^{\pi} [f_r(x) - f(x)] g(x)\, dx \\ &\quad = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \int_{-\pi}^{\pi} [f(x+t) - f(x)] g(x)\, dx \right\} P_r(t)\, dt \\ &\quad \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(x+t) - f(x)\|_{L^p(-\pi,\pi)} \|g\|_{L^q(-\pi,\pi)} P_r(t)\, dt \\ &\quad \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega_p(|t|, f) P_r(t)\, dt. \end{aligned}$$

Consequently

$$\|f_r - f\|_{L^p(-\pi,\pi)} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \omega_p(|t|, f)P_r(t)\,dt,$$

and if for a given $\varepsilon > 0$ we find a number $\delta = \delta(\varepsilon)$ such that $\omega_p(t, f) \leq \varepsilon/2$ for $t \in (0, \delta)$, then for r sufficiently close to 1 we obtain (see a)–c))

$$\begin{aligned}\|f_r - f\|_{L^p(-\pi,\pi)} &\leq \frac{1}{2\pi}\int_{|t|<\delta} \omega_p(|t|, f)P_r(t)\,dt \\ &\quad + \frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi} \omega_p(|t|, f)P_r(t)\,dt \\ &\leq \frac{\varepsilon}{2} + 2\|f\|_{L^p(-\pi,\pi)} \cdot \mu_\delta(r) < \varepsilon.\end{aligned}$$

Similarly, the second statement in Theorem 1 follows from the inequality (see (12), and a), b))

$$\|f_r - f\|_{C(-\pi,\pi)} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \omega(|t|, f)p_r(t)\,dt.$$

This completes the proof of Theorem 1.

THEOREM 2. (FATOU). *Let $f(x)$ be a complex-valued element of $L^1(-\pi,\pi)$. Then*

$$\lim_{r\to 1} f_r(x) = f(x) \quad \textit{for almost all } x \in [-\pi, \pi].$$

PROOF. Let us show that, for $x \in [-\pi, \pi]$ and $0 < r < 1$,

$$f_r(x) \leq CM(f, x), \tag{13}$$

where C is an absolute constant and $M(f, x)$ is the maximal function for $f(x)$(1) (see Definition 2 in Appendix 1). For this purpose we use the inequality

$$P_r(t) \leq K\frac{\varepsilon}{\varepsilon^2 + t^2}, \qquad \varepsilon = 1 - r,\ 0 < r < 1,\ t \in [-\pi, \pi]$$

which is easily derived from (5) (K is an absolute constant).

(1)We suppose that $f(x)$ has been continued with preservation of periodicity to $[-2\pi, 2\pi]$ (i.e., $f(x) = f(y)$ if $x, y \in [-2\pi, 2\pi]$, $x - y = 2\pi$, and $f(x) = 0$ if $|x| > 2\pi$.

Let $k_0 = k_0(\varepsilon)$ be a number such that $2^{k_0}\varepsilon < \pi \le 2^{k_0+1}\varepsilon$. Then for $x \in [-\pi, \pi]$

$$\begin{aligned}
|f_r(x)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(x+t)P_r(t)\,dt\right| \\
&\le \frac{1}{2\pi}\sum_{k=-\infty}^{k_0-1}\int_{2^k\varepsilon<|t|\le 2^{k+1}\varepsilon} |f(x+t)|P_r(t)\,dt \\
&\quad + \frac{1}{2\pi}\int_{2^{k_0}\varepsilon<|t|\le\pi} |f(x+t)|P_r(t)\,dt \\
&\le \frac{K}{2\pi}\sum_{k=-\infty}^{\infty}\int_{2^k\varepsilon<|t|\le 2^{k+1}\varepsilon} |f(x+t)|\frac{\varepsilon}{\varepsilon^2+t^2}\,dt \\
&\le \frac{K}{2\pi}\sum_{k=-\infty}^{\infty}\frac{\varepsilon}{\varepsilon^2+2^{2k}\varepsilon^2}\int_{0\le|t|\le 2^{k+1}\varepsilon} \varepsilon|f(x+t)|\,dt \\
&\le M(f,x)\frac{K}{2\pi}\sum_{k=-\infty}^{\infty}\frac{1}{\varepsilon(1+2^{2k})}2^{k+2}\varepsilon \le CM(f,x).
\end{aligned}$$

This completes the proof of (13). Now, using the operator $f(x) \to M(f,x)$ of weak type $(1,1)$ (see Theorem 2 in Appendix 1), we can find a sequence $\{f^{(n)}(x)\}_{n=1}^{\infty}$, $x \in R^1$, such that

$$\begin{gathered}
f^{(n)}(x) \in C(-2\pi, 2\pi), \\
f^{(n)}(x) = 0 \quad \text{if } |x| > 2\pi,\ n = 1, 2, \dots, \\
\lim_{n\to\infty} f^{(n)}(x) = f(x) \quad \text{and} \quad \lim_{n\to\infty} M(f^{(n)} - f, x) = 0 \\
\text{for almost all } x \in (-2\pi, 2\pi).
\end{gathered} \tag{14}$$

By (13), for $x \in (-\pi, \pi)$,

$$\begin{aligned}
|f_r(x) - f(x)| &\le |f_r(x) - f_r^{(n)}(x)| \\
&\qquad + |f_r^{(n)}(x) - f^{(n)}(x)| + |f^{(n)}(x) - f(x)| \\
&\le C\cdot M(f^{(n)} - f, x) + |f_r^{(n)}(x) - f^{(n)}(x)| \\
&\qquad + |f^{(n)}(x) - f(x)|.
\end{aligned}$$

We obtain the conclusion of Theorem 2 by using the preceding inequality and (14), and taking account of the fact that, by Theorem 1, $\lim_{r\to 1} f_r^{(n)}(x) = f^{(n)}(x)$ for each $x \in [-\pi, \pi]$.

REMARK. If we use, instead of (13), the stronger inequality (59) which we shall prove in §3, we can show that $f_r(t) \to f(x)$ for almost all $x \in [-\pi, \pi]$ when re^{it} approaches e^{ix} on a path that is not tangent to the circle $|z| = 1$ (for details see §3).

§2. H^p spaces

DEFINITION 1. The space H^p $(1 \le p \le \infty)$ is the collection of functions $F(z)$ that are analytic in the unit disk and have finite norm

$$\|F\|_{R^p} = \sup_{0<r<1} \left\{ \int_{-\pi}^{\pi} |F(re^{it})|^p \, dt \right\}^{1/p}. \tag{15}$$

Let the complex-valued function $\Phi(x) \in L^p(-\pi, \pi)$ satisfy the condition

$$\int_{-\pi}^{\pi} \Phi(t) e^{int} \, dt = 0, \qquad n = 1, 2, \dots, \tag{16}$$

then the function $F(z)$ defined by

$$F(z) = \Phi_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(x+t) P_r(t) \, dt \qquad (z = re^{ix}), \tag{17}$$

belongs to H^p, and

$$\|F\|_{H^p} = \|\Phi\|_{L^p(-\pi,\pi)}. \tag{18}$$

In fact, it follows from (16) and (2) that $F(z)$ is analytic. Moreover, by the inequality

$$\|f * g\|_{L^p(-\pi,\pi)} \le \frac{1}{2\pi} \|f\|_{L^p(-\pi,\pi)} \|g\|_{L^1(-\pi,\pi)}$$

(see, for example, the proof of 4.(16)), we have

$$\|F\|_{H^p} \le \frac{1}{2\pi} \|\Phi\|_{L^p(-\pi,\pi)} \cdot \int_{-\pi}^{\pi} P_r(t) \, dt = \|\Phi\|_{L^p(-\pi,\pi)}.$$

Finally, by Theorem 1 (and for $p = \infty$, by Theorem 2) $\|\Phi\|_{L^p(-\pi,\pi)} \le \|F\|_{H^p}$, and we obtain (18).

We shall show below that every function $F \in H^p$ $(1 \le p \le \infty)$ can be represented in the form (17). For this purpose we shall need the following theorem.

THEOREM 3. *Let the complex-valued function $\varphi(t)$ have bounded variation on $[-\pi, \pi]$, be continuous on the left on $(-\pi, \pi]$, and satisfy*

$$\int_{-\pi}^{\pi} e^{int} d\varphi(t) = 0 \qquad \textit{for } n = 1, 2, \dots. \tag{19}$$

Then $\varphi(t)$ is absolutely continuous on $[-\pi, \pi]$.

REMARK. In (19), and later, we use a Lebesgue–Stieltjes integral with respect to a complex-valued function $\varphi(t)$ of bounded variation. We say that $\varphi(t) = u(t)+iv(t)$ has bounded variation (or is absolutely continuous) if both $u(t)$ and $v(t)$ have bounded variation (or are absolutely continuous). Then the integral

$$\int_{-\pi}^{\pi} f(t)\,d\varphi(t) = \int_{-\pi}^{\pi} f(t)\,du(t) + i\int_{-\pi}^{\pi} f(t)\,dv(t)$$

is defined for every $f(t)$ which is continuous on $[-\pi,\pi]$, and also if $f(t) = \chi_F(t)$ is the characteristic function of a closed set $F \subset [-\pi,\pi]$.

PROOF OF THEOREM 3. It is enough to verify that

$$\int_{-\pi}^{\pi} \chi_F(t)\,d\varphi(t) = 0 \tag{20}$$

for every closed set $F \subset (-\pi,\pi)$ with $m(F) = 0$. (In fact, it follows from (20) that φ is absolutely continuous on $(-\pi,\pi)$, and consequently $\varphi(t) = \varphi_1(t) + s(t)$, where φ_1 is absolutely continuous on $[-\pi,\pi]$, $s(t) = 0$ for $t \in (-\pi,\pi]$, and $s(-\pi) = \lambda$. But then

$$0 = \int_{-\pi}^{\pi} e^{int}\,d\varphi(t) = \int_{-\pi}^{\pi} e^{int}\,d\varphi_1(t) - \lambda e^{-in\pi} = o(1) - \lambda e^{-in\pi}, \qquad n \to \infty;$$

that is, $\lambda = 0$ and $\varphi(t)$ is absolutely continuous on $[-\pi,\pi]$.)

We need the following lemma.

LEMMA 1. *Let F be a closed set, V an open set, and let $F \subset V \subset (-\pi,\pi)$ and $m(F) = 0$. Then for every ε, $0 < \varepsilon < 1/3$, there is a function $g(t) \in C^\infty(-\pi,\pi)$ of the form*

$$g(t) = \sum_{n=1}^{\infty} c_n(g)e^{int}, \tag{21}$$

with the following properties:

$$\begin{aligned} &\text{a)} \quad |g(t) - 1| \le 3\varepsilon \quad \textit{if } t \in F;\\ &\text{b)} \quad |g(t)| \le 3\varepsilon \quad \text{if } t \notin V;\\ &\text{c)} \quad \|g\|_{C(-\pi,\pi)} \le 3. \end{aligned} \tag{22}$$

We first deduce (20) from Lemma 1, and then prove Lemma 1.

Let $(-\pi,\pi)\backslash F = \bigcup_k (a_k,b_k)$, where (a_k,b_k), $k = 1,2,\ldots$, is the finite or infinite sequence of complementary intervals of F, and for $r = 1,2,\ldots$ and $\nu = 3,4,\ldots$, let

$$V_{r,\nu} = (-\pi,\pi)\backslash \bigcup_{k\le r} \left(a_k + \frac{1}{\nu}(b_k - a_k), b_k - \frac{1}{\nu}(b_k - a_k)\right).$$

It is evident that $V_{r,\nu}$ is an open set and $F \subset V_{r,\nu}$. For a given $\varepsilon > 0$ and $r, \nu = 3, 4, \ldots$, consider the function $g(t) = g_{\varepsilon,r,\nu}(t)$, constructed in Lemma 1 for the number ε and the set $V_{r,\nu}$. Then it is easy to verify that if $r \to \infty$, $\nu \to \infty$, and $\varepsilon \to 0$, then the difference

$$\int_{-\pi}^{\pi} \chi_F(t)\,d\varphi(t) - \int_{-\pi}^{\pi} g_{\varepsilon,r,\nu}(t)\,d\varphi(t) \to 0. \tag{23}$$

But by (19) and the uniform convergence of (21),

$$\int_{-\pi}^{\pi} g_{\varepsilon,r,\nu}(t)\,d\varphi(t) = \sum_{n=1}^{\infty} c_n(g_{\varepsilon,r,\nu}) \int_{-\pi}^{\pi} e^{int}\,d\varphi(t) = 0,$$

and (see (23)) we obtain (20).

