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Elements of the Theory of Elliptic Functions

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Elements of the Theory of Elliptic Functions

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Providence, Rhode Island

Н. И. АХИЕЗЕР

ЭЛЕМЕНТЫ ТЕОРИИ ЭЛЛИПТИЧЕСКИХ ФУНКЦИЙ

«НАУКА», МОСКВА, 1970

Translated from the Russian by H. H. McFaden
Translation edited by Ben Silver

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ABSTRACT. This book presents a systematic account of the theory of elliptic functions and some of its applications. The main content is intended for engineers who have to work with elliptic functions. Reading this book should not be difficult for people who know the elements of mathematical analysis and the theory of functions in the scope of the first five semesters in the physics-mathematics departments at universities and technical colleges with an advanced program in mathematics.
Bibliography: 29 titles. 24 figures, 27 tables.

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Foreword to the Second Russian Edition

The main content of this book, like that of the first edition in 1948, is intended for engineers who have to use elliptic functions.

In preparing the second edition, I improved the original text in several places.

Numerical tables were added.(*) Moreover, a short chapter (the tenth) was added on generalizations of the Tchebycheff polynomials. In it elliptic functions are applied to the solution of some problems in the constructive theory of functions, and hence this chapter is a continuation of the ninth chapter, in which the well-known investigations of P. L. Tchebycheff and E. I. Zolotarev are presented.

Here I want to remember with gratitude my late friend Vsevolod Konstantinovich Baltag, who read through the manuscript of the first edition and made a number of useful remarks.

The author

(*)Borrowed from the Polish translation of [14] (Warsaw, 1963).

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Tables of the Most Important Formulas

I. Basic trigonometric functions

$$\sin u = u \prod{}' \left(1 - \frac{u}{m\pi}\right) e^{\frac{u}{m\pi}}$$

$$\operatorname{ctg} u = \frac{1}{u} + \sum{}' \left(\frac{1}{u - m\pi} + \frac{1}{m\pi}\right) = \frac{d}{du} \ln \sin u$$

$$\frac{1}{\sin^2 u} = \frac{1}{u^2} + \sum{}' \frac{1}{(u - m\pi)^2} = -\frac{d}{du} \operatorname{ctg} u$$

II. The Weierstrass functions

$$\sigma(u) = u \prod{}' \left(1 - \frac{u}{s}\right) e^{\frac{u}{s} + \frac{u^2}{2s^2}}$$

$$\zeta(u) = \frac{1}{u} + \sum{}' \left(\frac{1}{u - s} + \frac{1}{s} + \frac{u}{s^2}\right) = \frac{d}{du} \ln \sigma(u)$$

$$\wp(u) = \frac{1}{u^2} + \sum{}' \left\{\frac{1}{(u - s)^2} - \frac{1}{s^2}\right\} = -\zeta'(u)$$

$$g_2 = 60 \sum{}' \frac{1}{s^4}; \quad g_3 = 140 \sum{}' \frac{1}{s^6}$$

$$(s = 2m\omega + 2m'\omega')$$

$$\sigma(-u) = -\sigma(u); \quad \zeta(-u) = -\zeta(u); \quad \wp(-u) = \wp(u)$$

$$\sigma(u) = u - \frac{g_2 u^5}{2^4 \cdot 3 \cdot 5} - \frac{g_3 u^7}{2^3 \cdot 3 \cdot 5 \cdot 7} - \cdots$$

$$\zeta(u) = \frac{1}{u} - \frac{g_2 u^3}{2^2 \cdot 3 \cdot 5} - \frac{g_3 u^5}{2^2 \cdot 5 \cdot 7} - \cdots$$

$$\wp(u) = \frac{1}{u^2} + \frac{g_2 u^2}{2^2 \cdot 5} + \frac{g_3 u^4}{2^2 \cdot 7} + \cdots$$

III. HOMOGENEITY RELATIONS

$$\sigma(u)=\sigma(u\mid\omega,\omega')=\sigma(u;g_2,g_3)$$
$$\zeta(u)=\zeta(u\mid\omega,\omega')=\zeta(u;g_2,g_3)$$
$$\wp(u)=\wp(u\mid\omega,\omega')=\wp(u;g_2,g_3)$$
$$g_2=g_2(\omega,\omega');\quad g_3=g_3(\omega,\omega')$$

$$\sigma(\lambda u\mid\lambda\omega,\lambda\omega')=\lambda\sigma(u\mid\omega,\omega')$$
$$\zeta(\lambda u\mid\lambda\omega,\lambda\omega')=\frac{1}{\lambda}\zeta(u\mid\omega,\omega')$$
$$\wp(\lambda u\mid\lambda\omega,\lambda\omega')=\frac{1}{\lambda^2}\wp(u\mid\omega,\omega')$$
$$g_2(\lambda\omega,\lambda\omega')=\frac{1}{\lambda^4}g_2(\omega,\omega')$$
$$g_3(\lambda\omega,\lambda\omega')=\frac{1}{\lambda^6}g_3(\omega,\omega')$$

IV. THE DIFFERENTIAL EQUATION OF THE FUNCTION $\wp$

$$\wp'^2(u)=4\wp^3(u)-g_2\wp(u)-g_3=$$
$$=4\{\wp(u)-e_1\}\{\wp(u)-e_2\}\{\wp(u)-e_3\}$$

$$e_1+e_2+e_3=0$$
$$e_1e_2+e_2e_3+e_3e_1=-\frac{1}{2}(e_1^2+e_2^2+e_3^2)=-\frac{1}{4}g_2$$
$$e_1e_2e_3=\frac{1}{4}g_3$$
$$G=\frac{1}{16}(g_2^3-27g_3^2)=(e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2$$

$$\omega_1=\omega,\quad \omega_2=-\omega-\omega',\quad \omega_3=\omega'$$
$$e_\alpha=\wp(\omega_\alpha)\qquad(\alpha=1,2,3)$$

V. ADDITION OF PERIODS

$$\wp(u+2\omega)=\wp(u+2\omega')=\wp(u)$$

$$\zeta(u+2\omega)=\zeta(u)+2\eta \qquad \eta=\zeta(\omega)=\eta_1$$
$$\zeta(u+2\omega')=\zeta(u)+2\eta' \qquad \eta'=\zeta(\omega')=\eta_3$$

$$\eta\omega'-\eta'\omega=\begin{cases}\dfrac{\pi i}{2} & \left(\Im\dfrac{\omega'}{\omega}>0\right)\\[2ex] -\dfrac{\pi i}{2} & \left(\Im\dfrac{\omega'}{\omega}<0\right)\end{cases}$$

$$\eta_2=-\eta-\eta'$$

$$\sigma(u+2\omega_\alpha)=-e^{2\eta_\alpha(u+\omega_\alpha)}\sigma(u) \quad (\alpha=1,\ 2,\ 3)$$

$$\sigma_\alpha(u)=-e^{\eta_\alpha u}\frac{\sigma(u-\omega_\alpha)}{\sigma(\omega_\alpha)} \quad (\alpha=1,\ 2,\ 3)$$

$$\sigma_\alpha(u+2\omega_\alpha)=-e^{2\eta_\alpha(u+\omega_\alpha)}\sigma_\alpha(u) \quad (\alpha=1,\ 2,\ 3)$$

$$\sigma_\alpha(u+2\omega_\beta)=e^{2\eta_\beta(u+\omega_\beta)}\sigma_\alpha(u) \quad (\beta\neq\alpha;\ \alpha,\ \beta=1,\ 2,\ 3)$$

