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Asymptotic Methods in the Theory of Stochastic Differential Equations

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Providence, Rhode Island

А. В. СКОРОХОД

АСИМПТОТИЧЕСКИЕ МЕТОДЫ ТЕОРИИ СТОХАСТИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

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ABSTRACT. The topics in this monograph are ergodic theory for Markov processes and for solutions of stochastic differential equations, stochastic differential equations containing a small parameter, and stability theory for solutions of systems of stochastic differential equations. The main part of the material is presented for the first time. The book is intended for specialists in the theory of random processes and its applications.

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Foreword

The 1982 book on stochastic differential equations written jointly by the author and Iosif Il'ich Gikhman did not include a number of areas in this theory that are important for applications. Therefore, we decided to write a book that would bring together material relating to applied areas in the theory of stochastic equations. We intended to treat equations in infinite-dimensional spaces, in particular, infinite systems of stochastic equations; the theory of linear equations in infinite-dimensional spaces and the semigroups connected with them, in particular, stochastic partial differential equations of evolution type; equations for conditionally Markov processes and the equations of nonlinear filtration connected with them; and the asymptotic behavior of solutions of stochastic equations, including ergodic theory, the method of averaging, and the theory of stability. The plan of the book was discussed for a fairly long time, and we convinced ourselves at last that it was impossible to present all these topics in a single book. We then decided to treat the last topic. This choice was made under the influence of the interests of Iosif Il'ich, who, as a student of Nikolaĭ Nikolaevich Bogolyubov, had directed much attention to the study of the asymptotic behavior of systems undergoing random perturbations.

A serious illness did not permit Iosif Il'ich to work on this book. Now he is no longer, but the book is published. It would certainly have been different if he had taken part in its writing—he had a better feeling for the "physical" aspects of mathematical theories and could convey this in his expositions, thus giving them more substance. Moreover, he knew far more than was written in his (and others') works.

While recognizing how far this book was from what we had envisioned, I wrote it nevertheless, hoping at least by the choice of topic to pay homage to the shining memory of my teacher and friend.

A. V. Skorokhod

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List of Notation

R—the real line.

R_+—the set of nonnegative numbers.

$a \wedge b$ and $a \vee b$—the smaller and larger of the respective numbers $a, b \in R$.

R^d—the d-dimensional Euclidean space.

$|x|$—the absolute value of a number $x \in R$ or the norm of a vector $x \in X$, where X is a Euclidean space.

(x, y)—the inner product in a Euclidean space.

$X \times Y$—the Cartesion product of sets X and Y.

(x, y)—an element of $X \times Y; x \in X,\ y \in Y$.

$\mathscr{B}_X, \mathscr{B}(X)$—the σ-algebra of Borel subsets of a metric space X.

$(R^a)m_S$—Lebesgue measure on a set S.

$\mathscr{A} \otimes \mathscr{B}$—the product of σ-algebras $\mathscr{A}$ and $\mathscr{B}$.

$\vee \mathscr{A}_n$—the smallest σ-algebra containing $\mathscr{A}_n$.

$\sigma(\xi_\alpha, \alpha \in A)$—the σ-algebra generated by the variables $\{\xi_\alpha, \alpha \in A\}$.

$L(X, Y)$—the linear space of linear operators from a linear space X to a linear space Y.

$\|A\|$—the norm of a linear operator $A \in L(X, Y)$.

A^*—the operator adjoint to A $(A^* \in L(Y, X))$.

$\{e_k\}$—an orthonormal basis in a Euclidean space X.

$\operatorname{tr} A = \sum_1^d (Ae_k, e_k)$.

$x \circ y \in L(X, X)$—defined by $(x \circ y)z = (x, z)y$, where X is a Euclidean space.

$\varphi'(x)$—the function in $L(X, Y)$ defined for $\varphi \colon X \to Y$ by the equality

$$\varphi'(x)y = \frac{d}{dt}\varphi(x + ty)\,|_{t=0}, \qquad x, y \in X,\ t \in R.$$

$\|\varphi\| = \sup |\varphi(x)|$.

C_X—the space of continuous functions on X.

Introduction

Asymptotic problems for stochastic differential equations arose and were solved simultaneously with the very beginnings of the theory of such equations, because the founder of this theory, I. I. Gikhman, was considering first and foremost problems on asymptotic behavior, and he constructed the equations themselves partly in order to be able to pose and solve these problems rigorously. In this he, as a student of N. N. Bogolyubov, was continuing the traditions of the new direction developed in the 1930's by N. M. Krylov and Bogolyubov in investigations on nonlinear mechanics—the study of systems subject to the action of random perturbations. A cycle of papers by Krylov and Bogolyubov [1]–[5] were devoted to these investigations. They established, in particular, ergodic theorems for Markov processes with a phase space of a very general form. Special mention should be made of [1], in which a study was made of the behavior of a system subject to the action of a rapidly variable random force that becomes a "white noise" in the limit. It is this paper that served as an impetus for the creation by Gikhman of the theory of stochastic differential equations. In [1]–[5] various approaches were considered to the rigorous definition of a dynamical system subject to the action of a random force of "white noise" type, as well as the definition of a stochastic differential equation in a random field of forces with independent values, and results were obtained on the asymptotic behavior of the system when the field varies (for example, when impulse actions become continuous actions). (Itô used the convenient concept of a stochastic integral to construct a stochastic equation in [1] and [2]; this form of the equation is more accepted at present.)

