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Volume 76

Introduction to Algebraic Curves

Phillip A. Griffiths

American Mathematical Society

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ABSTRACT. This book gives an introduction to the theory of compact Riemann surfaces and plane algebraic curves. It is a translation from the original Chinese text, which was based on a course of lectures delivered in 1982.

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"This book is a translation from the original Chinese text of notes based on a six-week course on algebraic curves and Riemann surfaces taught at Beijing University in 1982"—Pref.

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English Translation of the Preface to the Chinese Edition

In the summer of 1982, I had the honor of teaching a course on algebraic curves at Peking University. The course met six hours a week for a period of six weeks. This book is a thoroughly edited version of the notes taken during this course.

Algebraic curves and compact Riemann surfaces comprise the most developed and arguably the most beautiful portion of algebraic geometry. The majority of books written on the subject, however, discuss algebraic curves and compact Riemann surfaces separately as part of distinct general theories. Consequently they are forced to devote a great deal of time supplying the various background materials in these areas. Moreover, nearly all elementary texts and university courses on curve theory generally conclude with the Riemann–Roch theorem, whereas, in fact, it is only with the appearance of this theorem that the theory of algebraic curves begins to get really fascinating.

When I gave this course at Peking University, I decided to begin with minimal technical requirements and proceed as directly as possible to the basic Riemann–Roch and Abel theorems. I was especially hoping to have enough time at the end of the course to give various applications of these theorems, for it is only then that one can really appreciate the real texture of the study of algebraic curves. Due to the generally excellent preparation of the students together with their effort, I believe that it was possible to more or less achieve this goal. The table of contents in this book accurately reflects the contents of the course.

This is a book on algebraic curves. It only assumes on the part of the reader a working knowledge of elementary complex function theory and algebra together with some exposure to the topology of compact surfaces. It differs from a number of recent books on this subject in that it combines analytic and geometric methods at the outset, so that the reader can most effectively get at the basic results of the subject. While this enables us to discuss interesting geometric results very early (here, I emphasize once again that the Riemann–Roch theorem is the *beginning*, and not the *end* of algebraic curves), this approach also has its disadvantages, the most important of which is the failure to introduce the modern techniques of sheaf theory, cohomology, commutative algebra, complex funtions of several variables and Hermitian differential geometry, etc., which are needed in the general study of algebraic geometry. However, after achieving an understanding of this book, a university student will be able to proceed to more advanced texts in general algebraic geometry, complex manifolds, Riemann surfaces, as well as algebraic curves. The Annotated Bibliography includes some of the available texts.

In concluding this part of the preface, I wish to wholeheartedly thank: the Mathematics Department of Peking University for their invitation; the students for giving me the opportunity; and my assistants in the course Zhang Zhu-Sheng, Zhao Chun-Lai, and Zhou Qing for an outstanding job of transcribing my lectures into this book.

Preface to the English Edition

This book is a translation from the orginal Chinese text of notes based on a six-week course on algebraic curves and Riemann surfaces taught at Peking University in 1982. The audience consisted of third- and fourth-year undergraduates and beginning graduate students. Their preparation consisted mainly of a solid course in one complex variable; in addition some algebra (mostly elementary) and some basic topology of surfaces (explained in the text) was used.

The goals of the course were twofold. One was to develop an understanding of algebraic curves and compact Riemann surfaces as essentially equivalent objects. The second was to prove in as elementary a way as possible the main results of the theory for a fixed curve, and to show how these are used to study specific geometric questions by an ample discussion of examples. In this regard, there are numerous exercises in the text, and the two exams for the course are given at the end of the book.

Since this course was taught, the book *Geometry of Algebraic Curves*, I by E. Arbarello, M. Cornalba, J. Harris and this author has appeared. The present text may be viewed as an introduction to that book, in which more advanced topics and a guide to the literature may be found.

The original Chinese text was prepared from course notes by Zhang Zhu-Sheng, Zhao Chun-Lai, and Zhou Qing. This text was rather complete in its attention to detail, and naturally this is reflected in the translation. The excellent English translation was prepared by Ms. Kuniko Weltin, and the whole project was overseen by Professor T. Y. Lam. To all of these I express my gratitude.