$$\zeta_\alpha(u)=\frac{d}{du}\ln\sigma_\alpha(u) \quad (\alpha=1,\ 2,\ 3)$$

$$\zeta_\alpha(u)=\zeta(u+\omega_\alpha)-\eta_\alpha \quad (\sigma=1,\ 2,\ 3)$$

VI. Addition theorems of the Weierstrass functions

$$\wp(u)-\wp(v)=-\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)}$$

$$\zeta(u+v)=\zeta(u)+\zeta(v)+\frac{1}{2}\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}$$

$$\wp(u+v)=\wp(u)-\frac{1}{2}\frac{\partial}{\partial u}\left\{\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right\}$$

$$\wp(u+v)+\wp(u)+\wp(v)=\frac{1}{4}\left\{\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right\}^2$$

$$\wp(u+v)-\wp(u-v)=-\frac{\wp'(u)\,\wp'(v)}{[\wp(u)-\wp(v)]^2}$$

$$\begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(w) & \wp'(w) \end{vmatrix}=0 \qquad (u+v+w=0)$$

$$\wp(u+\omega_\alpha)-e_\alpha=\frac{(e_\alpha-e_\beta)(e_\alpha-e_\gamma)}{\wp(u)-e_\alpha} \qquad (\alpha,\ \beta,\ \gamma=1,\ 2,\ 3)$$

$$\sqrt{\wp(u)-e_\alpha}=\frac{\sigma_\alpha(u)}{\sigma(u)} \qquad (\alpha=1,\ 2,\ 3)$$

$$\wp'(u)=-2\frac{\sigma_1(u)\,\sigma_2(u)\,\sigma_3(u)}{[\sigma(u)]^3}$$

VII. Degeneration of the Weierstrass functions

$\omega' = \infty$, ω finite

$$\sigma(u) = \frac{2\omega}{\pi} e^{\frac{1}{3!}\left(\frac{\pi u}{2\omega}\right)^2} \sin\frac{\pi u}{2\omega}$$

$$\zeta(u) = \frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2 u + \frac{\pi}{2\omega}\operatorname{ctg}\frac{\pi u}{2\omega}$$

$$\wp(u) = -\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2 + \left(\frac{\pi}{2\omega}\right)^2 \frac{1}{\sin^2\frac{\pi u}{2\omega}}$$

$$g_2^3 - 27 g_3^2 = 0$$

$$e_1 = \frac{3g_3}{g_2}, \quad e_2 = e_3 = -\frac{3g_3}{2g_2}$$

$$\left(\frac{\pi}{2\omega}\right)^2 = \frac{9g_3}{2g_2}, \quad 2\eta\omega = \frac{\pi^2}{6}$$

$\omega = \infty$, $\omega' = \infty$

$$\sigma(u) = u$$

$$\zeta(u) = \frac{1}{u}$$

$$\wp(u) = \frac{1}{u^2}$$

$$g_2 = g_3 = 0$$

$$e_1 = e_2 = e_3 = 0$$

VIII. THETA FUNCTIONS. REDUCTION FORMULAS

$$h=e^{\pi i\tau},\quad h^{\frac{1}{4}}=e^{\frac{\pi i\tau}{4}},\quad \Im\tau>0,\quad z=e^{\pi iv}$$

$$\vartheta_1(v)=2h^{\frac{1}{4}}\sin\pi v-2h^{\frac{9}{4}}\sin 3\pi v+2h^{\frac{25}{4}}\sin 5\pi v-\dots$$
$$\vartheta_2(v)=2h^{\frac{1}{4}}\cos\pi v+2h^{\frac{9}{4}}\cos 3\pi v+2h^{\frac{25}{4}}\cos 5\pi v+\dots$$
$$\vartheta_3(v)=1+2h\cos 2\pi v+2h^4\cos 4\pi v+2h^9\cos 6\pi v+\dots$$
$$\vartheta_0(v)=1-2h\cos 2\pi v+2h^4\cos 4\pi v-2h^9\cos 6\pi v+\dots$$

$\vartheta_1(v\pm 1)=-\vartheta_1(v)$	$\vartheta_1\left(v\pm\frac{1}{2}\right)=\pm\vartheta_2(v)$
$\vartheta_2(v\pm 1)=-\vartheta_2(v)$	$\vartheta_2\left(v\pm\frac{1}{2}\right)=\mp\vartheta_1(v)$
$\vartheta_3(v\pm 1)=\vartheta_3(v)$	$\vartheta_3\left(v\pm\frac{1}{2}\right)=\vartheta_0(v)$
$\vartheta_0(v\pm 1)=\vartheta_0(v)$	$\vartheta_0\left(v\pm\frac{1}{2}\right)=\vartheta_3(v)$
$\vartheta_1(v\pm\tau)=-h^{-1}z^{\mp 2}\vartheta_1(v)$	$\vartheta_1\left(v\pm\frac{\tau}{2}\right)=\pm ih^{-\frac{1}{4}}z^{\mp 1}\vartheta_0(v)$
$\vartheta_2(v\pm\tau)=h^{-1}z^{\mp 2}\vartheta_2(v)$	$\vartheta_2\left(v\pm\frac{\tau}{2}\right)=h^{-\frac{1}{4}}z^{\mp 1}\vartheta_3(v)$
$\vartheta_3(v\pm\tau)=h^{-1}z^{\mp 2}\vartheta_3(v)$	$\vartheta_3\left(v\pm\frac{\tau}{2}\right)=h^{-\frac{1}{4}}z^{\mp 1}\vartheta_2(v)$
$\vartheta_0(v\pm\tau)=-h^{-1}z^{\mp 2}\vartheta_0(v)$	$\vartheta_0\left(v\pm\frac{\tau}{2}\right)=\pm ih^{-\frac{1}{4}}z^{\mp 1}\vartheta_1(v)$

All theta functions satisfy the differential equation

$$\frac{\partial^2\vartheta}{\partial v^2}=4\pi i\frac{\partial\vartheta}{\partial\tau}\qquad\{\vartheta=\vartheta(v\mid\tau)\}$$

IX. Expansion of theta functions in infinite products

$$h=e^{\pi i\tau},\quad h^{\frac{1}{4}}=e^{\frac{1}{4}\pi i\tau},\quad \Im\tau>0$$

$H_0=\prod_{k=1}^{\infty}(1-h^{2k})$	$H_1=\prod_{k=1}^{\infty}(1+h^{2k})$
$H_2=\prod_{k=1}^{\infty}(1+h^{2k-1})$	$H_3=\prod_{k=1}^{\infty}(1-h^{2k-1})$

$$H_1H_2H_3=1$$

$$\vartheta_1(v)=2H_0h^{\frac{1}{4}}\sin\pi v\prod_{k=1}^{\infty}(1-2h^{2k}\cos 2\pi v+h^{4k})$$

$$\vartheta_2(v)=2H_0h^{\frac{1}{4}}\cos\pi v\prod_{k=1}^{\infty}(1+2h^{2k}\cos 2\pi v+h^{4k})$$

$$\vartheta_3(v)=H_0\prod_{k=1}^{\infty}(1+2h^{2k-1}\cos 2\pi v+h^{4k-2})$$

$$\vartheta_0(v)=H_0\prod_{k=1}^{\infty}(1-2h^{2k-1}\cos 2\pi v+h^{4k-2})$$

Zeros of theta functions

$\vartheta_1(v)$	$\vartheta_2(v)$	$\vartheta_3(v)$	$\vartheta_0(v)$
$m+n\tau$	$m-\frac{1}{2}+n\tau$	$m-\frac{1}{2}+\left(n-\frac{1}{2}\right)\tau$	$m+\left(n-\frac{1}{2}\right)\tau$