Proof of Lemma 1. Let $f(t)$ be a continuous function on $[-\pi, \pi]$ for which $\|f\|_{C(-\pi,\pi)} \le 1$, $f(t) = 1$ if $t \in F$, and $f(t) = 0$ if $t \notin V$. Since the Fejér means $\sigma_N(f, t)$ (see (4.12)), $N = 1, 2, \ldots$, converge uniformly to $f(t)$ and $\|\sigma_N(f, t)\|_{C(-\pi,\pi)} \le \|f\|_{C(-\pi,\pi)}$, there is a trigonometric polynomial

$$G(t) = \sum_{|n| \le m} \alpha_n e^{int} \tag{24}$$

such that

$$\begin{gathered} \|G\|_{C(-\pi,\pi)} \le 1, \\ |G(t) - 1| < \varepsilon \quad \text{if } t \in F, \\ |G(t)| < \varepsilon \quad \text{if } t \notin V. \end{gathered} \tag{25}$$

Let $\varepsilon = e^{-A} (A > 1)$. For each $\delta > 0$ consider a function $h_\delta(t) \in C(-\pi, \pi)$ such that

$$\begin{gathered} h_\delta(t) = -2A + \frac{\varepsilon}{2} \quad \text{if } t \in F, \\ -2A + \frac{\varepsilon}{2} \le h_\delta(t) \le 0 \quad \text{for all } t \in (-\pi, \pi), \\ \|h_\delta(t)\|_{L^1(-\pi,\pi)} \le \delta. \end{gathered}$$

The function $h(t)$ can be constructed in the following way: select a closed set $B \subset [-\pi, \pi]$, $B \cap F = \varnothing$, π, $-\pi \in B$, with the measure $m(B)$ sufficiently close to 2π, and set

$$\begin{gathered} h_\delta(t) = \left(-2A + \frac{\varepsilon}{2}\right) \frac{\rho(t, B)}{\rho(t, F) + \rho(t, B)}, \\ \rho(t, E) = \inf_{y \in E} |t - y|). \end{gathered}$$

Since

$$\sum_{|n| \le m} |c_n(h_\delta)| \le (2m + 1)\|h_\delta\|_{L^1(-\pi,\pi)} \le (2m + 1)\delta$$

(here m is the same as in (24)), for sufficiently small $\delta > 0$ the function $h(t) = h_\delta(t) - \sum_{|n|\le m} c_n(h_\delta)e^{int}$ satisfies

$$\begin{aligned} |h(t) + 2A| &< \varepsilon \quad \text{if } t \in F, \\ -2A < h(t) &< \varepsilon, \qquad t \in [-\pi, \pi]. \end{aligned} \tag{26}$$

Moreover, $c_n(h) = 0$ if $|n| \le m$. Then the Fejér means $\sigma_N(h, t)$ of $h(t)$ have the form

$$P(t) \equiv \sigma_N(h, t) = \sum_{n=-N}^{-m-1} \beta_n e^{int} + \sum_{n=m+1}^{N} \beta_n e^{int}$$

and for sufficiently large N (see (26))

$$|P(t) + 2A| < \varepsilon, \qquad t \in F, \ -2A < P(t) < \varepsilon, \ t \in [-\pi, \pi]. \tag{27}$$

Set

$$P^+(t) = \sum_{n=m+1}^{N} \beta_n e^{int}, \ P^-(t) = \sum_{n=-N}^{-m-1} \beta_n e^{int}. \tag{28}$$

Since $h(t)$ is real-valued, $c_n(h) = \overline{c_{-n}(h)}$ and $\beta_n = \overline{\beta}_{-n}$, $n = 0, \pm 1, \pm 2, \ldots$. Therefore

$$P^+(t) = \overline{P^-(t)} \quad \text{and} \quad \operatorname{Re} P^+(t) = \frac{P(t)}{2}, \qquad t \in [-\pi, \pi]. \tag{29}$$

We can now define the required function $g(t)$:

$$g(t) = G(t)[1 - \exp\{P^+(t)\}] = -G(t) \sum_{\nu=1}^{\infty} \frac{[P^+(t)]^\nu}{\nu!}.$$

It is clear that $g(t) \in C^\infty(-\pi, \pi)$, and it follows from (24) and (28) that $c_n(g) = 0$ for $n < 0$, i.e.,

$$g(t) = \sum_{n=1}^{\infty} c_n(g) e^{int}. \tag{30}$$

For $t \in F$, by (25), (27), and (29) we have

$$\begin{aligned} |g(t) - 1| &\le |g(t) - G(t)| + |G(t) - 1| \\ &\le \exp\{\operatorname{Re} P^+(t)\} + \varepsilon \\ &\le \exp\left\{\frac{P(t)}{2}\right\} + \varepsilon \le \exp\left\{-A + \frac{\varepsilon}{2}\right\} + \varepsilon < 3\varepsilon, \end{aligned}$$

and for $t \in [-\pi, \pi] \setminus V$,

$$|g(t)| \le \varepsilon[1 + \exp\{\operatorname{Re} P^+(t)\}] = \varepsilon\left[1 + \exp\left\{\frac{P(t)}{2}\right\}\right] < 3\varepsilon.$$

Finally, for every $t \in [-\pi, \pi]$

$$|g(t)| \leq 1 + \exp\{\operatorname{Re} P^+(t)\} < 3.$$

Consequently $g(t)$ has the required properties (see (22)). Lemma 1, and with it Theorem 3, are established.

THEOREM 4. *Let $F \in H^p$ $(1 \leq p \leq \infty)$. Then the limit*

$$\lim_{r \to 1} F(re^{ix}) = \Phi(x). \tag{31}$$

exists for almost every $x \in [-\pi, \pi]$. Moreover,

1) $F(re^{ix}) = \Phi_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(x+t) P_r(t)\, dt,$
$0 \leq r < 1, \qquad x \in [-\pi, \pi];$

2) $\lim_{r \to 1} \|F(re^{ix}) - \Phi(x)\|_{L^p(-\pi,\pi)} = 0$ $(1 \leq p < \infty)$;

3) $\|F\|_{H^p} = \|\Phi\|_{L^p(-\pi,\pi)}$ $(1 \leq p \leq \infty)$.

PROOF. It is enough to show that for every $F \in H^1$ there is a function $\Phi \in L^1(-\pi, \pi)$ such that 1) holds. In fact, if $f \in H^p$, $1 \leq p \leq \infty$, then also $F \in H^1$ and (31) follows from 1) and Theorem 2 for almost all $x \in [-\pi, \pi]$. Moreover,

$$\|\Phi\|_{L^p(-\pi,\pi)} \leq \sup_{0<r<1} \|F(re^{ix})\|_{L^p(-\pi,\pi)} < \infty,$$

and by Theorem 1

$$\lim_{r \to 1} \|F(re^{ix}) - \Phi(x)\|_{L^p(-\pi,\pi)} = 0 \qquad (1 \leq p < \infty).$$

Finally, it follows from 1) (also see (2)) that

$$\int_{-\pi}^{\pi} \Phi(x) e^{inx}\, dx = 0, \qquad n = 1, 2, \ldots,$$

and then (see (18))

$$\|F\|_{H^p} = \|\Phi\|_{L^p(-\pi,\pi)}.$$

Let $F \in H^1$. To construct the required function $\Phi \in L^1(-\pi, \pi)$, we set

$$\varphi_r(t) = \int_{-\pi}^{\pi} F(re^{ix})\, dx, \qquad t \in [-\pi, \pi],\ 0 < r < 1.$$

The function $\varphi_r(t)$, $0 < r < 1$, has bounded variation on $[-\pi, \pi]$:

$$V_{[-\pi,\pi]}(\varphi_r) \leq \int_{-\pi}^{\pi} |F(re^{it})|\, dt \leq \|F\|_{H^1}.$$

By using the definition of the Stieltjes integral and Helly's theorem, it is easy to deduce from the preceding relation that there are a function $\varphi(t)$ of bounded variation, continuous on the left on $(-\pi, \pi]$, and a sequence

$\{r_k\}$, $0 < r_k < 1$, $k = 1, 2, \ldots, \lim_{k\to\infty} r_k = 1$, such that $\varphi_{r_k}(t) \to \varphi(t)$ at each $t \in [-\pi, \pi]$ and

$$\int_{-\pi}^{\pi} g(t)\, d\varphi(t) = \lim_{k\to\infty} \int_{-\pi}^{\pi} g(t)\, d\varphi_{r_k}(t) \equiv \lim_{k\to\infty} \int_{-\pi}^{\pi} g(t) F(r_k e^{it})\, dt \qquad (32)$$

for every function $g \in C(-\pi, \pi)$. Moreover (see (32)), for $n = 1, 2, \ldots$,

$$\int_{-\pi}^{\pi} e^{int}\, d\varphi(t) = \lim_{k\to\infty} \int_{-\pi}^{\pi} e^{int} F(r_k e^{it})\, dt = 0$$

(we used the analyticity of $F(z)$ in the unit disk) and consequently (see Theorem 3) $\varphi(t)$ is absolutely continuous: there is a function $\Phi \in L^1(-\pi, \pi)$ for which

$$\varphi(t) = \int_{-\pi}^{\pi} \Phi(u)\, du, \qquad t \in [-\pi, \pi].$$

Then (see (32))

$$\int_{-\pi}^{\pi} g(t)\Phi(t)\, dt = \lim_{k\to\infty} \int_{-\pi}^{\pi} g(t) F(r_k e^{it})\, dt, \qquad g \in C(-\pi, \pi). \qquad (33)$$

Select a number r, $0 < r < 1$. The functions $F_k(z) = F(r_k z)$, $k = 1, 2, \ldots$, are analytic in the disk $|z| < 1/r_k$ $(1/r_k > 1)$, and consequently, by Proposition 1,

$$F(r_k r e^{ix}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(r_k e^{it}) P_r(t - x)\, dt, \qquad x \in [-\pi, \pi].$$

In the limit as $k \to \infty$, it follows from the preceding equation that

$$\begin{aligned} F(re^{ix}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t - x)\Phi(t)\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(x + t) P_r(t)\, dt, \qquad 0 \le r < 1, \ x \in [-\pi, \pi]. \end{aligned}$$

This establishes equation 1), and with it also Theorem 4.

We denote by H^p $(1 \le p \le \infty)$ the class of functions $\Phi(x)$, $x \in [-\pi, \pi]$, which are boundary values of elements of H^p, i.e., can be represented in the form (see (31))

$$\Phi(x) = \lim_{r\to 1} F(re^{ix}) \quad \text{for almost all } x \in [-\pi, \pi], \ F \in H^p.$$

By parts 2) and 3) of Theorem 4, $\mathcal{H}^p \subset L^p(-\pi, \pi)$, and every function $\Phi \in \mathcal{H}^p$ satisfies (16). On the other hand, we showed above that, for any $\Phi \in L^p(-\pi, \pi)$, under condition (16) the Poisson integral (17) defines an element of H^p. Consequently (also the Theorem 2)

$$\mathcal{H}^p = \left\{ \Phi \in L^p(-\pi, \pi) \colon \int_{-\pi}^{\pi} \Phi(x) e^{inx}\, dx = 0, \ n = 1, 2, \ldots \right\}. \qquad (34)$$

It follows from (34) that $\mathcal{H}^p$ is a (closed) subspace of $L^p(-\pi,\pi)$, and H^p is a Banach space with the norm (15) (also see Theorem 4, 3).

Let $1 \le p \le \infty$. Set

$$H_0^p = \{F \in H^p : \operatorname{Im} F(0) = 0\},$$
$$\mathcal{H}_0^p = \left\{\Phi \in \mathcal{H}^p : \int_{-\pi}^{\pi} \operatorname{Im}\Phi(x)\,dx = 0\right\}$$

and

$$\operatorname{Re}\mathcal{H}^p = \{f(x) : f(x) = \operatorname{Re}\Phi(x),\ x \in [-\pi,\pi],\ \Phi \in \mathcal{H}^p\}. \tag{35}$$

We have the following theorem.

THEOREM 5. *For $1 \le p \le \infty$, the following conditions are equivalent*:

a) $\Phi(x) \in \mathcal{H}_0^p$;

b) $\Phi(x) \in L^p(-\pi,\pi)$, $\int_{-\pi}^{\pi} \operatorname{Im}\Phi(x)\,dx = 0$, $\int_{-\pi}^{\pi} \Phi(x)e^{inx}\,dx = 0$, $n = 1, 2, \dots$;

b) $F(re^{ix}) = \Phi * P_r(x) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(x+t)P_r(t)\,dt \in H_0^p$;

d) $\Phi(x) = f(x) + i\tilde{f}(x)$, *where* $f \in L^p(-\pi,\pi)$

is a real function whose conjugate $\tilde{f}(x)$ (*see Definition* 5.2) *also belongs to* $L^p(-\pi,\pi)$:

$$\tilde{f}(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)}{2\tan\frac{x-t}{2}}\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)-f(x)}{2\tan\frac{x-t}{2}}\,dt \in L^p(-\pi,\pi). \tag{36}$$

PROOF. The equivalence of a) and b) follows immediately from (34), and the equivalence of a) and c) follows from Theorems 4 and 2.

Let us show that d) implies b). It is enough to show that when the function and its conjugate are summable (i.e., $f, \tilde{f} \in L^1(-\pi,\pi)$) we have the equations

$$c_n(\tilde{f}) = i(\operatorname{sgn} n)c_n(f), \qquad n = 0, \pm 1, \pm 2, \dots \tag{37}$$

(see (9)). A direct calculation from (36) shows that $\widetilde{\cos nx} = \sin nx$, $n = 0, 1, \dots$, and that $\widetilde{\sin nx} = -\cos nx$, $n = 1, 2\dots$—in this case

$$\frac{f(t)-f(x)}{2\tan\frac{1}{2}(t-x)} \in L^1.$$

Consequently (37) is satisfied for every trigonometric polynomial.

Let the number $n = 0, \pm 1, \pm 2, \ldots$ be given. For an arbitrary $f \in L^1(-\pi, \pi)$ and $k = 1, 2, \ldots$, we set

$$I_{f,k}(t) = \frac{1}{2k} \sum_{j=-k+1}^{k} f(x_j + t), \qquad t \in [-\pi, \pi],$$

$$I^*_{f,k}(t) = \frac{1}{2k} \sum_{j=-k+1}^{k} \widetilde{f}(x_j + t) e^{-in(x_j+t)},$$

where $x_j = \frac{\pi j}{k}$, $j = -k+1, -k+2, \ldots, k$.

Let us show that for a given index n equation (37) is a consequence of the following properties of $I_{f,x}(t)$ and $I^*_{f,k}(t)$ (that $I_{f,k}(t)$ and $I^*_{f,k}(t)$ have these properties will be established later):

1) $m\{t \in (-\pi, \pi) \colon |I_{f,k}(t)| > \varepsilon\} \geq \frac{1}{\varepsilon} \|f\|_{L^1(-\pi,\pi)}$, $\varepsilon > 0$, $k = 1, 2, \ldots$.

2) As $k \to \infty$ the function $I_{f,k}(t)$, $t \in (-\pi, \pi)$, converges in measure to $I(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \text{const}$.

3) $m\{t \in (-\pi, \pi) \colon |I^*_{j,k}(t)| > \varepsilon\} \leq \frac{C}{\varepsilon} \|f\|_{L^1(-\pi,\pi)}$, $\varepsilon > 0$, $k = 1, 2, \ldots$, where $C > 0$ is an absolute constant.

Thus we assume that 1) and 3) hold.