Phillip A. Griffiths

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Contents

Appendix
On a "General Fact"

In the text, we made use of the following theorem.

EXISTENCE THEOREM. *Suppose C is a compact Riemann surface. Then there exists an immersion*

$$f\colon C \to \mathbb{P}^2$$

such that $f(C)$ has at most ordinary double points.

To say that this f is an *immersion* means that the differential of f is nowhere equal to zero, and to say that $f(C)$ has *at most double points* means:

a) for any point $p \in f(C)$, there are at most two points in C which map onto p;

b) if

$$f^{-1}(p) = \{p_1, p_2\},$$

then the two local curve components of $f(C)$ passing through the point p have distinct tangent lines. This is to say, the two curve components at the point of intersection should look as shown in Figure A and not as in Figure B.

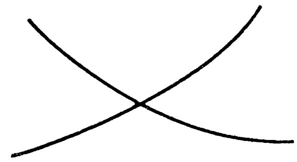

FIGURE A. Ordinary double point

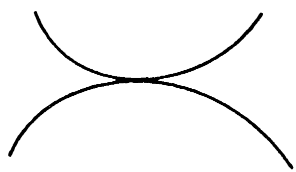

FIGURE B(a). Not an ordinary double point

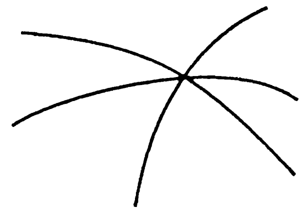

FIGURE B(b). Singular point of multiplicity greater than 2.

We point out that the preceding existence theorem is a corollary of the following theorem.

SECOND EXISTENCE THEOREM. *There exists a holomorphic mapping*

$$f\colon C \to \mathbb{P}^n,$$

which is injective and whose differential is nowhere zero.

In fact, if in this theorem $n > 3$, we can always choose a projection p from $\mathbb{P}^n$ onto a suitable $(n-1)$-dimensional subspace $(\simeq \mathbb{P}^{n-1})$ of $\mathbb{P}^n$ such that

$$p \circ f\colon C \to \mathbb{P}^{n-1}$$

is injective and whose differential is nowhere zero. If $n = 3$, we can choose a projection q from $\mathbb{P}^3$ onto a suitable subspace $(\simeq \mathbb{P}^2)$ of $\mathbb{P}^3$ such that

$$q \circ f\colon C \to \mathbb{P}^2$$

satisfies the requirements (for f) in the first existence theorem.

We shall now discuss (not prove) the second existence theorem.

We should point out, that this theorem is actually nothing but another result in this book in a different guise. The basic difficulty of this theorem lies in the following: given an abstract compact Riemann surface C, can we *a priori* decide whether or not there exists a nonconstant meromorphic function on C? Clearly, this problem must involve the use of some nontrivial analysis.

If we can prove the following result, we would be able to prove the second existence theorem.

$(*)$ There exists a subspace of finite dimension

$$V \subset K(C),$$

which possesses the following three properties:

a) the constant function $1 \in V$:

b) for any pair of different points $p_1, p_2 \in C$, there exists $f \in K(C)$ such that

$$f(p_1) \neq f(p_2);$$

c) for any $p \in C$, there exists $f \in K(C)$ such that $f(p) = 0$ and

$$(df)_p \neq 0.$$

Actually, if $f_0 = 1, f_1, \dots, f_n$ form a basis of V, then it is not difficult to see that the mapping

$$f\colon C \to \mathbb{P}^n$$

given by

$$f(p) = [f_0(p), f_1(p), \dots, f_n(p)]$$

satisfies the requirements of the second existence theorem.

Below we discuss the method of proving the existence of a nonconstant meromorphic function. A refinement of this discussion would enable us to arrive at $(*)$. It will become clear that sheaf theory and cohomology are implicit and therefore unavoidable in any discussion of this problem. Consequently, we shall assume some familiarity on the part of the reader with these topics.