$(m,\ n=0,\ \pm 1,\ \pm 2,\ \ldots)$

Zero values of theta functions

$\vartheta_1'=\vartheta_1'(0)=2\pi h^{\frac{1}{4}}H_0^3$ $\vartheta_3=\vartheta_3(0)=H_0H_2^2$	$\vartheta_2=\vartheta_2(0)=2h^{\frac{1}{4}}H_0H_1^2$ $\vartheta_0=\vartheta_0(0)=H_0H_3^2$
$\vartheta_1'=\pi\vartheta_2\vartheta_3\vartheta_0$	$\vartheta_3^4=\vartheta_0^4+\vartheta_2^4$

X. VARIOUS EXPANSIONS IN SIMPLE SERIES

$$\frac{\omega'}{\omega}=\tau \qquad \Im\tau>0 \qquad h=e^{\pi i\tau} \qquad v=\frac{u}{2\omega} \qquad e^{\pi i v}=z$$

$$\sigma(u)=2\omega e^{2\eta\omega v^2}\frac{\vartheta_1(v)}{\vartheta_1'} \qquad \sigma_1(u)=e^{2\eta\omega v^2}\frac{\vartheta_2(v)}{\vartheta_2}$$

$$\sigma_2(u)=e^{2\eta\omega v^2}\frac{\vartheta_3(v)}{\vartheta_3} \qquad \sigma_3(u)=e^{2\eta\omega v^2}\frac{\vartheta_0(v)}{\vartheta_0}$$

$$\zeta(u)=\frac{\eta}{\omega}u+\frac{\pi i}{2\omega}\left\{\frac{z+z^{-1}}{z-z^{-1}}+\sum_{k=1}^{\infty}\frac{2h^{2k}z^{-2}}{1-h^{2k}z^{-2}}-\sum_{k=1}^{\infty}\frac{2h^{2k}z^{2}}{1-h^{2k}z^{2}}\right\}$$

$$\wp(u)=-\frac{\eta}{\omega}-\left(\frac{\pi}{\omega}\right)^2\left\{\frac{1}{(z-z^{-1})^2}+\sum_{k=1}^{\infty}\frac{h^{2k}z^{-2}}{(1-h^{2k}z^{-2})^2}+\sum_{k=1}^{\infty}\frac{h^{2k}z^{2}}{(1-h^{2k}z^{2})^2}\right\}$$

$$2\eta\omega=-\frac{1}{6}\frac{\vartheta_1'''}{\vartheta_1'}=\frac{\pi^2}{6}\frac{1-3^3h^{1\times2}+5^3h^{2\times3}-\dots}{1-3h^{1\times2}+5h^{2\times3}-\dots}=$$

$$=\frac{\pi^2}{6}\left\{1-24\sum_{k=1}^{\infty}\frac{h^{2k}}{(1-h^{2k})^2}\right\}$$

$$e_1=-\frac{\eta}{\omega}+\left(\frac{\pi}{\omega}\right)^2\left\{\frac{1}{4}+2\sum_{k=1}^{\infty}\frac{h^{2k}}{(1+h^{2k})^2}\right\}$$

$$e_2=-\frac{\eta}{\omega}+2\left(\frac{\pi}{\omega}\right)^2\sum_{k=1}^{\infty}\frac{h^{2k-1}}{(1+h^{2k-1})^2}$$

$$e_3=-\frac{\eta}{\omega}-2\left(\frac{\pi}{\omega}\right)^2\sum_{k=1}^{\infty}\frac{h^{2k-1}}{(1-h^{2k-1})^2}$$

$$\sqrt{e_2-e_1}=i\sqrt{e_1-e_2}=i\frac{\pi}{2\omega}\vartheta_0^2$$

$$\sqrt{e_3-e_2}=i\sqrt{e_2-e_3}=-i\frac{\pi}{2\omega}\vartheta_2^2$$

$$\sqrt{e_1-e_3}=i\sqrt{e_3-e_1}=\frac{\pi}{2\omega}\vartheta_3^2$$

XI. Other notations for theta functions

$h=e^{\pi i\tau}$	$\Im\tau>0$

$$K=\frac{\pi}{2}\{1+2h+2h^4+\ldots\}^2=\frac{\pi}{2}\vartheta_3^2=\frac{\pi}{2}\vartheta_3^2(0\mid\tau)$$

$$iK'=\tau K$$

$\lambda=e^{-\frac{\pi i}{4K}(2u+iK')}$	$\mu=e^{-\frac{\pi i}{K}(u+iK')}$

$$\vartheta_\alpha(w)=\vartheta_\alpha\left(\frac{w}{2K}\right)\quad(\alpha=0,\ 1,\ 2,\ 3)$$

$H(w)=\vartheta_1(w)$ $H_1(w)=\vartheta_2(w)$	$\Theta(w)=\vartheta_0(w)$ $\Theta_1(w)=\vartheta_3(w)$

Zeros

$H(w)$	$H_1(w)$	$\Theta(w)$	$\Theta_1(w)$
$2mK+2niK'$	$(2m+1)K+$ $+2niK'$	$2mK+$ $+(2n+1)iK'$	$(2m+1)K+$ $+(2n+1)iK'$

$$(m,\ n=0,\ \pm1,\ \pm2,\ \ldots)$$

$H(u+K)=H_1(u)$ $\Theta(u+K)=\Theta_1(u)$ $H_1(u+K)=-H(u)$ $\Theta_1(u+K)=\Theta(u)$	$H(u+iK')=i\lambda\,\Theta(u)$ $\Theta(u+iK')=i\lambda\,H(u)$ $H_1(u+iK')=\lambda\,\Theta_1(u)$ $\Theta_1(u+iK')=\lambda\,H_1(u)$
$H(u+K+iK')=\lambda\,\Theta_1(u)$ $\Theta(u+K+iK')=\lambda\,H_1(u)$ $H_1(u+K+iK')=-i\lambda\,\Theta(u)$ $\Theta_1(u+K+iK')=i\lambda\,H(u)$	$H(u+2iK')=-\mu\,H(u)$ $\Theta(u+2iK')=-\mu\,\Theta(u)$ $H_1(u+2iK')=\mu\,H_1(u)$ $\Theta_1(u+2iK')=\mu\,\Theta_1(u)$

XII. JACOBI FUNCTIONS

$$\sqrt{\frac{2K}{\pi}} = 1 + 2h + 2h^4 + 2h^9 + \dots$$

$$\sqrt{k} = \frac{H_1(0)}{\Theta_1(0)} = \frac{2h^{\frac{1}{4}} + 2h^{\frac{9}{4}} + \dots}{1 + 2h + 2h^4 + \dots} \qquad (h^{\frac{1}{4}} = e^{\frac{\pi i \tau}{4}})$$

$$\sqrt{k'} = \frac{\Theta(0)}{\Theta_1(0)} = \frac{1 - 2h + 2h^4 - \dots}{1 + 2h + 2h^4 + \dots}$$

$$\operatorname{sn}(u; k) = \frac{1}{\sqrt{k}} \frac{H(u)}{\Theta(u)}$$

$$\operatorname{cn}(u; k) = \sqrt{\frac{k'}{k}} \frac{H_1(u)}{\Theta(u)}$$

$$\operatorname{dn}(u; k) = \sqrt{k'} \frac{\Theta_1(u)}{\Theta(u)}$$

$\operatorname{sn}(u+2K) = -\operatorname{sn} u$	$\operatorname{sn}(u+2iK') = \operatorname{sn} u$
$\operatorname{cn}(u+2K) = -\operatorname{cn} u$	$\operatorname{cn}(u+2iK') = -\operatorname{cn} u$
$\operatorname{dn}(u+2K) = \operatorname{dn} u$	$\operatorname{dn}(u+2iK') = -\operatorname{dn} u$