It is easily seen that $I^*_{f,k}(t) = I_{g,k}(t)$, where $g(x) = \widetilde{f}(x) e^{-inx} \in L^1(-\pi, \pi)$; hence 2) implies the convergence in measure of the sequence of functions $I^*_{f,k}(t)$, $k = 1, 2, \ldots$:

$$I^*_{f,k}(t) \to c_n(\widetilde{f}) \quad \text{in measure } (k \to \infty). \tag{38}$$

For an arbitrary $\varepsilon \in (0, 1)$, we can find a trigonometric polynomial $f_1(x)$ such that

$$f(x) = f_1(x) + f_2(x), \qquad \|f_2\|_{L^1(-\pi,\pi)} \leq \frac{\varepsilon^2}{4(C+1)}. \tag{39}$$

Then by 3)

$$m\left\{t \in (-\pi, \pi) \colon I^*_{f_2,k}(t)| > \frac{\varepsilon}{2}\right\} \leq \frac{2C}{\varepsilon} \|f_2\|_{L^1(-\pi,\pi)} \leq \frac{\varepsilon}{2} \tag{40}$$

and (see (38)) for $k > k_0$

$$m\left\{t \in (-\pi, \pi) \colon |I^*_{f_1,k}(t) - c_n(\widetilde{f_1})| > \frac{\varepsilon}{4}\right\} \leq \frac{\varepsilon}{2}. \tag{41}$$

Since $f_1(x)$ is a polynomial, we have $c_n(\widetilde{f_1}) = -i(\operatorname{sgn} n) c_n(f_1)$ and (see (39))

$$\begin{aligned} c_n(\widetilde{f_1}) + i(\operatorname{sgn} n) c_n(f)| &\leq |c_n(f_1) - c_n(f)| \\ &= |c_n(f_2)| \leq \|f_2\|_{L^1(-\pi,\pi)} \leq \frac{\varepsilon}{4}. \end{aligned} \tag{42}$$

Using the equation $I^*_{f,k} = I^*_{f_1,k} + I^*_{f_2,k}$, and inequalities (40)–(42), we obtain

$$m\{t \in (-\pi,\pi): |I^*_{f,k}(t) + i(\operatorname{sgn} n)c_n(f)| > \varepsilon\} \le \varepsilon, \qquad k > k_0,$$

which, with (38), establishes (37).

We now show that every function $f \in L^1(-\pi,\pi)$ satisfies 1)–3). Inequality 1) follows at once from Tchebycheff's inequality, since

$$\|I_{f,k}(t)\|_{L^1(-\pi,\pi)} \le \frac{1}{2k} \sum_{j=-k+1}^{k} \|f(x_j+t)\|_{L^1(-\pi,\pi)} = \|f\|_{L^1(-\pi,\pi)}. \tag{43}$$

To prove 2), we take an arbitrary $\varepsilon \in (0,1)$ and represent $f(x)$ in the form

$$f(x) = g(x) + h(x), \qquad g \in C(-\pi,\pi),\ \|h\|_{L^1(-\pi,\pi)} \le \frac{\varepsilon^2}{2}.$$

Since g is continuous, it follows easily that

$$\lim_{k\to\infty} I_{g,k}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx$$

uniformly in $t \in (-\pi,\pi)$. Therefore, for sufficiently large k ($k > k'$) we will have, taking account of (43),

$$\begin{aligned} \left|I_{g,k}(t) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx\right| &\le \left|I_{g,k}(t) - \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx\right| \\ &\quad + \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} h(x)\,dx\right| \le \frac{\varepsilon}{2}, \qquad t \in (-\pi,\pi). \end{aligned} \tag{44}$$

In addition, by 1) and (43)

$$m\left\{t \in (-\pi,\pi): |I_{h,k}(t)| > \frac{\varepsilon}{2}\right\} \le \frac{2}{\varepsilon}\|h\|_{L^1(-\pi,\pi)} \le \varepsilon;$$

it follows form this inequality and (44) that when $k > k'$

$$\begin{aligned} &m\left\{t \in (-\pi,\pi): \left|I_{f,k}(t) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx\right| > \varepsilon\right\} \\ &\qquad \le m\left\{t \in (-\pi,\pi): |I_{h,k}(t)| > \frac{\varepsilon}{2}\right\} \le \varepsilon. \end{aligned}$$

To prove 3), we observe that

$$|I^*_{f,k}(t)| = \left|\frac{1}{2k}\sum_{j=-k+1}^{k} \widetilde{f}(x_j+t)e^{-inx_j}\right| = |\widetilde{F}(t)|,$$

where

$$F(t) = \frac{1}{2k}\sum_{j=-k+1}^{k} f(x_j+t)e^{-inx_j}.$$

We obtain 3) by applying inequality a) from Theorem 5.3 to $F(t)$ and using the inequality $\|F\|_{L^1(-\pi,\pi)} \le \|f\|_{L^1(-\pi,\pi)}$.

Properties 1)–3) are now established. Hence we have established that condition d) of Theorem 5 implies b). To complete the proof of Theorem 5, it is enough to show that c) implies d).

Let $\Phi(x) = f(x) + ig(x)$ ($x \in (-\pi,\pi)$, $f = \operatorname{Re}\Phi$, $g = \operatorname{Im}\Phi$) and $F(re^{ix}) = \Phi * P_r(x) \in H_0^p$. Then $f, g \in L^p(-\pi,\pi)$ (see Theorem 4), and we need only prove that $g(x) = \widetilde{f}(x)$ for almost all $x \in (-\pi,\pi)$.

Since the Poisson kernel is a real function, we may suppose that for $0 \le r < 1$ and $x \in [-\pi,\pi]$ (see (3))

$$u(re^{ix}) = \operatorname{Re} F(re^{ix}) = f_r(x), \qquad v(re^{ix}) = \operatorname{Im} F(re^{ix}) = g_r(x).$$

On the other hand, it follows from (2), (8), and (37) that for every r, $0 < r < 1$,

$$g_r(x) = \widetilde{f_r}(x), \qquad x \in (-\pi,\pi). \tag{45}$$

By Theorem 1,

$$\lim_{r\to 1} \|f - f_r\|_{L^1(-\pi,\pi)} = \lim_{r\to 1} \|g - g_r\|_{L^1(-\pi,\pi)} = 0. \tag{46}$$

In addition, by Theorem 5.3, it follows from the limit $f_r \xrightarrow{L_1} f\,(r \to 1)$ that $\widetilde{f_r}(x)$ converges to $\widetilde{f}(x)$ in measure. Therefore (see (45)) $g_r(x) \to \widetilde{f}(x)$ in measure ($r \to 1$), and hence (see (46)) $g(x) = \widetilde{f}(x)$ for almost all $x \in [-\pi,\pi]$. This completes the proof of Theorem 5.

COROLLARY 1. a) *If* $f \in \operatorname{Re}\mathcal{H}^1$, *then* $\widetilde{f} \in \operatorname{Re}\mathcal{H}^1$.

b) *If* $f \in \operatorname{Re}\mathcal{H}^1$ *and* $\int_{-\pi}^{\pi} f(x)\,dx = 0$ *then* $f = -(\widetilde{\widetilde{f}})$.

c) *If* $f \in \operatorname{Re}\mathcal{H}^p$ *and* $g \in \operatorname{Re}\mathcal{H}^q$, *where* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $1 \le p \le \infty$, *then*

$$\int_{-\pi}^{\pi} f(x)\widetilde{g}(x)\,dx = -\int_{-\pi}^{\pi} \widetilde{f}(x) g(x)\,dx. \tag{47}$$

PROOF. Relations a) and b) follow at once from the equivalence of conditions a) and d) of Theorem 5.

To obtain c), we set

$$F(re^{ix}) = \{f + i\widetilde{f}\} * P_r(x),$$
$$G(re^{ix}) = \{g + i\widetilde{g}\} * P_r(x).$$

According to Theorem 5, we have $F \in H_0^p$ and $G \in H_0^q$, and consequently $F \cdot G \in H_0^1$. But then (for almost all $x \in (-\pi,\pi)$)

$$\Phi(x) = (f + i\widetilde{f})(g + i\widetilde{g}) = \lim_{r\to 1} F(re^{ix})G(re^{ix}) \in \mathcal{H}_0^1,$$

and we obtain (see the definition of $\mathcal{H}_0^p$)

$$\int_{-\pi}^{\pi} \operatorname{Im} \Phi(x)\, dx = 0. \tag{48}$$

Equation (47) follows immediately from (48).

REMARK 1. If $1 < p < \infty$, then by, Theorem 5, d) and Theorem 5.3, the space $\operatorname{Re} \mathcal{H}^p$ coincides with $L^p(-\pi, \pi)$. For $p = 1$ this is not the case. The space $\operatorname{Re} \mathcal{H}^1$ is narrower than $L^1(-\pi, \pi)$ and consists (see Theorem 5, d)) of the functions for which $\widetilde{f} \in L^1(-\pi, \pi)$. The space $\operatorname{Re} \mathcal{H}^1$ is a Banach space with the norm

$$\|f\|_{\operatorname{Re} \mathcal{H}^1} = \|f\|_{L^1(-\pi,\pi)} + \|\widetilde{f}\|_{L^1(-\pi,\pi)}. \tag{49}$$

(The completeness of $\operatorname{Re} \mathcal{H}^1$ with the norm (49) follows from Theorem 5.3 and the completeness of $L^1(-\pi, \pi)$: if $\|f_n - f_m\|_{\operatorname{Re} \mathcal{H}^1} \to 0$ as $n, m \to \infty$, then $f_n \xrightarrow{L_1} f$, $\widetilde{f}_n \xrightarrow{L^1} g$, and $f, g \in L^1(-\pi, \pi)$; and since $f_n \to \widetilde{f}$ in measure as $n \to \infty$, we have $g = \widetilde{f}$ and $\|f_n - f\|_{\operatorname{Re} \mathcal{H}^1} \to 0$ as $n \to \infty$.)

REMARK 2. According to Remark 1, equation (47) is satisfied, in particular, in the case when $f \in L^p(-\pi, \pi)$ and $g \in L^q(-\pi, \pi)$, $1/p + 1/q = 1$, $1 < p < \infty$.

We also notice that if we take $\widetilde{f}(x)$ in (47) instead of $f(x)$ and use b), we obtain

$$\int_{-\pi}^{\pi} f(x) g(x)\, dx = \int_{-\pi}^{\pi} \widetilde{f}(x) \widetilde{g}(x)\, dx, \quad \text{if } \int_{-\pi}^{\pi} f(x)\, dx = 0. \tag{50}$$

§3. The Blaschke product and the nontangential maximal function

Let a sequence $\{a_n\}_{n=1}^{\infty}$ of nonzero complex numbers (not necessarily all different) satisfy the condition

$$|a_n| < 1, \qquad n = 1, 2, \ldots, \qquad \sum_{n=1}^{\infty} (1 - |a_n|) < \infty. \tag{51}$$

We consider the product (*Blaschke product*)

$$B(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - \overline{a}_n z} \cdot \frac{\overline{a}_n}{|\overline{a}_n|} \equiv \prod_{n=1}^{\infty} b(z, a_n). \tag{52}$$

For a given r, $0 < r < 1$, we have, for $|z| < r$,

$$|1 - b(z, a_n) \cdot |a_n|| = \frac{1 - |a_n|^2}{|1 - \overline{a}_n z|} \le \frac{2(1 - |a_n|)}{1 - r}. \tag{53}$$

Since the series in (51) converges, it is easily seen from (53) that the product (52) converges absolutely and uniformly in the disk $|z| \leq r$, i.e., $B(z)$ is analytic in the unit disk and has zeros at the points a_n, $n = 1, 2, \ldots$, and only at these points. Moreover, by using the inequality $|b(z, a_n)| \leq 1$ ($|z| < 1, n = 1, 2, \ldots$) we obtain

$$|B(z)| \leq 1, \qquad |z| \leq 1. \tag{54}$$

Suppose now that $a_1, a_2, \ldots$ ($|a_n| < 1$) are the zeros of a function $F(z) \in H^1$ with $F(0) \neq 0$, and that each a_n is repeated according to its multiplicity. Let us show that then (51) converges. Set

$$B_m(z) = \prod_{n=1}^{m} \frac{z - a_n}{1 - \bar{a}_n z}, \qquad m = 1, 2, \ldots.$$

The function $B_m(z)$ ($m = 1, 2, \ldots$) is analytic in a disk of radius greater than 1, and $|B_m(z)| = 1$ if $|z| = 1$. Consequently $F_m(z) = F(z) \cdot (B_m(z))^{-1} \in H^1$ and (see Theorem 4, 3)) $\|F_m\|_{H^1} = \|F\|_{H^1}$. But then

$$\left| \frac{F(0)}{\prod_{n=1}^{m} a_n} \right| = |F_m(0)| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} F_m\left(\frac{1}{2} e^{it}\right) dt \right| \leq \|F_m\|_{H^1} = \|F\|_{H^1}$$

and

$$\prod_{n=1}^{m} |a_n| \geq \frac{|F(0)|}{\|F\|_{H^1}} > 0, \qquad m = 1, 2, \ldots. \tag{55}$$

Since $|a_n| < 1$, $n = 1, 2, \ldots$, it follows from (55) that the product $\pi_1^{\infty} |a_n|$ converges, and therefore that the series (51) converges.

Let $F(z)$ be analytic in the disk $|z| < 1$ and let a_n, $n = 1, 2, \ldots$ ($0 < |a_n| < 1$) be its zeros, counted according to multiplicity. Also let $p \geq 0$ be the multiplicity of the zero of F at $z = 0$. The product (see (52))

$$B(z) = z^p \prod_n b(z, a_n) \tag{56}$$

is the Blaschke product of $F(z)$.

We have the following theorem.

THEOREM 6. *Every function $F \in H^1$ can be represented in the form*

$$F(z) = B(z) \cdot G(z),$$

where $G(z)$ has no zeros in the disk $|z| < 1$ and

$$G \in H^1, \|G\|_{H^1} = \|F\|_{H^1},$$

and $B(z)$ is the Blaschke product of $F(z)$.