We indicate two directions in the asymptotic investigation of systems with random actions: 1) investigation of the behavior of systems as $t \to \infty$, and 2) investigation of systems depending on a small parameter as this parameter tends to zero. The mixed problem also relates here—investigation of a system as a parameter tends to zero and t tends to infinity simultaneously.

The main systems considered are those describable by Markov processes that are, in turn, solutions of stochastic differential equations. However, many of the results are simpler to formulate and prove for Markov processes, and even for processes of a more general form. It is often considerations of convenience that dictate the choice of the form of a system. We remark also that, in addition to problems on the behavior of a system, new problems connected with the study of the asymptotic behavior of distributions (transition probabilities) arise for stochastic systems.

In considering the asymptotic behavior of a system as $t \to \infty$ we are primarily interested in a definite "stabilization" of the system. This term can be used to characterize any regularity that manifests itself in the behavior of the system. The crudest type of such stabilization is boundedness in probability. Under fairly natural assumptions about the probabilistic properties of the system, boundedness in probability implies ergodicity—this property characterizes more precisely the behavior of the system on the whole unbounded interval of variation. Even when the system is not bounded in probability, it can fail to diverge to infinity but instead return to a neighborhood of the original state with probability 1. Then it has an infinite invariant measure, and we can judge the qualitative behavior of the system on the basis of exact quantitative laws.

Although ergodic theory (including ergodic theory for Markov processes) is very well developed, some questions connected with this theory, as well as some results relating specifically to solutions of stochastic equations, are appearing here for the first time in a monograph. Shurenkov's book [1] contains the most complete reflection of the state of ergodic theory for Markov processes, along with a detailed bibliography.

Questions involving (asymptotic) stability of a system in a neighborhood of an equilibrium state or involving instability of the system arise naturally in the study of the behavior of systems on an infinite interval. Under very general assumptions, stability implies asymptotic stability for stochastic systems, and instability with positive probability implies instability with probability 1. Linear systems for which the point 0 is the only equilibrium point are of special interest. Such systems are either stable or unstable. In the latter case the system either diverges to infinity, or oscillates and hence has an invariant measure.

Gikhman founded the theory of stability for solutions of stochastic differential equations in [6] and [7], and then Khas′minskiĭ developed it further in [1]–[5]. We note that the study of stability of linear systems is closely connected with the study of products of independent identically distributed matrices (about this see Bellman, Kesten, and Furstenberg (see Furstenberg [1]), Tutubalin [1], and Sazonov and Tutubalin [1]).

We mention also results of Kulinich [1] that have not appeared in a book: for recurrent processes he found conditions for the existence of a limit distribution for a solution of a stochastic equation under a suitable normalization.

Carrying results relating to stochastic equations in finite-dimensional spaces over to the infinite-dimensional case is far from trivial. Although the form of stochastic equation proposed by Gikhman is insensitive to a change in the dimension of the space, the more natural form based on the Itô integral needed a certain reinterpretation (Daletskiĭ [1], [2]). The study of linear systems led to the concept of a stochastic semigroup (Skorokhod [1], [2], [4], and Butsan [1]). Mean-square stability of solutions of linear equations involves stability of certain now nonrandom semigroups in the Banach space of linear operators acting in a Hilbert space. There is a fairly complete exposition of the theory of stability of such semigroups in Daletskiĭ and Kreĭn's book [1]. A small parameter in the equation has the effect that some terms in the equation become large in comparison with others, and since a stochastic differential equation contains four different terms (the differential of the unknown solution, the drift, the diffusion, and the jumps), we obtain different problems with a small parameter by placing the small parameter as a coefficient of different groups of terms. Most natural is the problem when the system is determined by an ordinary differential equation with a small random perturbation. Then under a mixing condition for the process on the right-hand side it behaves like a solution of a stochastic equation of diffusion type on large time intervals. Another class of problems is connected with the presence of rapidly varying components in the system. If these components have ergodic properties, then their effect on the remaining components is "averaged", i.e., for the latter a closed equation is obtained whose coefficients are the coefficients of the original equation, averaged with respect to an ergodic distribution. These kinds of theorems generalize the Bogolyubov method of averaging to random systems. Gikhman and Khas′minskiĭ occupied themselves with the justification of the Bogolyubov method of averaging in various degrees of generality in the case of stochastic equations (see also Stratonovich [1], [2], V. V. Sarafyan [1], and Sarafyan and Skorokhod [1]).

We remark that for finite Markov chains and semi-Markov processes such a method of averaging was developed by Korolyuk and Turbin [1] (see also Turbin [1]) as a method of asymptotic phase amalgamation.

A special place is occupied by the class of problems on the behavior of a dynamical system under the influence of a small diffusion. They have been investigated by Venttsel′ and Freĭdlin [1] (see also Venttsel′ [1], and Sarafyan [1]), and relate to the determination of an asymptotic expression for the probability of unlikely events (large deviations) such as, for example, the system reaching the boundary of a domain whose interior contains a point of stable equilibrium, due to a small diffusion or a transition of the system from one stable state to another.

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