Periods

$\operatorname{sn} u$	$\operatorname{cn} u$	$\operatorname{dn} u$
$4K,\ 2iK'$	$4K,\ 2K+2iK'$	$2K,\ 4iK'$

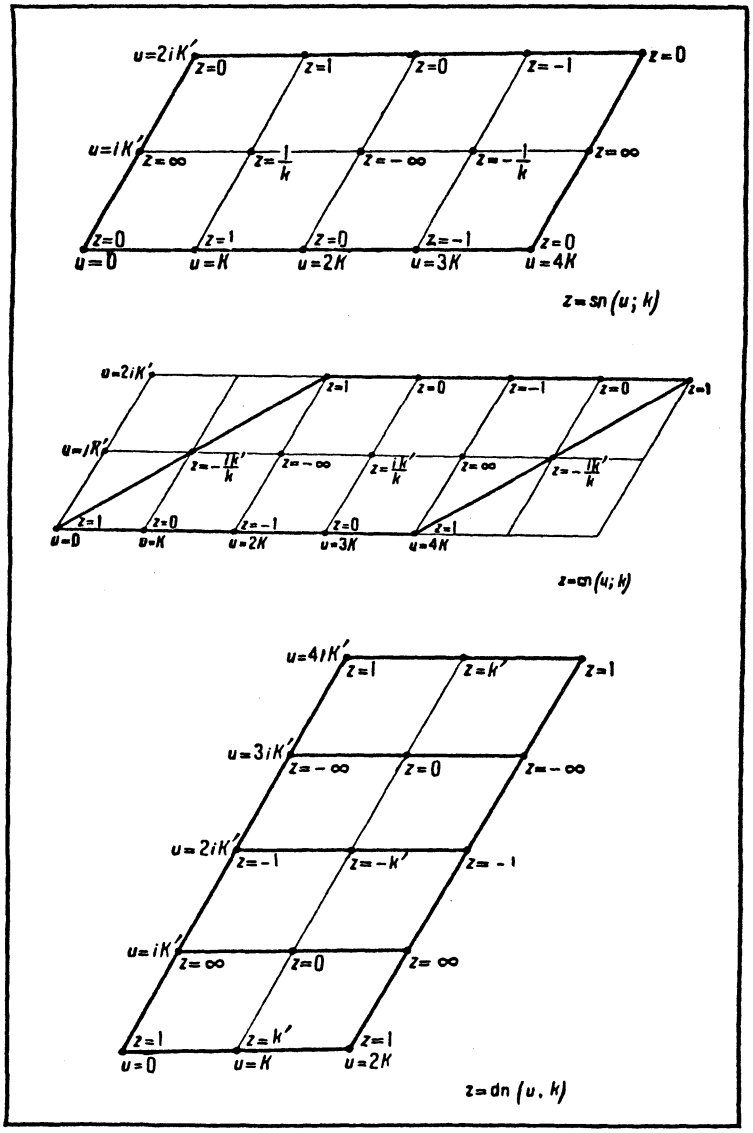
$\operatorname{sn}(u+K) = \frac{\operatorname{cn} u}{\operatorname{dn} u}$	$\operatorname{sn}(u+iK') = \frac{1}{k \operatorname{sn} u}$
$\operatorname{cn}(u+K) = -k' \frac{\operatorname{sn} u}{\operatorname{dn} u}$	$\operatorname{cn}(u+iK') = -i \frac{\operatorname{dn} u}{k \operatorname{sn} u}$
$\operatorname{dn}(u+K) = \frac{k'}{\operatorname{dn} u}$	$\operatorname{dn}(u+iK') = -i \frac{\operatorname{cn} u}{\operatorname{sn} u}$

$$\operatorname{sn}(u+K+iK') = \frac{1}{k} \frac{\operatorname{dn} u}{\operatorname{cn} u}$$

$$\operatorname{cn}(u+K+iK') = -\frac{ik'}{k \operatorname{cn} u}$$

$$\operatorname{dn}(u+K+iK') = ik' \frac{\operatorname{sn} u}{\operatorname{cn} u}$$

XIII. Some values of Jacobi functions



XIV. Differentiation of Jacobi functions. Addition theorems

$$z = \operatorname{sn} u \qquad z'^2 = (1-z^2)(1-k^2z^2)$$

$$z = \operatorname{cn} u \qquad z'^2 = (1-z^2)(1-k^2+k^2z^2)$$

$$z = \operatorname{dn} u \qquad z'^2 = (1-z^2)(z^2-1+k^2)$$

$$\frac{d}{du}\operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u$$

$$\frac{d}{du}\operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u$$

$$\frac{d}{du}\operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u$$

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

$$\operatorname{cn}(u+v) = \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

$$\operatorname{dn}(u+v) = \frac{\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{sn} v \operatorname{cn} v}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

$$\operatorname{sn}(u+v)\operatorname{sn}(u-v) = \frac{\operatorname{sn}^2 u - \operatorname{sn}^2 v}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

$$\operatorname{cn}(u+v)\operatorname{cn}(u-v) = \frac{\operatorname{cn}^2 v - \operatorname{dn}^2 v \operatorname{sn}^2 u}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

$$\operatorname{dn}(u+v)\operatorname{dn}(u-v) = \frac{\operatorname{dn}^2 v - k^2 \operatorname{cn}^2 v \operatorname{sn}^2 u}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

XV. CERTAIN VALUES OF THE JACOBI FUNCTIONS (CONTINUATION)

$$\operatorname{sn}\frac{K}{2}=\frac{1}{\sqrt{1+k'}}$$

$$\operatorname{cn}\frac{K}{2}=\frac{\sqrt{k'}}{\sqrt{1+k'}}$$

$$\operatorname{dn}\frac{K}{2}=\sqrt{k'}$$

$$\operatorname{sn}\frac{iK'}{2}=\frac{i}{\sqrt{k}}$$

$$\operatorname{cn}\frac{iK'}{2}=\frac{\sqrt{1+k}}{\sqrt{k}}$$

$$\operatorname{dn}\frac{iK'}{2}=\sqrt{1+k}$$

$$\operatorname{sn}\frac{K+iK'}{2}=\frac{1}{\sqrt{2}\sqrt{k}}(\sqrt{1+k}+i\sqrt{1-k})$$

$$\operatorname{cn}\frac{K+iK'}{2}=\frac{\sqrt{k'}(1-i)}{\sqrt{2}\sqrt{k}}$$

$$\operatorname{dn}\frac{K+iK'}{2}=\frac{\sqrt{k'}}{\sqrt{2}}(\sqrt{1+k'}-i\sqrt{1-k'})$$

In the normal case ($0 < k < 1$) all the roots are arithmetic

XVI. ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

If the points e_1, e_2, and e_3 lie on a single line, then e_2 denotes the middle one of these points.

$$k^2=\frac{e_2-e_3}{e_1-e_3} \qquad k'^2=\frac{e_1-e_2}{e_1-e_3}$$

$$K=\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \qquad K'=\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}$$

$$E=\int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt \qquad E'=\int_0^1 \sqrt{\frac{1-k'^2t^2}{1-t^2}}\,dt$$

The integral is taken along a rectilinear path with a positive circuit over a small semicircle about the point $t = 1/|k|$ if $1 < k^2 < \infty$, and about the point $t = 1/|k'|$ if $1 < k'^2 < \infty$. In the normal case ($0 < k < 1$) all the integrals are positive.)