PROOF. Let a_n, $n = 1, 2, \dots$ $(0 < |a_n| < 1)$, be the zeros of F (or, equivalently, of $F/z^p \in H^1$). Then, as we noticed above, $B(z)$ is analytic in the disk $|z| < 1$, and

$$|B(z)| < 1, \qquad |z| < 1. \tag{57}$$

Moreover, $G(z) = F(z)/B(z)$ is also analytic in the unit disk and has no zeros there, and (see (57)) $\|G\|_{H^1} \geq \|F\|_{H^1}$.

To prove the inequality in the opposite direction, we consider the partial products of (56):

$$B_m(z) = z^p \prod_{n=1}^{m} b(z, a_n), \qquad m = 1, 2, \dots, \ |z| \leq 1.$$

Since $|B_m(e^{ix})| = 1$ for every $x \in [-\pi, \pi]$, then by Theorem 4

$$\left\| \frac{F}{B_m} \right\|_{H^1} = \|F\|_{H^1}$$

and

$$\int_{-\pi}^{\pi} \left| \frac{F(re^{ix})}{B_m(re^{ix})} \right| dx \leq \|F\|_{H^1}, \quad \text{if } 0 < r < 1.$$

If we let m tend to infinity in the preceding inequality and use the fact that $B_m(re^{ix}) \to B(re^{ix})$ as $m \to \infty$, uniformly for $x \in [-\pi, \pi]$, we obtain

$$\int_{-\pi}^{\pi} \left| \frac{F(re^{ix})}{B(re^{ix})} \right| dx \leq \|F\|_{H^1}, \qquad 0 < r < 1,$$

i.e., $G \in H^1$ and $\|G\|_{H^1} = \|F\|_{H^1}$. This completes the proof of Theorem 6.

Let σ, $0 \leq \sigma < 1$, be an arbitrary number. We denote by $\Omega_\sigma(x)$, $x \in [-\pi, \pi]$, the region bounded by two tangents to the circle $|z| = \sigma$ from e^{ix} and the longer arc of the circle included between the points of tangency (when $\sigma = 0$, $\Omega_\sigma(x)$ degenerates to a radius of the unit disk). For $f \in L^1(-\pi, \pi)$ we set

$$f_\sigma^*(x) = \sup_{re^{i\theta} \in \Omega_\sigma(x)} |f_r(\theta)|, \qquad x \in [-\pi, \pi],$$

where $f_r(x)$ is the Poisson integral of $f(x)$ (see (3)).

The function $f^*(x)$ is the nontangential maximal function of $f(x)$. By Theorem 2,

$$|f(x)| \leq f_\sigma^*(x) \quad \text{for almost all } x \in [-\pi, \pi]. \tag{58}$$

We shall show that for any $f \in L^1(-\pi, \pi)$ the number $f_\sigma^*(x)$ does not exceed (in order) the value of the maximal function $M(f)$(2) at x, i.e.,

$$f_\sigma^*(x) \le C_\sigma M(f, x), \qquad x \in [-\pi, \pi]. \tag{59}$$

Let $re^{i\theta} \in \Omega_\sigma(x)$ and $x - \theta = \xi$. By the definition of the Poisson integral,

$$f_r(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta + t)P_r(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)P_r(t+\xi)\,dt.$$

Let $F(t) = \int_0^t f(x+u)\,du$. Then we will have

$$\begin{aligned} 2\pi f_r(\theta) = \int_{-\pi}^{\pi} P_r(t+\xi)\,dF(t) &= P_r(\pi+\xi)F(\pi) \\ &- P_r(-\pi+\xi)F(-\pi) - \int_{-\pi}^{\pi} F(t)P_r'(t+\xi)\,dt \end{aligned}$$

and, by the inequality $|F(t)| \le |t| \cdot M(f, x)$, $t \in [-\pi, \pi]$, and the periodicity of $P_r(t)$,

$$2\pi|f_r(\theta)| \le M(f, x)\left[2\pi P_r(\pi+\xi) + \int_{-\pi}^{\pi} |tP_r'(t+\xi)|\,dt\right]. \tag{60}$$

Since the functions $g(t) = -t$ and $g_1(t) = P_r'(t)$ are positive for $t \in (-\pi, 0)$ and negative for $t \in (0, \pi)$ (see (5)), then, assuming without loss of generality that $\xi > 0$, we obtain

$$\int_{-\pi}^{\pi} |tP_r'(t+\xi)|\,dt = -\int_{-\pi}^{-\xi} + \int_{-\xi}^{0} - \int_{0}^{\pi-\xi} + \int_{\pi-\xi}^{\pi} tP_r'(t+\xi)\,dt. \tag{61}$$

For $-\pi \le a < b \le \pi$ we have the inequalities

$$2\pi \ge \int_a^b P_r(t+\xi)\,dt = P_r(t+\xi)t|_a^b - \int_a^b tP_r'(t+\xi)\,dt,$$

$$\left|\int_a^b tP_r'(t+\xi)\,dt\right| \le 2\pi + |b|P_r(b+\xi) + |a|P_r(a+\xi).$$

Consequently (see (60) and (61)), in order to prove (59) it is enough to verify that

$$|t|P_r(t+\xi) \le C_\sigma \quad \text{for } t = -\pi, -\xi, 0, \pi - \xi, \text{ and } \pi \tag{62}$$

if $re^{i\xi} \in \Omega_\sigma(x)$. Let $t = -\xi$; then

$$|t|P_r(t+\xi) = \xi P_r(0) \le 2\frac{x-\theta}{1-r} \le C_\sigma.$$

(2)Since $M(f, x)$ was defined in Appendix 1 for a function f defined on R^1, we also set $f(x) = 0$ if $|x| > 2\pi$, $f(x) = f(x + 2\pi)$ for $-2\pi \le x < \pi$, and $f(x) = f(x - 2\pi)$ for $\pi < x \le 2\pi$.

In the remaining cases, (62) is evident. It follows from (58), (59), and Theorem 2 of Appendix 1 that for every $f \in L^p(-\pi, \pi)$, $1 < p < \infty$,

$$\|f\|_{L^p(-\pi,\pi)} \le \|f_\sigma^*\|_{L^p(-\pi,\pi)} \le C_{\sigma,p}\|f\|_{L^p(-\pi,\pi)}, \tag{63}$$

where $C_{\sigma,p}$ is a constant depending only on σ and p.

THEOREM 7. *Let $F(z) \in H^p$ $(1 \le p < \infty)$, $0 \le \sigma < 1$, and*

$$F_\sigma^*(x) = \sup_{z \in \Omega_\sigma(x)} |F(z)|, \qquad x \in [-\pi, \pi].$$

Then $F_\sigma^*(x) \in L^p(-\pi, \pi)$ and

$$\|F\|_{H^p} \le \|F_\sigma^*\|_{L^p(-\pi,\pi)} \le C_{p,\sigma}\|F\|_{H^p}. \tag{64}$$

PROOF. In the case $1 < p < \infty$, Theorem 7 is a direct corollary of (63) and Theorem 4. Now let $F(z) \in H^1$. By Theorem 6 we have $F(z) = B(z) \cdot G(z)$, where $|B(z)| \le 1$, $G(z) \ne 0$ if $|z| < 1$, and $\|G\|_{H^1} = \|F\|_{H^1}$. We may take the square root of $G(z)$: there is a function $E(z) \in H^2$ such that $E^2(z) = G(z)$ and consequently (see (64) for $p = 2$)

$$\int_{-\pi}^{\pi} F_\sigma^*(x)\,dx \le \int_{-\pi}^{\pi} G_\sigma^*(x)\,dx = \int_{-\pi}^{\pi} [E_\sigma^*(x)]^2\,dx$$
$$\le C_{\sigma,2}^2\|E\|_{H^2}^2 = C_\sigma\|G\|_{H^1} = C_\sigma\|F\|_{H^1}.$$

The lower bound for $\|F_\sigma^*\|_{L^1(-\pi,\pi)}$ follows from (58). This completes the proof of Theorem 7.

Notes

First of all we observe that the theory of orthogonal series is treated in a number of books: Kaczmarz and Steinhaus [64] (first edition in 1935, in German; Russian translation, supplemented by a wider survey of the literature, by R. S. Guter and P. L. Ul'yanov, in 1958), Alexits [2] and Olevskiĭ [38]. These books throw light on several topics on which we hardly touch (for example, summability of orthogonal series, and series and Fourier coefficients of functions in L^p, $p \neq 2$, with respect to general O.N.S.). In [38] there is also a survey of a number of directions in the theory of orthogonal series that were developed between 1960 and 1975. In this connection, we should say that the monographic literature on the theory of general orthogonal series can hardly be called extensive. It does not fill all the gaps in the material that has been investigated, and this book, in particular, represents only to a small extent the work of the Hungarian school of the theory of orthogonal series.

We now give bibliographic references to the material that we do consider.

Chapter 1

This chapter is introductory; the results presented here we obtained before 1935. We restrict ourselves to brief indications.

Theorem 1 was proved by Orlicz [143]. Theorem 2 was probably first published in [64] (see [64], p. 264). Theorems 3 and 4 belong to the foundations of the theory of real functions and are always included in university curricula; for their history see [2]. Theorem 5 was obtained as early as 1909 by Lebesgue (see [2], Chapter IV, §1). Theorem 6 was proved by Banach [9], to whom also belongs (in essentials) Theorem 7 (for details see [170] and [35]). Theorem 8 was actually proved by Haar [54] as early as 1910, although the concept of a basis was itself introduced later by Schauder (see [9] and [170]). A special case of Theorem 9 (for the Haar system; see Chapter 3) was discussed by Schauder [164]; the proof in the general case is similar. Theorem 10 is a simple corollary of Theorems 6 and 1. Finally, for Theorem 11 see the notes on Chapter 2.

Chapter 2

In §1 and §2 we introduce and study sequences of independent functions. In the language of probability theory, the fundamental definition of this chapter, Definition 1, is nothing but the definition of a set of independent random variables $\{f_n\}$ defined on a probability space (Ω, F, P), where $\Omega = (0,1)$, F is the system of Borel sets, and P is Lebesgue measure on $(0,1)$. All the theorems of §1 and §2 are very often formulated in probabilistic language. For example, Theorem 4 states that the expectation of the product of two independent random variables is the product of their expectations. The method that we used for constructing sequences of independent functions were used in the 1930's in the work of the Polish school (Steinhaus, Marcinkiewicz, Zygmund, etc.; for details see [105], p. 235, and [64]); and for the discrete case by Khinchin and Kolmogorov [83]. This method (possibly somewhat out of date) is closely related to the theory of functions and has the advantage that it does not require any additional knowledge of measure theory. Theorems 1–10 of §1 and §2 are related to the general theory of probability; for detailed comments on them we refer the reader to monographs on this theory. We restrict ourselves to a few remarks.

The Rademacher functions $r_n(x)$, $n = 1, 2, \dots$, were introduced in 1922 by Rademacher [157], but the behavior of the sums $\sum_1^N r_n(x)$ as $N \to \infty$ and for $x \in (0,1)$ was investigated considerably earlier in connection with binary decompositions of the real line (see, for example, [97], p. 42). Theorems 5 and 6 were obtained for the special case of the Rademacher system by Khinchin [82] in 1923, and later were extended to other systems of independent functions (in the most general form, Khinchin's inequality (Theorem 6) was proved in 1937 by Marcinkiewicz and Zygmund (see [105], p. 257). Theorem 7 was obtained by Paley and Zygmund [147] as an immediate corollary of Khinchin's inequality and the simple, but important, Lemma 1 (see Theorem 7), which was probably first formulated in [147]. Theorem 8 is due to Kolmogorov [83] (inequality (47) was not mentioned in [83], but, like the statement of Theorem 7, is easily deduced form the results of §3 of [83]). Theorem 9 was also proved by Kolmogorov [87] (the case of the Rademacher system had already been discussed in

[157]). In [87] there appears the inequality (Kolmogorov's inequality)

$$m\{x \in (0,1): S_\psi^*(\{a_n\}) > y\} \leq \frac{1}{y^2} \sum a_n^2; y > 0, \qquad (*)$$

which is hardly less precise than (47). It should be noticed that the method of estimating the majorant, proposed in [87] for proving (*), has turned out to be very important in both probability theory and function theory. We often use it in Chapter 3. Inequality (47) in Theorem 9 was proved by Marcinkiewicz and Zygmund (see [105], p. 238), but follows at once from a version of (*) that was obtained in [88].

The application of properties of systems of independent functions to the study of series of functions was initiated almost simultaneously by Orlicz [140]–[142] and Paley and Zygmund [146], [147] (see also Littlewood [100], [101]). At the present time, random series play a fundamental role in the theory of orthogonal series and in functional analysis (see in particular [65], [17], [169], and [107]), and are frequently used in the present book.

In Theorem 11 the implication 2) $\Rightarrow$ 3) was proved by Orlicz [142], and 3) $\Rightarrow$ 1), by Paley and Zygmund [146]. Corollary 4 was given by Orlicz [142]. In Theorem 12 the necessity of condition (72) for the convergence of the series $\sum_1^\infty f_n(x)$ in L^p for almost all choices of signs was obtained by Orlicz [143], and the sufficiency (in the slightly weaker form of Corollary 5) by Paley and Zygmund [146]. Theorem 1.11, stated in Chapter 1 and proved in §2.3, follows easily from Theorems 12 and 1.10. Theorem 12 is Pełczyński's [151] (our proof is close to that given in [122]). Theorem 14 and Corollary 6 were proved by Kashin [70]. The similar result about convergence in measure (see the remark at the end of §3) was obtained in [70] and independently by Maurey and Pisier [113]. Lemma 1 from Theorem 14 (with an inexact constant in (84)) is contained in Drobot [38] (also see Theorem 10.5). Our proof provides the exact constant in (84).

The study of random rearrangements of series of functions (see §4) was initiated by Garsia [48], [49]. Theorem 16 was proved by Garsia in [48], and Corollary 8 and part b) of Corollary 7, in [49]. Theorem 15 and Corollary 7, a) (see [80], Russian p. 386, English p. 51), represent a sharpening of estimates in [49]. The approach used here (and in [49]) for the study of random permutations of sets of numbers, based on analogy with the proof of Kolmogorov's inequality (*), was suggested by Rosén [159].