Suppose $D = p_1 + \cdots + p_d$ is a divisor on C and z_i is a local holomorphic coordinate in a neighborhood of p_i. In a neighborhood of p_i, any function $f \in \mathscr{L}(D)$ is necessarily of the following form:

$$f = \frac{a_i}{z_i} + h(z_i),$$

where $h(z_i)$ is a holomorphic function. We consider the linear mapping

$$\mathscr{L}(D) \xrightarrow{\lambda} \mathbb{C}^d$$

given by

$$f \mapsto (a_1, \dots, a_d).$$

Since this linear function has as its kernel the set $\mathbb{C}$ of constant functions, we have the exact sequence

$$0 \to \mathbb{C} \to \mathscr{L}(D) \xrightarrow{\lambda} \mathbb{C}^d, \tag{1}$$

whence

$$\dim \mathscr{L}(D) = d + 1 - \dim(\operatorname{coker} \lambda). \tag{2}$$

Then the problem of finding $\dim \mathscr{L}(D)$ is equivalent to that of finding $\dim(\operatorname{coker}\lambda)$. In particular, suppose we can prolong (1) to an exact sequence of vector spaces

$$0 \to \mathbf{C} \to \mathscr{L}(D) \xrightarrow{\lambda} \mathbb{C}^d \to H^1(C, \mathscr{O}) \tag{3}$$

such that

$$\left.\begin{array}{ll} \text{a)} & H^1(C, \mathscr{O}) \text{ is independent of } D; \\ \text{b)} & \dim H^1(C, \mathscr{O}) < +\infty. \end{array}\right\} \tag{4}$$

Then, we get

$$\dim \mathscr{L}(D) \geq d + 1 - \dim H^1(C, \mathscr{O}) \tag{5}$$

and thus, as $d \to +\infty$,

$$\dim \mathscr{L}(D) \to +\infty.$$

Using this kind of method, we shall be able to prove the existence of a nonconstant meromorphic function.

Now (3) and (4a) are standard consequences of the formal theory of sheaf cohomology. In fact, if we let

$\mathscr{O}$ = the sheaf of holomorphic functions on C;

$\mathscr{O}(D)$ = the sheaf of meromorphic functions f such that $(f) + D \geq 0$;

$\mathscr{O}(D)/\mathscr{O}$ = the quotient sheaf of $\mathscr{O}(D)$ modulo $\mathscr{O}$,

then we have the following exact sequence of sheaves

$$0 \to \mathscr{O} \to \mathscr{O}(D) \to \mathscr{O}(D)/\mathscr{O} \to 0. \tag{6}$$

Since

$$H^0(C, \mathscr{O}) = \mathbb{C}, \ H^0(C, \mathscr{O}(D)) = \mathscr{L}(D),$$

and

$$H^0(C, \mathscr{O}(D)/\mathscr{O}) = \mathbb{C}^d,$$

we see that (3) is nothing but the first four terms of the long cohomology exact sequence associated to (6). In other words, whether one likes it or not, the formal theory of sheaf cohomology is going to show up in the study of complex manifolds and algebraic geometry. To a student, the important thing is to learn this formalism without losing sight of geometric intuition.

Part b) of (4) is a basic analytic result: the cohomology groups of a coherent sheaf are always finite dimensional. We shall not discuss this further except to point out that the basic facts underlying this assertion are the normal families of Montel and the compactness of C.

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Algebraic curves and compact Riemann surfaces comprise the most developed and arguably the most beautiful portion of algebraic geometry. This book is based on a six-week series of lectures presented by the author to third- and fourth-year undergraduates and graduate students at Beijing University in 1982. The lectures began with minimal technical requirements (a working knowledge of elementary complex function theory and algebra together with some exposure to topology of compact surfaces) and proceeded directly to the Riemann–Roch and Abel theorems. This book differs from a number of other books on this subject in that it combines analytic and geometric methods at the outset, so that the reader can grasp the basic results of the subject. Although such modern techniques of sheaf theory, cohomology, and commutative algebra are not covered here, the book provides a solid foundation to proceed to more advanced texts in general algebraic geometry, complex manifolds, and Riemann surfaces, as well as algebraic curves. Containing numerous exercises and two exams, this book would make an excellent introductory text.

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