$$\eta_1=\sqrt{e_1-e_3}\left\{E-\frac{e_1}{e_1-e_3}K\right\} \qquad \eta_3=-i\sqrt{e_1-e_3}\left\{E'+\frac{e_3}{e_1-e_3}K'\right\}$$

$$EK'+E'K-KK'=\frac{1}{2}\pi$$

$$\omega_1=\frac{K}{\sqrt{e_1-e_3}} \qquad \omega_3=\frac{iK'}{\sqrt{e_1-e_3}}$$

$$\frac{\sigma(u)}{\sigma_3(u)}=\frac{1}{\sqrt{e_1-e_3}}\,\mathrm{sn}\left(\sqrt{e_1-e_3}\,u;\ k\right)$$

$$\frac{\sigma_1(u)}{\sigma_3(u)}=\mathrm{cn}\left(\sqrt{e_1-e_3}\,u;\ k\right) \qquad \frac{\sigma_2(u)}{\sigma_3(u)}=\mathrm{dn}\left(\sqrt{e_1-e_3}\,u;\ k\right)$$

$$E(u)=\int_0^u \mathrm{dn}^2 v\,dv \qquad Z(u)=E(u)-\frac{E}{K}u=\frac{\Theta'(u)}{\Theta(u)}$$

$$\int \frac{du}{\mathrm{sn}^2 u}=u\,Z'(0)-\frac{H'(u)}{H(u)} \qquad \int \frac{du}{\mathrm{cn}^2 u}=\frac{u}{k'^2}Z'(K)-\frac{1}{k'^2}\frac{H_1'(u)}{H_1(u)}$$

$$\int \frac{du}{\mathrm{dn}^2 u}=\frac{u}{k'^2}\frac{E}{K}+\frac{1}{k'^2}\frac{\Theta_1'(u)}{\Theta_1(u)}$$

XVII. TRANSFORMATION OF THETA FUNCTIONS (OF THE FIRST DEGREE)

$$h = e^{\pi i \tau},\ \Im\tau > 0,\ \tau' = -\frac{1}{\tau},\ \Re\sqrt{-i\tau'} > 0,\ i^{-\frac{1}{2}} = e^{-\frac{\pi i}{4}}$$

$$\vartheta_1(v \mid \tau) = i^{-\frac{1}{2}}\vartheta_1(v \mid \tau+1)$$
$$\vartheta_2(v \mid \tau) = i^{-\frac{1}{2}}\vartheta_2(v \mid \tau+1)$$
$$\vartheta_3(v \mid \tau) = \vartheta_0(v \mid \tau+1)$$
$$\vartheta_0(v \mid \tau) = \vartheta_3(v \mid \tau+1)$$

$$\vartheta_1(v \mid \tau) = -i\sqrt{-i\tau'}\, e^{\tau'\pi i v^2}\vartheta_1(\tau' v \mid \tau')$$
$$\vartheta_2(v \mid \tau) = \sqrt{-i\tau'}\, e^{\tau'\pi i v^2}\vartheta_0(\tau' v \mid \tau')$$
$$\vartheta_3(v \mid \tau) = \sqrt{-i\tau'}\, e^{\tau'\pi i v^2}\vartheta_3(\tau' v \mid \tau')$$
$$\vartheta_0(v \mid \tau) = \sqrt{-i\tau'}\, e^{\tau'\pi i v^2}\vartheta_2(\tau' v \mid \tau')$$

XVIII. THE FIRST PRINCIPAL FIRST-DEGREE TRANSFORMATION

$$\lambda = \frac{ik}{k'} \qquad M = \frac{1}{k'}$$
$$L = \frac{K}{M} \qquad iL' = \frac{iK'+K}{M}$$
$$\operatorname{sn}\left(\frac{u}{M};\lambda\right) = \frac{1}{M}\frac{\operatorname{sn}(u;k)}{\operatorname{dn}(u;k)}$$
$$\operatorname{cn}\left(\frac{u}{M};\lambda\right) = \frac{\operatorname{cn}(u;k)}{\operatorname{dn}(u;k)}$$
$$\operatorname{dn}\left(\frac{u}{M};\lambda\right) = \frac{1}{\operatorname{dn}(u;k)}$$

XIX. THE SECOND PRINCIPAL FIRST-DEGREE TRANSFORMATION

$$\lambda = k' \qquad M = \frac{1}{i}$$
$$L = \frac{iK'}{M} \qquad iL' = -\frac{K}{M}$$
$$\operatorname{sn}\left(\frac{u}{M};\lambda\right) = \frac{1}{M}\frac{\operatorname{sn}(u;k)}{\operatorname{cn}(u;k)}$$
$$\operatorname{cn}\left(\frac{u}{M};\lambda\right) = \frac{1}{\operatorname{cn}(u;k)}$$
$$\operatorname{dn}\left(\frac{u}{M};\lambda\right) = \frac{\operatorname{dn}(u;k)}{\operatorname{cn}(u;k)}$$

XX. LANDEN'S TRANSFORMATION

$$\lambda = \frac{1-k'}{1+k'} \qquad M = \frac{1}{1+k'}$$

$$L = \frac{K}{2M} \qquad L' = \frac{K'}{M}$$

$$\operatorname{sn}\left(\frac{u}{M}\,;\,\lambda\right) = \frac{1}{M}\,\frac{\operatorname{sn}(u;k)\operatorname{cn}(u;k)}{\operatorname{dn}(u;k)}$$

$$\operatorname{cn}\left(\frac{u}{M}\,;\,\lambda\right) = \frac{1-(1+k')\operatorname{sn}^2(u;k)}{\operatorname{dn}(u;k)}$$

$$\operatorname{dn}\left(\frac{u}{M}\,;\,\lambda\right) = \frac{1-(1-k')\operatorname{sn}^2(u;k)}{\operatorname{dn}(u;k)}$$

XXI. GAUSS'S TRANSFORMATION

$$\lambda = \frac{2\sqrt{k}}{1+k} \qquad M = \frac{1}{1+k}$$

$$L = \frac{K}{M} \qquad L' = \frac{K'}{2M}$$

$$\operatorname{sn}\left(\frac{u}{M}\,;\,\lambda\right) = \frac{1}{M}\,\frac{\operatorname{sn}(u;k)}{1+k\operatorname{sn}^2(u;k)}$$

$$\operatorname{cn}\left(\frac{u}{M}\,;\,\lambda\right) = \frac{\operatorname{cn}(u;k)\operatorname{dn}(u;k)}{1+k\operatorname{sn}^2(u;k)}$$

$$\operatorname{dn}\left(\frac{u}{M}\,;\,\lambda\right) = \frac{1-k\operatorname{sn}^2(u;k)}{1+k\operatorname{sn}^2(u;k)}$$

$$\vartheta_1\left(v\,\middle|\,\frac{\tau}{2}\right) = \frac{2\vartheta_1(v\,|\,\tau)\,\vartheta_0(v\,|\,\tau)}{\vartheta_2\left(0\,\middle|\,\frac{\tau}{2}\right)}$$

$$\vartheta_2\left(v\,\middle|\,\frac{\tau}{2}\right) = \frac{2\vartheta_2(v\,|\,\tau)\,\vartheta_3(v\,|\,\tau)}{\vartheta_2\left(0\,\middle|\,\frac{\tau}{2}\right)}$$

$$\vartheta_3\left(v\,\middle|\,\frac{\tau}{2}\right) = \frac{\vartheta_0^2(v\,|\,\tau)-\vartheta_1^2(v\,|\,\tau)}{\vartheta_0\left(0\,\middle|\,\frac{\tau}{2}\right)}$$

$$\vartheta_0\left(v\,\middle|\,\frac{\tau}{2}\right) = \frac{\vartheta_0^2(v\,|\,\tau)+\vartheta_1^2(v\,|\,\tau)}{\vartheta_3\left(0\,\middle|\,\frac{\tau}{2}\right)}$$