Chapter 3

The Haar system was introduced in Haar's dissertation (see [54]) and is now widely used in the theory of functions, as well as in probability theory and numerical mathematics. From the point of view of probability theory the partial sums $S_N(x)$, $N = 1, 2, \ldots$, of an arbitrary series $\sum_1^\infty a_n\chi_n(x)$ in the Haar system are a special case of a sequence of random variables that form a martingale (on martingales see, for example, [36], [169]). Martingales are extensively used in probability theory, and some of the theorems proved in Chapter 3 were first established in probabilistic terms for martingales. We also notice that in many cases propositions on properties of the Haar system can be generalized to martingales without the introduction of essentially new ideas.

The formulas for the partial sums of Fourier–Haar series were obtained by Haar himself [54]. Inequalities for Fourier–Haar coefficients (see Theorem 1) were noticed for $f \in C(0, 1)$ by Ciesielski [24], and for $f \in L^p(0, 1)$, $1 \le p < \infty$, by Ul′yanov [189], who initiated the systematic study of the Haar system in the USSR. The uniform convergence of the Fourier–Haar series of continuous functions (see Theorem 2) was established by Haar (the construction of an O.N.S. with this property was one of Haar's original objectives; see [54]); inequality (14) in Theorem 2 was noticed by B. Szökefalvi–Nagy in [175], which was devoted to general O.N.S. That the Haar system forms a basis in $L^p(0, 1)$, $1 \le p < \infty$ (see Theorem 3) was established by Schauder [164], and inequality (17) by Ul′yanov [189]. The convergence a.e. of arbitrary Fourier–Haar series (see Theorem 4) was established by Haar [54], and the properties of the majorants of the partial sums of these series were observed by Marcinkiewicz (see [105], p. 310). Theorem 5 was proved by Golubov [50], and Theorem 6 by Bochkarev [11] (also see [17]).

The study of unconditional convergence of Fourier–Haar series in $L^p(0, 1)$ (see §3) was initiated by Marcinkiewicz (see [105], p. 308), who obtained the statement of Theorem 8 and inequality 2) of Theorem 9 as direct consequences of Paley's results on the Walsh system. Theorem 7 and inequality 1) in Theorem 9 were established by Yano [200]; our proof of Theorem 7 was given by Watari [196] (also see [61]). In Theorem 10, statement 2) follows from Theorem 2.13, and 3) was obtained by Ul′yanov [191] and then extended by Olevskiĭ (see [138], p. 75) to general C.O.N.S. Theorem 11 was proved at the same time for matrices by Burkholder and Gundy [18] and Davis [34]. We note that the method of proof of Theorem 11 (see, in particular, Lemma 1) is a typical method of martingale theory, and has features in common with the classical proof of Kolmogorov's inequality (see Theorem 2.9 and the notes to Chapter 2). Theorem 12 can be obtained as a corollary of general interpolation theorems; the direct proof in the text was suggested by Saakyan. Theorem 13 was proved independently by Arutyunyan [4] and (for martingales) by Gundy [53]. Theorem 14 follows from results of Chow [23] on martingales and was also obtained by Arutyunyan [4] (also see [53] and [180]).

The first result on the unconditional convergence a.e. of Fourier–Haar series was obtained by Ul'yanov: in [184] he proved Corollaries 6 and 7 (see §5). Theorem 15 was proved by Nikishin and Ul'yanov [130], and Theorem 16 by Olevskiĭ [132], [133]. Theorem 17 and Corollary 8 were established by Ul'yanov [187], [188], [192]; and Theorem 18, by Bochkarev [12].

Transformations of the Haar system similar to those considered in §6 were already used by Schauder [164] for the construction of bases in $L^p(\Omega)$ (where Ω is a bounded domain in R^n). Corollary 9 was obtained by Olevskiĭ (see [138], p. 61, and also the remarks on Chapter 9, below).

In conclusion, we note that a number of results on the Haar system, not included here, are given in the survey by Golubov [51].

Chapter 4

We consider the trigonometric system in §§1–4. There is immense literature on trigonometric series, including the important monographs of Bari [10] and Zygmund [201], both of which contain extensive bibliographies.

In Chapter 4, §1 consists of standard material. Theorem 5 (see §2) was proved by Jackson in 1911. Jackson actually considered only continuous functions, but his proof applies without essential changes to functions in $L^p(-\pi, \pi)$, $1 \leq p < \infty$ (for details see [201] and [156]). Our proof of Theorem 3 is close to that in [22], and is based on properties of the Vallée-Poussin means. We note that the Vallée-Poussin means, introduced by Vallée–Poussin in 1918, play an extremely important role in approximation theory (see [202], "Vallée–Poussin method of summation," and [39]).

Theorem 4 was proved by M. Riesz (see [10], Chapter VIII, §19). Corollary 3 was proved independently by Konyagin [93] and McGehee, Pigno and Smith [114]. Theorem 5 was also obtained in [114], whose method we follow. Carleson's theorem was proved in [19]. Later, Hunt [60] proved by Carleson's method that the Fourier series of every function in $L^p(-\pi, \pi)$, $p > 1$, converges almost everywhere. The existence of a function $f \in L^1(-\pi, \pi)$ whose Fourier series diverges a.e. was discovered by Kolmogorov as early as 1923 (see [10], Chapter V, §17). Theorem 6 was obtained in 1925 by Kolmogorov and Seliverstov, and also by Plessner (see [10], Chapter V, §2).

Theorem 7 was obtained in the last century by Dini and Lipschitz (see [202], "Dini–Lipschitz test"). The study of the strong summability of Fourier series was initiated by Hardy and Littlewood; the fundamental results on strong summability were obtained by Marcinkiewicz; for details see [10], Chapter VII, §6; Theorem 8 was proved by Alexits and Králik [3] (also see [66]). Theorem 9 was proved by H. Bohr by complex-variable methods; our proof was suggested by Saakyan [160] (also see [161] and [66]). Theorem 10 was obtained by Paley and Zygmund (see [10], Chapter IV, §13); generalizations of Theorem 10 and Proposition 2 were given in [106]. Theorem 11 was proved independently by Shapiro and by Rudin (see [67], p. 133). Corollary 5 was established by S. N. Bernstein, and Corollary 6 by Zygmund; generalizations of the results of Theorem 12 were obtained by Szász (for details see [10], Chapter IX, §2).

The Walsh system was introduced by Walsh [195] in 1923; at present, it is used in many branches of mathematics, and especially in applied problems. The enumeration of the Walsh functions used in §5 (and in most of the work on the Walsh system) was suggested by Paley [145], who proved many of the fundamental properties of the Walsh system (in particular, Theorem 15). Inequality (115) for the Lebesgue functions of the Walsh system was obtained by Vilenkin [194], and Corollary 7 by Fine [44]. Many properties of the Walsh system (see, for example, Theorems 13 and 14) are analogs of properties of the trigonometric system, a fact which is often explained by the observation that both systems are character systems of locally compact Abelian groups. More information on the Walsh system and character systems can be found in the survey by Balashov and Rubinshteĭn [8]; also in [1].

Chapter 5

The Hilbert transform and matrix (see (27)) were introduced by Hilbert around 1900. Hilbert himself (see [57], [56]) proved that the Hilbert transform is a bounded operator from $L^2(R^1)$ to $L^2(R^1)$. Subsequently the concept of conjugate function, which is closely related to the Hilbert transform, was introduced by Luzin [102] in the theory of trigonometric series. In the problems that we consider, the study of the Hilbert transform and of the operator of conjugation are equivalent.

The existence and finiteness for a.e. $x \in R^1$ of the Hilbert transform $T(f,x)$ of every $f \in L^1(R^1)$ (see Theorem 1) was proved by Privalov [155], and inequality (5), by Kolmogorov [86] (also see [10] and [201]).

Our proof of Theorem 1 resembles that in [49]; we note that the essential point in the proof, Lemma 1, was obtained by Boole in 1857 (see [99], p. 68).

Theorem 2 was proved by M. Riesz [158]. Theorem 3 is a combination of the results of Privalov, Kolmogorov, and Riesz already cited.

The space BMO was introduced by John and Nirenberg in [62] and is much used in analysis at present; Theorem 7 is also proved in [62]. Theorems 4 and 5 were obtained by Fefferman (see [42] and [31]), and are probably better known in connection with the long series of results on the Hardy spaces H^p and their higher-dimensional analogs that were established in the seventies (we touch on this extensive topic mainly in connection with the Franklin system in Chapter 6; for details see [32] and [43]). In the presentation of Theorem 4 we use the reasoning of [198], simplified for the one-dimensional case by Oswald.

Theorem 6 is due to Coifman and Weiss [32].

Chapter 6

The Faber–Schauder system was introduced in 1910 by Faber [41], who constructed it by integrating the Haar functions (see equation (2)). Theorem 1 was also proved in [41]. In 1927 Schauder [163] rediscovered the Faber–Schauder system: this system is the simplest member of the family of bases of $C(0,1)$ that were constructed in [163].

In Corollary 1, inequality a) follows immediately from Theorem 1, and b) was obtained by Matveev [108].

Theorem 2 was established by Karlin [69]; our proof was suggested by Arutyunyan [5]. Theorem 3 was proved by Ciesielski [25] (see also [162]). Proposition 1 and Corollary 2 were obtained by Saakyan [160]. For the systems of Faber–Schauder type discussed in §2, see [64], p. 50, and [163].

The Franklin system was introduced in 1928 by Franklin as the first example of an orthonormal basis in the space of continuous functions (see Theorem 6). Afterwards the Franklin system was not studied until the work of Ciesielski, who originated the systematic investigation of this system. Ciesielski in [26] and [27] proved Theorems 5 and 7–9, and Corollary 4. The proofs of Theorems 8 and 9 in the text are simpler than those in [27] and were suggested by Ciesielski in 1977; they use the functions $N_j(t)$ (see (30)), which are special cases of the B-splines which were introduced in [33]. We note that Theorem 9 plays a fundamental role in many papers on the Franklin system.

Theorems 10 and 11, and Corollary 5, were established by Wojtaszczyk [199], making essential use of Carleson's arguments [20]. Before Carleson and Wojtaszchyk, Maurey [112] had discovered (by a nonconstructive method) that there is an unconditional basis in H^1. It should be noted in connection with Wojtaszczyk's theorem that as early as 1969 Ciesielski [28] and Schonefeld [166] had applied the Franklin system to the construction of a basis in the space $C^1(I^2)$ of continuously differentiable functions on a square. Then Bochkarev [14] (by transforming the functions in the Franklin system into analytic functions by the rule given in Corollary 5.6; see 5.(90)) applied this system to the construction of a basis in the space C_A of functions that are analytic in the disk $|z| < 1$ and continuous up to the boundary (the question of the existence of bases in $C^1(l^2)$ and C_A was mentioned in [9]). Hence the Franklin system has proved to be very useful in constructing bases in various function spaces.

Theorem 12 was proved by Bochkarev [14]. The stronger form of the theorem, mentioned after the proof of Theorem 12, was obtained in [30]. Theorem 13 was proved by Chang and Ciesielski [21].

Finally we note that at present there are rather detailed investigations and generalizations of the Franklin system, namely the orthogonal splines; for details, see [29] and [168].

Chapter 7

The orthogonalization theorems discussed in §1 are usually helpful in the construction of various orthonormal systems. Theorem 1 was proved by I. Schur ([167]; see also [64]). Theorem 2 was obtained by Men′shov [121], who in its proof a method of orthogonalization which was probably first applied by Kolmogorov and Men′shov [90]. Criteria for the extendability of a system of functions to a complete orthonormal system (see Theorem 3) were given by Kozlov [94] and (in the form given in the text) by Olevskiĭ (see [137] and [138], p. 57).

Corollary 1 has probably been known for thirty years; Hobby and Rice [58] obtained a more precise result: they showed that the required function $\varepsilon(x)$ (see Corollary 1) can be chosen to have $\leq m$ changes of sign. Theorem 4 follows immediately from Corollary 1.

In §§2–4 we consider factorization theorems and their applications to the theory of orthogonal series. This subject was opened by the work of Grothendieck [52]; one of his results is mentioned at the beginning of this chapter. In addition, here we must mention Kolmogorov [86]. In [86] it was shown by the explicit example of the Hilbert operator (transformation) $f(x) \rightarrow T(f,x)$ (see §5.1) how by using the information that $T(f,x)$ is finite almost everywhere for every $f \in L^1$ one can obtain substantially more: the inequality 5.5 for the weak type. Later Kolmogorov's approach was generalized by Stein [172].

Essential developments of factorization theorems were obtained by Nikishin in [126], [127], and [129], and later by Maurey, Pisier and others (see [110]). Nikishin, in contrast to Grothendieck, considered operators that act not only on Banach spaces, but also on the space of all measurable, a.e. finite functions; this significantly extends the domain of applicability of factorization theorems. At present, factorization theorems have found a variety of applications in the theory of orthogonal series, probability theory, and functional analysis (see, in particular, the bibliographies in [68] and [84]).

Theorems 5 and 6 were proved by Nikishin in [127] and [126]. Theorem 7 was published by Maurey, but can be immediately obtained by successive applications of theorems of Nikishin (see [127], Theorem 4) and Grothendieck [52]. In connection with Theorem 7, we notice the following result (see [113], [144], and also [111]):

For the series $\sum_1^\infty f_n(x)$, $x \in (0,1)$, to be unconditionally convergent in measure, it is necessary and sufficient that for every $\varepsilon > 0$ there are a set $E_\varepsilon \subset (0,1)$, $m(E_\varepsilon) > 1-\varepsilon$, a constant K_ε, and a series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x), \qquad \sum_{n=1}^{\infty} a_n^2 < \infty, \quad \{\varphi_n(x)\}_{n=1}^{\infty} \quad \text{an O.N.S. (on } (0,1))$$

such that $f_n(x) = K_\varepsilon a_n \varphi_n(x)$ for $x \in E_\varepsilon$ and $n = 1, 2, \ldots$ (we did not include the proof of this proposition in the text, since it depends on methods that are somewhat remote from the subject matter of the book).