XXII. THE FIRST PRINCIPAL nTH-DEGREE TRANSFORMATION

$$L=\frac{K}{nM}, \quad L'=\frac{K'}{M}$$

$$c_r=\operatorname{sn}^2\left(\frac{rK}{n}; k\right)$$

$$\lambda=k^n\prod_{r=1}^{[\frac{n}{2}]}c_{2r-1}^2 \qquad M=\prod_{r=1}^{[\frac{n}{2}]}\frac{c_{2r-1}}{c_{2r}}$$

A. n odd

$$\operatorname{sn}\left(\frac{u}{M}; \lambda\right)=\frac{1}{M}\operatorname{sn}(u; k)\prod_{r=1}^{\frac{n-1}{2}}\frac{1-\frac{\operatorname{sn}^2(u; k)}{c_{2r}}}{1-k^2c_{2r}\operatorname{sn}^2(u; k)}$$

$$\operatorname{cn}\left(\frac{u}{M}; \lambda\right)=\operatorname{cn}(u; k)\prod_{r=1}^{\frac{n-1}{2}}\frac{1-\frac{\operatorname{sn}^2(u; k)}{c_{2r-1}}}{1-k^2c_{2r}\operatorname{sn}^2(u; k)}$$

$$\operatorname{dn}\left(\frac{u}{M}; \lambda\right)=\operatorname{dn}(u; k)\prod_{r=1}^{\frac{n-1}{2}}\frac{1-k^2c_{2r-1}\operatorname{sn}^2(u; k)}{1-k^2c_{2r}\operatorname{sn}^2(u; k)}$$

B. n even

$$\operatorname{sn}\left(\frac{u}{M}+L; \lambda\right)=\prod_{r=1}^{\frac{n}{2}}\frac{1-\frac{\operatorname{sn}^2(u; k)}{c_{2r-1}}}{1-k^2c_{2r-1}\operatorname{sn}^2(u; k)}$$

$$\operatorname{cn}\left(\frac{u}{M}+L; \lambda\right)=-\frac{\lambda'}{M}\frac{\operatorname{sn}(u; k)}{\operatorname{cn}(u; k)}\prod_{r=1}^{\frac{n}{2}}\frac{1-\frac{\operatorname{sn}^2(u; k)}{c_{2r}}}{1-k^2c_{2r-1}\operatorname{sn}^2(u; k)}$$

$$\operatorname{dn}\left(\frac{u}{M}+L; \lambda\right)=\frac{\lambda'}{\operatorname{dn}(u; k)}\prod_{r=1}^{\frac{n}{2}}\frac{1-k^2c_{2r}\operatorname{sn}^2(u; k)}{1-k^2c_{2r-1}\operatorname{sn}^2(u; k)}$$

XXIII. THE SECOND PRINCIPAL nTH-DEGREE TRANSFORMATION

$$L=\frac{K}{M}, \quad L'=\frac{K'}{nM}$$

$$\lambda=\prod_{r=1}^{n}\frac{\Theta^2\left(\frac{2r}{n}K';\,k'\right)}{\Theta^2\left(\frac{2r-1}{n}K';\,k'\right)} \qquad M=\prod_{r=1}^{\left[\frac{n}{2}\right]}\frac{\operatorname{sn}^2\left(\frac{2r-1}{n}K',\,k'\right)}{\operatorname{sn}^2\left(\frac{2r}{n}K';\,k'\right)}$$

$$c_r=\frac{\operatorname{sn}^2\left(\frac{rK'}{n};\,k'\right)}{\operatorname{cn}^2\left(\frac{rK'}{n};\,k'\right)} \qquad \delta_r=\operatorname{dn}^2\left(\frac{rK'}{n};\,k'\right)$$

$$\operatorname{sn}\left(\frac{u}{M};\,\lambda\right)=\frac{1}{M}\operatorname{sn}(u;\,k)\prod_{r=1}^{\left[\frac{n}{2}\right]}\frac{1+\frac{\operatorname{sn}^2(u;\,k)}{c_{2r}}}{1+\frac{\operatorname{sn}^2(u;\,k)}{c_{2r-1}}}$$

A. n odd

$$\operatorname{cn}\left(\frac{u}{M};\,\lambda\right)=\operatorname{cn}(u;\,k)\prod_{r=1}^{\frac{n-1}{2}}\frac{1-\delta_{2r}\operatorname{sn}^2(u;\,k)}{1+\frac{\operatorname{sn}^2(u;\,k)}{c_{2r-1}}}$$

$$\operatorname{dn}\left(\frac{u}{M};\,\lambda\right)=\operatorname{dn}(u;\,k)\prod_{r=1}^{\frac{n-1}{2}}\frac{1-\delta_{2r-1}\operatorname{sn}^2(u;\,k)}{1+\frac{\operatorname{sn}^2(u;\,k)}{c_{2r-1}}}$$

B. n even

$$\operatorname{cn}\left(\frac{u}{M};\,\lambda\right)=\operatorname{cn}(u;\,k)\operatorname{dn}(u;\,k)\frac{\prod_{r=1}^{\frac{n}{2}-1}1-\delta_{2r}\operatorname{sn}^2(u;\,k)}{\prod_{r=1}^{\frac{n}{2}}1+\frac{\operatorname{sn}^2(u;\,k)}{c_{2r-1}}}$$

$$\operatorname{dn}\left(\frac{u}{M};\,\lambda\right)=\prod_{r=1}^{\frac{n}{2}}\frac{1-\delta_{2r-1}\operatorname{sn}^2(u;\,k)}{1+\frac{\operatorname{sn}^2(u;\,k)}{c_{2r-1}}}$$

XXIV. SOME INTEGRALS

$$\int \operatorname{sn} u\, du = -\frac{1}{k} \ln(\operatorname{dn} u + k \operatorname{cn} u)$$

$$\int \operatorname{cn} u\, du = \frac{i}{k} \ln(\operatorname{dn} u - ik \operatorname{sn} u)$$

$$\int \operatorname{dn} u\, du = i \ln(\operatorname{cn} u - i \operatorname{sn} u)$$

$$\int \frac{du}{\operatorname{sn} u} = \ln \frac{\operatorname{dn} u - \operatorname{cn} u}{\operatorname{sn} u}$$

$$\int \frac{du}{\operatorname{cn} u} = \frac{1}{k'} \ln \frac{\operatorname{dn} u + k' \operatorname{sn} u}{\operatorname{cn} u}$$

$$\int \frac{du}{\operatorname{dn} u} = \frac{1}{ik'} \ln \frac{\operatorname{cn} u + ik' \operatorname{sn} u}{\operatorname{dn} u}$$

$$\int \frac{\operatorname{sn} u}{\operatorname{cn} u} du = \frac{1}{k'} \ln \frac{\operatorname{dn} u + k'}{\operatorname{cn} u}$$

$$\int \frac{\operatorname{cn} u}{\operatorname{dn} u} du = -\frac{1}{k} \ln \frac{1 - k \operatorname{sn} u}{\operatorname{dn} u}$$

$$\int \frac{\operatorname{dn} u}{\operatorname{sn} u} du = \ln \frac{1 - \operatorname{cn} u}{\operatorname{sn} u}$$

$$\int \frac{\operatorname{sn} u}{\operatorname{dn} u} du = \frac{i}{kk'} \ln \frac{ik' - k \operatorname{cn} u}{\operatorname{dn} u}$$

$$\int \frac{\operatorname{cn} u}{\operatorname{sn} u} du = \ln \frac{1 - \operatorname{dn} u}{\operatorname{sn} u}$$

$$\int \frac{\operatorname{dn} u}{\operatorname{cn} u} du = \ln \frac{1 + \operatorname{sn} u}{\operatorname{cn} u}$$

$$\int \frac{\operatorname{sn} u}{\operatorname{cn}^2 u} du = \frac{1}{k'^2} \frac{\operatorname{dn} u}{\operatorname{cn} u}$$

$$\int \frac{\operatorname{sn} u}{\operatorname{dn}^2 u} du = -\frac{1}{k'^2} \frac{\operatorname{cn} u}{\operatorname{dn} u}$$

$$\int \frac{\operatorname{cn} u}{\operatorname{sn}^2 u} du = -\frac{\operatorname{dn} u}{\operatorname{sn} u}$$

$$\int \frac{\operatorname{cn} u}{\operatorname{dn}^2 u} du = \frac{\operatorname{sn} u}{\operatorname{dn} u}$$

$$\int \frac{\operatorname{dn} u}{\operatorname{sn}^2 u} du = -\frac{\operatorname{cn} u}{\operatorname{sn} u}$$

$$\int \frac{\operatorname{dn} u}{\operatorname{cn}^2 u} du = \frac{\operatorname{sn} u}{\operatorname{cn} u}$$

XXV. COMPUTATION OF ELLIPTIC INTEGRALS IN THE REAL CASE

$$\int R(x, y)\,dx$$

I	$y=\sqrt{(a^2-x^2)(b^2-x^2)} \qquad a^2>b^2$ $k^2=\frac{b^2}{a^2} \qquad x=\begin{cases} b\,\mathrm{sn}\,u & (x^2<b^2) \\ \frac{a}{\mathrm{sn}\,u} & (x^2>a^2) \end{cases}$
II	$y=\sqrt{(a^2-x^2)(x^2-b^2)} \qquad a^2>b^2 \quad b^2<x^2<a^2$ $k^2=\frac{a^2-b^2}{a^2} \qquad x=a\,\mathrm{dn}\,u$
III	$y=\sqrt{(a^2-x^2)(b^2+x^2)} \qquad x^2<a^2$ $k^2=\frac{a^2}{a^2+b^2} \qquad x=a\,\mathrm{cn}\,u$
IV	$y=\sqrt{(x^2-a^2)(b^2+x^2)} \qquad x^2>a^2$ $k^2=\frac{b^2}{a^2+b^2} \qquad x=\frac{a}{\mathrm{cn}\,u}$
V	$y=\sqrt{(a^2+x^2)(b^2+x^2)} \qquad a^2>b^2$ $k^2=\frac{a^2-b^2}{a^2} \qquad x=a\frac{\mathrm{cn}\,u}{\mathrm{sn}\,u}$

Tables of Values of Elliptic Integrals

I. Complete elliptic integrals

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad E = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt$$

α°	$k^2 = \sin^2 \alpha$	K	E	α°	$k^2 = \sin^2 \alpha$	K	E
0	0,00000	1,57080	1,57080	28	0,22040	1,67006	1,48029
1	0,00030	1,57092	1,57068	29	0,23504	1,67773	1,47397
2	0,00122	1,57127	1,57032	30	0,25000	1,68575	1,46746
3	0,00274	1,57187	1,56972	31	0,26526	1,69411	1,46077
4	0,00487	1,57271	1,56888	32	0,28081	1,70284	1,45391
5	0,00760	1,57379	1,56781	33	0,29663	1,71192	1,44687
6	0,01093	1,57511	1,56650	34	0,31270	1,72139	1,43966
7	0,01485	1,57668	1,56495	35	0,32899	1,73125	1,43229
8	0,01937	1,57849	1,56316	36	0,34549	1,74150	1,42476
9	0,02447	1,58054	1,56114	37	0,36218	1,75217	1,41707
10	0,03015	1,58284	1,55889	38	0,37904	1,76326	1,40924
11	0,03641	1,58539	1,55640	39	0,39604	1,77479	1,40126
12	0,04323	1,58820	1,55368	40	0,41318	1,78677	1,39314
13	0,05060	1,59125	1,55073	41	0,43041	1,79922	1,38489
14	0,05853	1,59457	1,54755	42	0,44774	1,81216	1,37650
15	0,06699	1,59814	1,54415	43	0,46512	1,82560	1,36800
16	0,07598	1,60198	1,54052	44	0,48255	1,83957	1,35938
17	0,08548	1,60608	1,53667	45	0,50000	1,85407	1,35064
18	0,09549	1,61045	1,53260	46	0,51745	1,86915	1,34181
19	0,10599	1,61510	1,52831	47	0,53488	1,88481	1,33287
20	0,11698	1,62003	1,52380	48	0,55226	1,90108	1,32384
21	0,12843	1,62523	1,51908	49	0,56959	1,91800	1,31473
22	0,14033	1,63073	1,51415	50	0,58682	1,93558	1,30554
23	0,15267	1,63652	1,50901	51	0,60396	1,95386	1,29628
24	0,16543	1,64260	1,50366	52	0,62096	1,97288	1,28695
25	0,17861	1,64900	1,49811	53	0,63782	1,99267	1,27757
26	0,19217	1,65570	1,49237	54	0,65451	2,01327	1,26815
27	0,20611	1,66272	1,48643	55	0,67101	2,03472	1,25868

CONTINUATION

$\alpha°$	$k^2 = \sin^2\alpha$	K	E	$\alpha°$	$k^2 = \sin^2\alpha$	K	E
56	0,68730	2,05706	1,24918	82,4	0,98251	3,41994	1,02558
57	0,70337	2,08036	1,23966	82,6	0,98341	3,44601	1,02447
58	0,71919	2,10466	1,23013	82,8	0,98429	3,47282	1,02338
59	0,73474	2,13002	1,22059	83,0	0,98515	3,50042	1,02231
60	0,75000	2,15652	1,21106	83,2	0,98598	3,52884	1,02126
61	0,76496	2,18421	1,20154	83,4	0,98680	3,55814	1,02023
62	0,77960	2,21319	1,19205	83,6	0,98757	3,58837	1,01921
63	0,79389	2,24355	1,18259	83,8	0,98834	3,61959	1,01821
64	0,80783	2,27538	1,17318	84,0	0,98907	3,65186	1,01724
65	0,82139	2,30879	1,16383	84,2	0,98979	3,68525	1,01628
66	0,83457	2,34390	1,15455	84,4	0,99048	3,71984	1,01534
67	0,84733	2,38087	1,14535	84,6	0,99114	3,75572	1,01443
68	0,85967	2,41984	1,13624	84,8	0,99178	3,79298	1,01354
69	0,87157	2,46100	1,12725	85,0	0,99240	3,83174	1,01266
70,0	0,88302	2,50455	1,11838	85,2	0,99300	3,87211	1,01181
70,5	0,88857	2,52729	1,11399	85,4	0,99357	3,91423	1,01099
71,0	0,89401	2,55073	1,10964	85,6	0,99411	3,95827	1,01018
71,5	0,89932	2,57490	1,10533	85,8	0,99464	4,00437	1,00940
72,0	0,90451	2,59982	1,10106	86,0	0,99513	4,05276	1,00865
72,5	0,90958	2,62555	1,09683	86,2	0,99561	4,10366	1,00792
73,0	0,91452	2,65214	1,09265	86,4	0,99606	4,15736	1,00721
73,5	0,91934	2,67962	1,08851	86,6	0,99648	4,21416	1,00653
74,0	0,92402	2,70807	1,08443	86,8	0,99688	4,27444	1,00588
74,5	0,92858	2,73752	1,08039	87,0	0,99726	4,33865	1,00526
75,0	0,93301	2,76806	1,07641	87,2	0,99761	4,40733	1,00466
75,5	0,93731	2,79975	1,07248	87,4	0,99794	4,48115	1,00410
76,0	0,94147	2,83267	1,06861	87,6	0,99825	4,56090	1,00356
76,5	0,94550	2,86691	1,06480	87,8	0,99854	4,64765	1,00306
77,0	0,94940	2,90256	1,06106	88,0	0,99878	4,74272	1,00258
77,5	0,95315	2,93974	1,05738	88,2	0,99901	4,84785	1,00215
78,0	0,95677	2,97857	1,05378	88,4	0,99922	4,96542	1,00174
78,5	0,96025	3,01918	1,05024	88,6	0,99940	5,09876	1,00137
79,0	0,96359	3,06173	1,04679	88,8	0,99956	5,25274	1,00104
79,5	0,96679	3,10640	1,04341	89,0	0,99970	5,43491	1,00075
80,0	0,96985	3,15339	1,04011	89,1	0,99975	5,54020	1,00062
80,2	0,97103	3,17288	1,03882	89,2	0,99981	5,65792	1,00050
80,4	0,97219	3,19280	1,03754	89,3	0,99985	5,79140	1,00039
80,6	0,97332	3,21317	1,03628	89,4	0,99989	5,94550	1,00030
80,8	0,97444	3,23400	1,03503	89,5	0,99992	6,12778	1,00021
81,0	0,97553	3,25530	1,03379	89,6	0,99995	6,35088	1,00014
81,2	0,97660	3,27711	1,03257	89,7	0,99997	6,63854	1,00008
81,4	0,97764	3,29945	1,03136	89,8	0,99999	7,04398	1,00004
81,6	0,97866	3,32234	1,03017	89,9	1,00000	7,73711	1,00001
81,8	0,97966	3,34580	1,02900	90,0	1,00000	∞	1,00000
82,0	0,98063	3,36987	1,02784				
82,2	0,98158	3,39457	1,02670				