Theorem 8 is a corollary of results of Olevskiĭ [135], and Theorem 9 was established by Nikishin [128], [129].

Chapter 8

Theorem 1 is the famous Men′shov–Rademacher theorem ([117], [157]); Men′shov [117] also proved Theorem 2. Theorem 3 was obtained by Kashin [72]; the simpler proof given in the text was suggested by Tandori [183]. Theorem 4 was established by Kaczmarz [63], but the main idea of the proof appeared in papers of Kolmogorov and Seliverstov, and of Plessner, on the convergence a.e. of trigonometric series (see the notes to Chapter 4 and [2], Chapter III, §1, Theorem 3.1.5). In the proof of Theorem 4 we follow [2]; on the definitiveness of this result, see [182]. On Theorems 1 and 4, see also Schipp [165].

Corollary 3 was established by Men′shov [118]; this was probably the first result on the unconditional convergence of series of functions. Corollary 2 was obtained by Orlicz [139]; Theorem 5 is also proved by the method of [139]. The definitive form of the sufficient conditions for unconditional convergence a.e. of orthogonal series (conditions imposed on the coefficients of the series) was given by Tandori [181]: see Theorem 5.

Theorem 6 and Corollary 4 were proved by Tandori [181]. Strictly speaking, Tandori proved somewhat more, namely that for a given sequence $a_1 \geq a_2 \geq \cdots$ the series (47) converges a.e. for every O.N.S. $\{\varphi_n(x)\}_{n=1}^{\infty}$ if and only if condition (48) is satisfied. Later, Tandori gave a simpler proof; in the text we present Tandori's discussion in a modified form.

Corollary 5 was obtained by Men′shov [119] and Marcinkiewicz [104] (also see [105], p. 164). Theorem 7 was proved by Marcinkiewicz [104] and a little later by Men′shov [120]; actually [120] practically repeats the discussion in [119]. We should say that the methods of proof in [119] and [104], in spite of superficial differences, are very similar.

In connection with Corollary 5 we note a result of Komlós [92], where it was proved that from every O.N.S. one can select a subsystem with unconditional convergence.

Theorem 8 and Corollary 6 were established by Erdös [40] for the special case of subsystems of the trigonometric system, and in the general case (by a similar method) by Stechkin (see [47] and [190]).

The inequality for the density of a Sidon subsystem of the trigonometric system (see (99)) was obtained by Stechkin [171]. The result mentioned in §4 on the possibility of selecting a Sidon subsystem from every collectively bounded O.N.S. was established by Gaposhkin [47]. Theorem 9 was proved by Kashin [79] (for applications of propositions of the type of Theorem 9 to problems on the geometry of normed spaces, see [123]). Theorem 10 was also obtained by Kashin and is a generalization of a theorem of Nisio (see [97], p. 102; Nisio proved that every series of the form (120)2) converges, for almost all t, uniformly (in y) on $(0, 1)$). Finally, proposition 1 was obtained by McLaughlin [115].

Chapter 9

The subject matter of Chapter 9 follows the work of Ul′yanov [186], [185] and Olevskiĭ [131], in which, using Corollary 3.6, established earlier in [184], Theorem 1 was proved. Among earlier work on divergence of orthogonal series and series of Fourier coefficients in general systems, we note the papers of Orlicz [139], [141], and Kozlov [95].

The systematic study of problems of divergence of orthogonal series and series of Fourier coefficients in general systems was begun by Bochkarev (see [17], [12]), who obtained many definitive results.

Corollary 1 was established by Kozlov [95] and strengthens a result of Orlicz [137], according to which, for every complete O.N.S. $\{\varphi_n(x)\}_{n=1}^{\infty}$, the sum $\sum_1^{\infty} \varphi_n^2(x) = \infty$ for almost all x. Theorems 2 and 3 were proved by Olevskiĭ [132], [133]. Theorem 4 was first established by Makhmudov [103], although his proof practically repeats the discussion by Orlicz ([141]; see also [10], Chapter IV, §16), who obtained a similar result for uniformly bounded O.N.S.

The statement of Theorem 5, in the special case when $\{\varphi_n\}$ is the trigonometric system, was proved in [98]; then S. V. Khrushchev noticed that the method of [98] is completely sufficient for obtaining results on general O.N.S., including Theorem 5 (also see [85]). Theorem 6 was established by Kashin [71], [75]; Corollary 2 had been obtained earlier by Mityagin [124] and generalizes a result of S. N. Bernstein (see [10], Chapter IX, §4) for the case of the trigonometric system. The proof of Theorem 6 in the text is different from that in [75] and is close to the work of Bochkarev (see, for example, [17]).

In §3 of Chapter 9 we consider uniformly bounded O.N.S. The first lower bounds for the Lebesgue functions of uniformly bounded systems, with which §3 begins, were obtained by Olevskiĭ [134] (there he established Theorem 8 and Corollary 5). Subsequently in [136], [138] he also obtained Corollary 3. Bochkarev [15], [16] proved a more precise result, Theorem 7, by a different method. Bochkarev's approach was modified in [77], following which we established Proposition 1 and inequality (63), which play a fundamental role in §3. In connection with Corollary 5 we note the weaker result of Szarek [174], who proved (using, in particular, Olevskiĭ's method) that for an arbitrary basis $\{\psi_n(x)\}_{n=1}^{\infty}$ in $L^1(0,1)$ with $\|\psi_n\|_1 = 1$, $n = 1, 2, \dots$, and for every $p > 1$, there is the relation

$$\varlimsup_{n\to\infty} \|\psi_n\|_p = \infty.$$

Theorem 9 was proved by Krantsberg [96]. Theorem 10 was proved by Bochkarev [16]; later, Kazaryan [81] showed that under the additional requirement of completeness one cannot assert the existence of a Fourier series that diverges almost everywhere.

Concerning inequality (113), see [189]. Theorem 11 was obtained by Bochkarev [12]. In [13] (also see [17]) Bochkarev also established a more precise result: under the hypotheses of Theorem 11, for every modulus of continuity $\omega(\delta)$ with

$$\sum_{n=1}^{\infty} \frac{\sqrt{\omega(1/n)}}{n} = \infty,$$

the required function $F(x)$ of bounded variation whose series of Fourier coefficients does not converge absolutely can be found in the class H_ω (as the example of the trigonometric system shows (see [10], Chapter IX, §3) a further increase in the smoothness of $F(x)$ is in general not possible).

Our presentation of the proof of Theorem 11 follows [75] and [77]; also see Wik [197], where the most important special case (the trigonometric system) is considered.

Chapter 10

In Chapter 10 we discuss metric theorems on the representation of functions by general orthogonal series (we do not say much about the old subject of the representation of functions by everywhere convergent trigonometric series).

The development of this direction of the theory of orthogonal series was initiated by Luzin (see [102]) and continued by Men′shov, who in 1916, in [116], constructed a null series for the trigonometric system. Originally considered only for trigonometric series, the problem of the representation of functions by general orthogonal series was developed in the middle thirties by Marcinkiewicz (see [105], p. 312). At present, theorems on the representation of functions occupy an appreciable part of the theory of general orthogonal series. For more details on this topic see the surveys by Talalyan [178] and Ul′yanov [193], and also [78], [153], [179], [6], and [125].

Theorem 1 and Corollary 1 were proved by Talalyan (see [176], [177]). Theorems 2, 2′, and 3 were proved by Kashin (see [73], [74], and [76]). The proof of Theorem 2′ clarified (see [74]) the existence of a difference, for problems of representing functions by series, between general convergence systems and the slightly narrower class Q of such systems (a C.O.N.S. $\Phi \in Q$ if the operator S_Φ^* of majorants of the partial sums is a bounded operator from L^2 to L^2). In this connection it is natural to raise the question of whether one can always represent a measurable function, finite almost everywhere, by an almost everywhere convergent series in a system of class Q. Theorem 4 (Arutyunyan and Pogosyan) provides a positive answer to this question. Theorem 4 was announced by Arutyunyan [7] and Pogosyan [154] in notes that appeared simultaneously, and proved by them in seminars. A proof of this result is published here for the first time. In our exposition of Theorem 4 we follow, in the main, Arutyunyan's presentation. Lemma 1 in Theorem 4 follows immediately from the results of [91], and was first applied to related problems by Pogosyan [152]. The use of the Haar system in problems on the representation of functions by general orthogonal series was initiated by Arutyunyan (see [6]).

We should remark that the possibility of representing every measurable function, finite a.e., by an almost everywhere convergent trigonometric series, and by a Haar series, had been established much earlier by the proofs of Theorems 1 and 4, respectively, by Men'shov and Bari (for details see [178]).

Theorem 5 was proved by Drobot [38]. A more general approach to similar theorems was suggested by Pecherskiĭ [149], [150]; also see [45]. For Lemma 1 and Theorem 5 see the notes on Theorem 2.14.

Theorem 6 was established by Pogosyan [153], and his paper [154] contains a proposition on universal series for an arbitrary C.O.N.S. (without the supplementary condition (115)). The case of the trigonometric and Haar systems was considered earlier by G. M. Mushegyan (see [154]).

Appendices

The concept of the maximal function was introduced by Hardy and Littlewood [55] in 1930. In that paper they also gave the first application of this concept to problems on the convergence of series of functions. In §2 of Appendix 1 we follow [172]; also see [201].

For the minimax theorem see, for example, [148].

More details on the contents of Appendix 2 are given in [59] and [201]. Theorem 7 in this appendix, which was established by the brothers F. and M. Riesz in 1916, is presented along the lines of [37].

This page intentionally left blank

Bibliography

1. G. N. Agaev et al., *Multiplicative systems of functions and harmonic analysis on zero-dimensional groups*, "Èlm", Baku, 1981. (Russian)

2. G. Alexits, *Konvergenzprobleme der Orthogonalreihen*, Akad. Kiadó, Budapest, 1960; English transl., Pergamon Press, 1961.

3. G. Alexits and D. Králik, *Über die Approximation mit starken de la Vallée-Poussinschen Mitteln*, Acta Math. Acad. Sci. Hungar. **16** (1965), 43–49.

4. F. G. Arutyunyan, *Series with respect to the Haar system*, Akad. Nauk Armyan. SSR Dokl. **42** (1966), 134–140. (Russian)

5. ——, *Bases of the spaces* $L_1[0,1]$ *and* $C[0,1]$, Mat. Zametki **11** (1972), 241–249; English transl. in Math. Notes **11** (1972).

6. ——, *Representation of measurable functions almost everywhere by convergent series*, Mat. Sb. **90(132)** (1973), 483–520; English transl. in Math. USSR Sb. **19** (1973).

7. ——, *Representation of functions by multiple series*, Akad. Nauk Armyan. SSR Dokl. **64** (1977), 72–76. (Russian)

8. L. A. Balashov and A. I. Rubinshteĭn, *Series in the Walsh system, and their generalizations*, Itogi Nauki i Tekhniki: Mat. Anal. 1970, VINITI, Moscow, 1971, pp. 147–202; English transl. in J. Soviet Math. **1** (1973), no. 6.

9. Stefan Banach, *Théorie des opérations linéaires*, PWN, Warsaw, 1932.

10. N. K. Bari, *Trigonometric series*, Fizmatgiz, Moscow, 1961. (Russian)

11. S. V. Bochkarev, *On the coefficients of Fourier series with respect to the Haar system*, Mat. Sb. **80(122)** (1969), 97–116; English transl. in Math. USSR Sb. **9** (1969).

12. ——, *Absolute convergence of Fourier series with respect to complete orthonormal systems*, Uspekhi Mat. Nauk **27** (1972), no. 2 (164), 53–76; English transl. in Russian Math. Surveys **27** (1972).

13. ——, *On the absolute convergence of Fourier series in bounded complete orthonormal systems of functions*, Mat. Sb. **93(135)** (1974), 203–217; English transl. in Math. USSR Sb. **22** (1974).

14. ——, *Existence of a basis in the space of functions analytic in the disk, and some properties of Franklin's system*, Mat. Sb. **95(137)** (1974), 3–18; English transl. in Math. USSR Sb. **24** (1974).

15. ——, *Logarithmic growth of arithmetic means of Lebesgue functions of bounded orthonormal systems*, Dokl. Akad. Nauk SSSR **223** (1975), 16–19; English transl. in Soviet Math. Dokl. **16** (1975).

16. ——, *A Fourier series in a bounded orthonormal system that diverges on a set of positive measure*, Mat. Sb. **98(140)** (1975), 436–449; English transl. in Math. USSR Sb. **27** (1975).

17. ——, *A method of averaging in the theory of orthogonal series, and some problems in the theory of bases*, Trudy Mat. Inst. Steklov. **146** (1978); English transl., Proc. Steklov Inst. Math. **1980**, no. 3(146).

18. D. L. Burkholder and R. F. Gundy, *Extrapolation and interpolation of quasi-linear operators on martingales*, Acta Math. **124** (1970), 249–304.

19. Lennart Carleson, *On covergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135–157.

20. ——, *An explicit unconditional basis in H^1*, Bull. Sci. Math. (2) **104** (1980), 405–416.

21. Sun-Yung, A. Chang and Z. Ciesielski, *Spline characterizations of H^1*, Studia Math. **75** (1983), 183–192.

22. E. W. Cheney, *Introduction to approximation theory*, McGraw-Hill, New York, 1966.

23. Y. S. Chow, *Convergence theorems of martingales*, Z. Wahrsch. Verw. Gebiete **1** (1962/63), 340–346.

24. Z. Ciesielski, *On Haar functions and on the Schauder basis of the space $C_{\langle 0,1\rangle}$*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 227–232.

25. ——, *Some properties of Schauder basis of the space $C_{\langle 0,1\rangle}$*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 141–144.

26. ——, *Properties of the orthonormal Franklin system*. I, Studia Math. **23** (1963), 141–157.

27. ——, *Properties of the orthonormal Franklin system*. II, Studia Math. **27** (1966), 289–323.

28. ——, *A construction of basis in $C^{(1)}(I^2)$*, Studia Math. **33** (1969), 243–247.

29. ——, *Convergence of spline expansions*, Linear Spaces and Approximation (Proc. Conf., Oberwolfach, 1977; P. L. Butger and B. Sz.-Nagy, editors), Birkhäuser, 1978, pp. 433–448.

30. Z. Ciesielski, P. Simon, and P. Sjölin, *Equivalence of Haar and Franklin bases in Lp spaces*, Studia Math. **60** (1977), 195–210.

31. Ronald R. Coifman, *A real variable characterization of H^p*, Studia Math. **51** (1974), 269–274.

32. Ronald R. Coifman and Guido Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.

33. H. B Curry and I. J. Schoenberg, *On Pólya frequency functions.* IV, J. Analyse Math. **17** (1966), 71–107.

34. Burgess Davis, *On the integrability of the martingale square function*, Israel J. Math. **8** (1970), 187–190.

35. Mahlon M. Day, *Normed linear spaces*, Springer-Verlag, Berlin, 1958.

36. J. L. Doob, *Stochastic processes*, Wiley, New York, and Chapman & Hall, London, 1953.

37. Raouf Doss, *Elementary proof of the Rudin-Carleson and the F. and M. Riesz theorems*, Proc. Amer. Math. Soc. **82** (1981), 599–602.

38. Vladimir Drobot, *Rearrangements of series of functions*, Trans. Amer. Math. Soc. **142** (1969), 239–248.

39. V. K. Dzyadik, *Introduction to the theory of uniform approximation of functions by polynomials*, "Nauka", Moscow, 1977. (Russian)

40. P. Erdős, *On the convergence of trigonometric series*, J. Math. and Phys. **22** (1943), 37–39.

41. Georg Faber, *Über die Orthogonalfunktionen des Herrn Haar*, Jber. Deutsch. Math.-Verein. **19** (1910), 104–112.

42. Charles Fefferman, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. **77** (1971), 587–588.

43. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

44. N. J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc. **65** (1949), 372–414.

45. V. P. Fonf, *Conditionally convergent series in a uniformly smooth Banach space*, Mat. Zametki **11** (1972), 209–214; English transl. in Math. Notes **11** (1972).

46. Philip Franklin, *A set of continuous orthogonal functions*, Math. Ann. **100** (1928), 522–529.

47. V. F. Gaposhkin, *Lacunary series and independent functions,* Uspekhi Mat. Nauk **21** (1966), no. 6 (132), 3–82; English transl. in Russian Math. Surveys **21** (1966).

48. Adriano M. Garsia, *Existence of almost everywhere convergent rearrangements for Fourier series of L_2 functions,* Ann. of Math. (2) **79** (1964), 623–629.

49. ——, *Topics in almost everywhere convergence,* Markham, Chicago, Ill., 1970.

50. B. I. Golubov, *On Fourier series of continuous functions in the Haar system,* Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 1271–1296. (Russian)

51. ——, *Series in the Haar system,* Itogi Nauki i Tekhniki: Mat. Anal. 1970, VINITI, Moscow, 1971, pp. 109–146; English transl. in J. Soviet Math. **1** (1973), no. 6.

52. A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques,* Bol. Soc. Mat. São Paulo **8** (1953), 1–79 (1956).

53. Richard F. Gundy, *Martingale theory and pointwise convergence of certain orthogonal series,* Trans. Amer. Math. Soc. **124** (1966), 228–248.

54. Alfred Haar, *Zur Theorie der orthogonalen Funktionensysteme.* I, Math. Ann. **69** (1910), 331–371.

55. G. H. Hardy and J. E. Littlewood, *A maximal theorem with function-theoretic applications,* Acta Math. **54** (1930), 81–116.

56. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities,* Cambridge Univ. Press, 1934.

57. David Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen,* Teubner, Leipzig, 1912.

58. Charles R. Hobby and John R. Rice, *A moment problem in L_1 approximation,* Proc. Amer. Math. Soc. **16** (1965), 665–670.

59. Kenneth Hoffman, *Banach spaces of analytic functions,* Prentice-Hall, Englewood Cliffs, N.J., 1962.

60. Richard A. Hunt, *On the convergence of Fourier series,* Orthogonal Expansions and Their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967; D. T. Haimo, editor), Southern Illinois Univ. Press, Edwardsville, Ill., and Feffer & Simons, London, 1968, pp. 235–255.

61. Satoru Igari, *An extension of the interpolation theorem of Marcinkiewicz.* II, Tôhoku Math. J. (2) **15** (1963), 343–358.

62. F. John and L. Nirenberg, *On functions of bounded mean oscillation,* Comm. Pure Appl. Math. **14** (1961), 415–426.

63. S. Kaczmarz, *Sur la convergence et la sommabilité des développements orthogonaux,* Studia Math. **1** (1929), 87–121.

64. Stefan Kaczmarz and Hugo Steinhaus, *Theorie der Orthogonalreihen*, PWN, Warsaw, 1935.

65. Jean-Pierre Kahane, *Some random series of functions*, Heath, Lexington, Mass., 1968.

66. J.-P. Kahane and Y. Katznelson, *Séries de Fourier des fonctions bornées*, Studies in Pure Math.: To the Memory of Paul Turán (P. Erdős et al., editors), akad. Kiadó, Budapest, and Birkhäuser, Basel, 1983, pp. 395–410.

67. Jean-Pierre Kahane and Raphael Salem, *Ensembles parfaits et séries trigonométriques*, Actualités Sci. Indust., no. 1301, Hermann, Paris, 1963.

68. N. J. Kalton, N. T. Peck, and J. W. Roberts, *L_0-valued vector measures are bounded*, Proc. Amer. Math. Soc. **85** (1982), 575–582.

69. S. Karlin, *Bases in Banach spaces*, Duke Math. J. **15** (1948), 971–985.

70. B. S. Kashin, *The stability of unconditional almost everywhere convergence*, Mat. Zametki **14** (1973), 645–654; English transl. in Math. Notes **14** (1973).

71. ____, *On some properties of series of functions and orthogonal series*, Candidate's Dissertation, Steklov Math. Inst. Acad. Sci. USSR, Moscow, 1976. (Russian)

72. ____, *On Weyl's multipliers for almost everywhere convergence of orthogonal series*, Anal. Math. **2** (1976), 249–266.

73. ____, *On a complete orthonormal system*, Mat. Sb. **99(141)** (1976), 356–365; English transl. in Math. USSR Sb. **28** (1976).

74. ____, *On orthogonal systems of convergence*, Dokl. Akad. Nauk SSSR **228** (1976), 285–286; English transl. in Soviet Math. Dokl. **17** (1976).

75. ____, *On the coefficients in the expansion of a certain class of functions in terms of complete systems*, Sibirsk. Mat. Zh. **18** (1977), 122–131; English transl. in Siberian Math. J. **18** (1977).

76. ____, *On some properties of orthogonal systems of convergence*, Trudy Mat. Inst. Steklov. **143** (1977), 68–87; English transl. in Proc. Steklov Inst. Math. **1980**, no. 1 (143).

77. ____, *Remarks on estimating the Lebesgue functions of an orthonormal system*, Mat. Sb. **106(148)** (1978), 380–385; English transl. in Math. USSR Sb. **35** (1979), no. 1.

78. ____, *General orthonormal systems and certain problems of approximation theory* (author's summary of doctoral dissertation), Mat. Zametki **26** (1979), 299–315; English transl. in Math. Notes **26** (1979).

79. ____, *On some properties of the space of trigonometric polynomials with uniform norm*, Trudy Mat. Inst. Steklov. **145** (1980), 111–116; English transl. in Proc. Steklov Inst. Math. **1981**, no. 1 (145).

80. ____, *Some properties of matrices of bounded operators l_2^n to l_2^m*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. **15** (1980), 379–394; English transl. in Soviet J. Contemp. Math. Anal. **15** (1980), no. 5.

81. K. S. Kazaryan, *On some questions in the theory of orthogonal series*, Mat. Sb. **119(161)** (1982), 278–294; English transl. in Math. USSR Sb. **47** (1984).

82. A. Khintchine [Khinchin], *Über dyadische Brüche*, Math. Z. **18** (1923), 109–116.

83. A. Khintchine [Khinchin] and A. Kolmogoroff [Kolmogorov], *Über Konwergenz von Reihen, deren Glieder durch den Zufall bestimmt werden*, Mat. Sb. **23** (1924/25), 668–677.

84. S. V. Kislyakov, *p-absolutely summing operators*, Geometry of Linear Spaces and Operator Theory (B. S. Mityagin, editor), Yaroslav. Gos. Univ., Yaroslavl, 1977, pp. 114–174. (Russian)

85. ____, *The Fourier coefficients of the boundary values of functions analytic in the disk and in the bidisk*, Trudy Mat. Inst. Steklov. **155** (1981), 77–94; English transl. in Proc. Steklov Inst. Math. **1983**, no. 1 (155).

86. A. Kolmogoroff [A. N. Kolmogorov], *Sur les fonctions harmoniques conjuguées et les séries de Fourier*, Fund. Math. **7** (1925), 24–29.

87. ____, *Über die Summen durch den Zufall bestimmter unabhängiger Grossen*, Math. Ann. **99** (1928), 309–319.

88. ____, *Über das Gesetz des iterierten Logarithmus*, Math. Ann. **101** (1929), 126–135.

89. A. N. Kolmogorov and S. V. Fomin, *Elements of the theory of functions and of functional analysis*, 3rd ed., "Nauka", Moscow, 1972; English transl. of 1st ed., Vols. 1, 2, Graylock Press, Albany, N.Y., 1957, 1961.

90. A. Kolmogoroff [Kolmogorov] and D. Menchoff [Men'shov], *Sur la convergence des séries de fonctions orthogonales*, Math. Z. **26** (1927), 432–441.

91. János Komlós, *A generalization of a problem of Steinhaus*, Acta Math. Acad. Sci. Hungar. **18** (1967), 217–229.

92. ____, *Every sequence converging to O weakly in L_2 contains an unconditional convergence sequence*, Ark. Mat. **12** (1974), 41–49.

93. S. V. Konyagin, *On a problem of Littlewood*, Izv. Akad. Nauk SSSR **45** (1981), 243–265; English transl. in Math. USSR Izv. **18** (1982).

94. V. Ya. Kozlov, *On a local property of a complete orthonormal system of functions*, Mat. Sb. **23(65)** (1948), 441–474. (Russian)

95. ——, *On the distribution of positive and negative values of orthonormal functions that form a complete system*, Mat. Sb. **23(65)** (1948), 475–480. (Russian)

96. A. S. Krantsberg, *On convergence systems in C and bases in L*, Mat. Zametki **26** (1979), 183–200; English transl. in Math. Notes **26** (1979).

97. John Lamperti, *Probability*, Benjamin, New York, 1966.

98. Karel de Leeuw, Yitzhak Katznelson, and Jean-Pierre Kahane, *Sur les coefficients de Fourier des fonctions continues*, C. R. Acad. Sci. Paris Sér. A-B**285** (1977), A1001–A1003.

99. Norman Levinson, *Gap and density theorems*, Amer. Math. Soc., Providence, R.I., 1940.

100. J. E. Littlewood, *On the mean values of power series*, Proc. London Math. Soc. (2) **25** (1926), 328–337.

101. ——, *On bounded bilinear forms in an infinite number of variables*, Quart. J. Math. Oxford Ser. **1** (1930), 164–174.

102. N. N. Luzin, *Integral and trigonometric series*, Lissner i Sovko, Moscow, 1915; 2nd ed., GITTL, Moscow, 1951. (Russian)

103. A. S. Makhmudov, *On Fourier coefficients of continuous functions. III*, Some problems of Functional Analysis and Its Applications, Izdat. Akad. Nauk Azerbaĭdzhan. SSR, Baku, 1965, pp. 103–117. (Russian)

104. Józef Marcinkiewicz, *Sur la convergence des séries orthogonales*, Studia Math. **6** (1936), 39–45.

105. ——, *Collected papers*, PWN, Warsaw, 1964.

106. Michael B. Marcus and Gilles Pisier, *Random Fourier series with applications to harmonic analysis*, Princeton Univ. Press, Princeton, N.J., and Univ. of Tokyo Press, Tokyo, 1981.

107. A. I. Markushevich, *The theory of analytic functions, a short course*, 4th ed., "Nauka", Moscow, 1978; rev. English transl., "Mir", Moscow, 1983.

108. V. A. Matveev, *Series in a Schauder system*, Mat. Zametki **2** (1967), 267–278; English transl. in Math. Notes **2** (1967).

109. Bernard Maurey, *Sur une application de la théorie des opérateurs p-sommants*, C. R. Acad. Sci. Paris Sér. A-B**274** (1982), A1304–A1307.

110. ——, *Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p*, Astérisque, no. 11, Soc. Math. France, Paris, 1974.

111. ——, *Quelques problèmes de factorisation d'opérateurs linéaires*, Proc. Internat. Congr. Math. (Vancouver, 1974), Vol. 2, Canad. Math. Congr., Montréal, 1975, pp. 75–79.

112. ——, *Isomorphismes entre espaces H_1*, Acta Math. **145** (1980), 79–120.

113. Bernard Maurey and Gilles Pisier, *Un théorème d'extrapolation et ses conséquences*, C.R. Acad. Sci. Paris Sér. A-B **277** (1973), A39–A42.

114. O. Carruth McGehee, Louis Pigno, and Brent Smith, *Hardy's inequality and the L^1 norm of exponential sums*, Ann. of Math. (2) **113** (1981), 613–618.

115. James R. McLaughlin, *Integrated orthonormal series*, Pacific J. Math. **42** (1972), 469–475.

116. D. Menchoff [D. E. Men'shov], *Sur l'unicité du développement trigonométrique*, C. R. Acad. Sci. Paris **163** (1916), 433–436.

117. ——, *Sur les séries de fonctions orthogonales*, Fund. Math. **4** (1923), 82–105.

118. ——, *Sur la convergence des séries de fonctions orthogonales*, C. R. Acad. Sci. Paris **178** (1924), 301–303.

119. ——, *Sur la convergence et la sommation des séries de fonctions orthogonales*, Bull. Soc. Math. France **64** (1936), 147–170.

120. ——, *Sur la sommation des séries de fonctions orthogonales par les méthodes linéaires*, Izv. Akad. Nauk SSSR Ser. Mat. **1** (1937), 203–229. (Russian; French summary)

121. ——, *Sur les séries de fonctions orthogonales bornées dans leur ensemble*, Mat. Sb. **3(45)** (1938), 103–120.

122. V. D. Mil'man, Geometric theory of Banach spaces. II, Uspekhi Mat. Nauk **26** (1971), no. 6 (162), 73–149; English transl. in Russian Math. Surveys **26** (1971).

123. V. D. Mil'man and H. Wolfson, *Minkowski spaces with extremal distance from the Euclidean space*, Israel J. Math. **29** (1978), 113–131.

124. B. S. Mityagin, *Absolute convergence of the series of Fourier coefficients*, Dokl. Akad. Nauk SSSR **157** (1964), 1047–1050; English transl. in Soviet Math. Dokl. **5** (1964).

125. G. M. Mushegyan, *On the coefficients of everywhere convergent series in some rearranged orthonormal systems*, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), 807–832; English transl. in Math. USSR Izv. **13** (1979).

126. E. M. Nikishin, *Systems of absolute convergence*, Mat. Sb. **74(116)** (1967), 544–553; English transl. in Math. USSR Sb. **3** (1967).

127. ——, *Resonance theorems and superlinear operators*, Uspekhi Mat. Nauk **25** (1970), no. 6 (156), 129–191; English transl. in Russian Math. Surveys **25** (1970).

128. ——, *Resonance theorems and series of functions*, Doctoral Dissertation, Moscow State Univ., Moscow, 1971; Author's summary published in Mat. Zametki **10** (1971), 583–595; English transl. in Math. Notes **10** (1971).

129. ——, *A resonance theorem and series in eigenfunctions of the Laplacian*, Izv. Akad. Nauk SSSR Ser. Math. **36** (1972), 795–813; English transl. in Math. USSR Izv. **6** (1972).

130. E. M. Nikishin and P. L. Ul'yanov, *On absolute and unconditional convergence*, Uspekhi Mat. Nauk **22** (1967), no. 3 (135), 240–242. (Russian)

131. A. M. Olevskiĭ, *Divergent series in L^2 with respect to complete systems*, Dokl. Akad. Nauk SSSR **138** (1961), 545–548; English transl. in Soviet Math. Dokl. **2** (1961).

132. ——, *Divergent Fourier series for continuous functions*, Dokl. Akad. Nauk SSSR **141** (1961), 28–31; English transl. in Soviet Math. Dokl. **2** (1961).

133. ——, *Divergent Fourier series*, Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 343–366. (Russian)

134. ——, *Fourier series of continuous functions relative to bounded orthonormal systems*, Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 387–432; English transl. in Amer. Math. Soc. Transl. (2) **67** (1968).

135. ——, *On an orthonormal system and its applications*, Mat. Sb. **71(113)** (1966), 297–336; English transl. in Amer. Math. Soc. Transl. (2) **76** (1968).

136. ——, *Fourier series and Lebesgue functions*, Uspekhi Mat. Nauk **22** (1967), no. 3 (135), 237–239. (Russian)

137. ——, *Extension of a sequence of functions to a complete orthonormal system*, Mat. Zametki **6** (1969), 737–747; English transl. in Math. Notes **6** (1969).

138. ——, *Fourier series with respect to general orthogonal systems*, Springer-Verlag, Berlin, 1975.

139. Władysław Orlicz, *Zur Theorie der Orthogonalreihen*, Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A Sci. Math. **1927**, 81–115.

140. ——, *Beiträge zur Theorie der Orthogonalentwicklungen*. II, Studia Math. **1** (1929), 241–255.

141. ____, *Beiträge zur Theorie der Orthogonalentwicklungen.* III, Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A Sci. Math. **1932**, 229–238.

142. ____, *Über die Divergenz von allgemeinen Orthogonalreihen*, Studia Math. **4** (1933), 27–32.

143. ____, *Über unbedingte Konvergenz in Funktionenräumen.* I, Studia Math. **4** (1933), 33–37.

144. Peter Ørno, *A note on unconditionally converging series in L_p*, Proc. Amer. Math. Soc. **59** (1976), 252–254.

145. R. E. A. C. Paley, *A remarkable series of orthogonal functions.* I, II, Proc. London Math. Soc. (2) **34** (1932), 241–264, 265–279.

146. R. E. A. C. Paley and A. Zygmund, *On some series of functions.* I, II, Proc. Cambridge Philos. Soc. **26** (1930), 337–357, 458–474.

147. ____, *On some series of functions.* III, Proc. Cambridge Philos. Soc. **28** (1932), 190–205.

148. T. Parthasarathy and T. E. S. Raghavan, *Some topics in two-person games*, Amer. Elsevier, New York, 1971.

149. D. V. Pecherskiĭ, *On rearrangements of terms in series of functions*, Dokl. Akad. Nauk SSSR **209** (1973), 1285–1287; English transl. in Soviet Math. Dokl. **14** (1973).

150. ____, *A theorem on projections of rearranged series with terms in L_p*, Izv. Akad. Nauk SSSR Ser. Mat. **11** (1977), 203–214; English transl. in Math. USSR Izv. **11** (1977).

151. A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–228.

152. N. B. Pogosyan, *On a property of complete orthonormal systems*, Mat. Zametki **17** (1975), 681–690; English transl. in Math. Notes **17** (1975).

153. ____, *Representation of measurable functions by orthogonal series*, Mat. Sb. **98(140)** (1975), 102–112; English transl. in Math. USSR Sb. **27** (1975).

154. ____, *Representation of measurable functions by bases in $L_p[0, 1]$ ($p \geq 2$)*, Akad. Nauk Armyan. SSR Dokl. **63** (1976), 205–209. (Russian)

155. I. I. Privalov, *The Cauchy integral*, Izv. Saratov. Univ. Fiz.-Mat. Fak. **2** (1918), no. 1; reprinted, under separate covers, as his dissertation, Saratov, 1919. (Russian)

156. E. S. Quade, *Trigonometric approximation in the mean*, Duke Math. J. **3** (1937), 529–543.

157. Hans Rademacher, *Einige Sätze über Reihen von allgemeinen Orthogonalfunktionen*, Math. Ann. **87** (1922), 111–138.

158. Marcel Riesz, *Sur les fonctions conjuguées*, Math. Z. **27** (1927/28), 218–244.

159. Bengt Rosén, *Limit theorems for sampling from finite populations*, Ark. Mat. **5** (1963/65), 383–424.

160. A. A. Saakyan, *On properties of Fourier coefficients of a composition of functions*, Dokl. Akad. Nauk SSSR **248** (1979), 302–306; English transl. in Soviet Math. Dokl. **20** (1979).

161. ____, *Integral moduli of smoothness and the Fourier coefficients of the composition of functions*, Mat. Sb. **110(152)** (1979), 597–608; English transl. in Math. USSR Sb. **38** (1981).

162. T. N. Saburova, *Some properties of Fourier coefficients with respect to a Faber-Schauder system*, Soobshch. Akad. Nauk Gruzin. SSR **82** (1976), 297–300. (Russian)

163. Julius Schauder, *Zur Theorie stetiger Abbildungen in Funktionalräumen*, Math. Z. **26** (1927), 47–65.

164. ____, *Eine Eigenschaft des Haarschen Orthogonalsystems*, Math. Z. **28** (1928), 317–320.

165. F. Schipp, *Maximal inequalities*, Approximation and Function Spaces (Proc. Internat. Conf., Gdánsk, 1979; Z. Ciesielski, editor), PWN, Warsaw, and North-Holland, Amsterdam, 1981, pp. 629–644.

166. Steven Schonefeld, *Schauder bases in spaces of differentiable functions*, Bull. Amer. Math. Soc. **75** (1969), 586–590.

167. I. Schur, *Über endliche Gruppen und Hermitesche Formen*, Math. Z. **1** (1918), 184–207.

168. Zbigniew Semadeni, *Schauder bases in Banach spaces of continuous functions*, Lecture Notes in Math., vol. 918, Springer-Verlag, Berlin, 1982.

169. A. N. Shiryaev, *Probability*, "Nauka", Moscow, 1980; English transl., Springer-Verlag, Berlin, 1984.

170. Ivan Singer, *Bases in Banach spaces.* I, Springer-Verlag, Berlin, 1970.

171. S. B. Stechkin, *On absolute convergence of Fourier series*, Izv. Akad. Nauk SSSR Ser. Mat. **20** (1956), 385–412. (Russian)

172. Elias M. Stein, *On limits of sequences of operators*, Ann. of Math. (2) **74** (1961), 140–170.

173. ____, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press Princeton, N.J., 1970.

174. S. J. Szarek, *Bases and biorthogonal systems in the spaces C and* L^1, Ark. Mat. **17** (1979), 255–271.

175. Béla Sz.-Nagy, *Approximation properties of orthogonal expansions*, Acta Sci. Math. (Szeged) **15** (1953/54), 31–37.

176. A. A. Talalyan, *On convergence of orthogonal series*, Dokl. Akad. Nauk SSSR **110** (1956), 515–516. (Russian)

177. ——, *Representation of measurable functions by series*, Uspekhi Mat. Nauk **15** (1960), no. 5 (95), 77–141; English transl. in Russian Math. Surveys **15** (1960).

178. ——, *Questions of representation and uniqueness in the theory of orthogonal series*, Itogi Nauki i Tekhniki: Mat. Anal. 1970, VINITI, Moscow, 1971, pp. 5–64; English transl. in J. Soviet Math. **1** (1973), no. 6.

179. ——, *On approximation properties of certain incomplete systems*, Mat. Sb. **115(157)** (1981), 499–531; English transl. in Math. USSR Sb. **43** (1982).

180. A. A. Talalyan and F. G. Arutyunyan, *On the convergence of Haar series to* $+\infty$, Mat. Sb. **66(108)** (1965), 240–247; English transl. in Amer. Math. Soc. Transl. (2) **72** (1968).

181. Károly Tandori, *Über die orthogonalen Funktionen*. X: *Unbedingte Konvergenz*, Acta Sci. Math. (Szeged) **23** (1962), 185-221.

182. ——, *Über die Lebesgueschen Funktionen*, II. Acta Math. Acad. Sci. Hungar. **29** (1977), 177–184.

183. ——, *Einfacher Beweis eines Satzes von B. S. Kašin*, Acta Sci. Math. (Szeged) **39** (1977), 175–178.

184. P. L. Ul'yanov, *Divergent Fourier series of class* L^p $(p \geq 2)$, Dokl. Akad. Nauk SSSR **137** (1961), 786–789; English transl. in Soviet Math. Dokl. **2** (1961).

185. ——, *Divergence of series in the Haar system and in bases*, Dokl. Akad. Nauk SSSR **138** (1961), 556–559; English transl. in Soviet Math. Dokl. **2** (1961).

186. ——, *Divergent Fourier series*, Uspekhi Mat. Nauk **16** (1961), no. 3(99), 61–142; English transl. in Russian Math. Surveys **16** (1961).

187. ——, *Exact Weyl multipliers for unconditional convergence*, Dokl. Akad. Nauk SSSR **141** (1961), 1048–1049; English transl. in Soviet Math. Dokl. **2** (1961).

188. ——, *Weyl multipliers for unconditional convergence*, Mat. Sb. **60(102)** (1963), 39–62. (Russian)

189. ——, *On Haar series*, Mat. Sb. **63(105)** (1964), 356–391. (Russian)

190. ——, *Solved and unsolved problems in the theory of trigonometric and orthogonal series*, Uspekhi Mat. Nauk **19** (1964), no. 1 (115), 3–69; English transl. in Russian Math. Surveys **19** (1964).

191. ——, *Series with respect to a Haar system with monotone coefficients*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 925–950. (Russian)

192. ——, *Some properties of series with respect to the Haar system*, Mat. Zametki **1** (1967), 17–24; English transl. in Math. Notes **1** (1970).

193. ——, *Representation of functions by series and the classes $\varphi(L)$*, Uspekhi Mat. Nauk **27** (1972), no. 2 (164), 3–52; English transl. in Russian Math. Surveys **27** (1972).

194. N. Ya. Vilenkin, *On a class of complete orthonormal systems*, Izv. Akad. Nauk SSSR Ser. Mat. **11** (1947), 363–400; English transl. in Amer. Math. Soc. Transl. (2) **28** (1963).

195. J. L. Walsh, *A closed set of normal orthogonal functions*, Amer. J. Math. **45** (1923), 5–24.

196. Chinami Watari, *Mean convergence of Walsh Fourier series*, Tôhoku Math. J. (2) **16** (1964), 183–188.

197. Ingemar Wik, *Criteria for absolute convergence of Fourier series of functions of bounded variation*, Trans. Amer. Math. Soc. **163** (1972), 1–24.

198. J. Michael Wilson, *A simple proof of the atomic decomposition for $H^p(\mathbf{R}^n)$, $0 < p \leq 1$*, Studia Math. **74** (1982), 25–33.

199. P. Wojtaszczyk, *The Franklin basis is an unconditional basis in H_1*, Ark. Mat. **20** (1982), 293–300.

200. Shigeki Yano, *On a lemma of Marcinkiewicz and its applications to Fourier series*, Tôhoku Math. J. (2) **11** (1959), 191–215.

201. A. Zygmund, *Trigonometric series*. 2nd rev. ed., Vols. 1, 2, Cambridge Univ. Press, 1959.

202. *Mathematical encyclopedia* (I. M. Vinogradov et al., editors). Vols. 1–5, Izdat. "Sovet. Èntsiklopediya", Moscow, 1977–1985; English transl., *Encyclopedia of mathematics*. Vols. 1-?, Reidel, 1988 ff.

This page intentionally left blank

Index(¹)

(¹)This index does not cover the Notes, pp. 421–433

ISBN 0-8218-3789-3

9 780821 837894

MMONO/75.S