k^2	K	K'	K'/K	K/K'	$\ln k$	$\ln k'$	k'^2
0,00	1,57080	∞	∞	0,00000	$-\infty$	0,00000	1,00
0,01	1,57475	3,69564	2,34682	0,42611	0,79806-4	0,41863-1	0,99
0,02	1,57874	3,35411	2,12457	0,47068	0,10129-3	0,35781-1	0,98
0,03	1,58278	3,15587	1,99388	0,50153	0,27960-3	0,31572-1	0,97
0,04	1,58687	3,01611	1,90067	0,52613	0,40677-3	0,28216-1	0,96
0,05	1,59100	2,90834	1,82799	0,54705	0,50393-3	0,25362-1	0,95
0,06	1,59519	2,82075	1,76828	0,56552	0,58738-3	0,22842-1	0,94
0,07	1,59942	2,74707	1,71754	0,58223	0,65663-3	0,20562-1	0,93
0,08	1,60371	2,68355	1,67334	0,59761	0,71693-3	0,18464-1	0,92
0,09	1,60805	2,62777	1,63414	0,61194	0,77042-3	0,16508-1	0,91
0,10	1,61244	2,57809	1,59887	0,62544	0,81853-3	0,14666-1	0,90
0,11	1,61689	2,53333	1,56680	0,63825	0,86230-3	0,12919-1	0,89
0,12	1,62139	2,49264	1,53734	0,65047	0,90249-3	0,11251-1	0,88
0,13	1,62595	2,45534	1,51009	0,66221	0,93967-3	0,09649-1	0,87
0,14	1,63058	2,42093	1,48471	0,67353	0,97430-3	0,08105-1	0,86
0,15	1,63526	2,38902	1,46094	0,68449	0,00672-2	0,06610-1	0,85
0,16	1,64000	2,35926	1,43858	0,69513	0,03724-2	0,05158-1	0,84
0,17	1,64481	2,33141	1,41744	0,70550	0,06608-2	0,03743-1	0,83
0,18	1,64968	2,30523	1,39738	0,71562	0,09344-2	0,02362-1	0,82
0,19	1,65462	2,28055	1,37829	0,72553	0,11949-2	0,01010-1	0,81
0,20	1,65962	2,25721	1,36007	0,73526	0,14435-2	0,99683-2	0,80
0,21	1,66470	2,23507	1,34262	0,74481	0,16816-2	0,98380-2	0,79
0,22	1,66985	2,21402	1,32588	0,75422	0,19099-2	0,97097-2	0,78
0,23	1,67507	2,19397	1,30978	0,76349	0,21297-2	0,95831-2	0,77
0,24	1,68037	2,17483	1,29425	0,77265	0,23415-2	0,94582-2	0,76
0,25	1,68575	2,15652	1,27926	0,78171	0,25461-2	0,93347-2	0,75
0,26	1,69121	2,13897	1,26476	0,79066	0,27439-2	0,92124-2	0,74
0,27	1,69675	2,12213	1,25070	0,79955	0,29356-2	0,90911-2	0,73
0,28	1,70237	2,10595	1,23707	0,80836	0,31218-2	0,89709-2	0,72
0,29	1,70809	2,09037	1,22381	0,81712	0,33026-2	0,88514-2	0,71
0,30	1,71389	2,07536	1,21091	0,82583	0,34787-2	0,87326-2	0,70
0,31	1,71978	2,06088	1,19834	0,83449	0,36502-2	0,86144-2	0,69
0,32	1,72577	2,04689	1,18607	0,84312	0,38175-2	0,84967-2	0,68
0,33	1,73186	2,03336	1,17409	0,85172	0,39810-2	0,83793-2	0,67
0,34	1,73805	2,02028	1,16238	0,86030	0,41408-2	0,82622-2	0,66
0,35	1,74435	2,00760	1,15091	0,86887	0,42972-2	0,81453-2	0,65
0,36	1,75075	1,99530	1,13986	0,87744	0,44504-2	0,80284-2	0,64
0,37	1,75727	1,98337	1,12867	0,88600	0,46007-2	0,79116-2	0,63
0,38	1,76390	1,97178	1,11786	0,89457	0,47482-2	0,77947-2	0,62
0,39	1,77065	1,96052	1,10723	0,90315	0,48932-2	0,76776-2	0,61
0,40	1,77752	1,94957	1,09679	0,91175	0,50356-2	0,75603-2	0,60
0,41	1,78452	1,93891	1,08652	0,92037	0,51758-2	0,74426-2	0,59
0,42	1,79165	1,92853	1,07640	0,92903	0,53139-2	0,73246-2	0,58
0,43	1,79892	1,91841	1,06642	0,93771	0,54500-2	0,72061-2	0,57
0,44	1,80633	1,90855	1,05659	0,94644	0,55841-2	0,70870-2	0,56
0,45	1,81388	1,89892	1,04688	0,95522	0,57166-2	0,69673-2	0,55
0,46	1,82159	1,88953	1,03730	0,96404	0,58474-2	0,68468-2	0,54
0,47	1,82946	1,88036	1,02782	0,97293	0,59766-2	0,67256-2	0,53
0,48	1,83749	1,87140	1,01845	0,98188	0,61045-2	0,66035-2	0,52
0,49	1,84569	1,86264	1,00918	0,99090	0,62310-2	0,64804-2	0,51
0,50	1,85407	1,85407	1,00000	1,00000	0,63562-2	0,63562-2	0,50

k'^2	K'	K	K/K'	K'/K	$\ln k'$	$\ln k$	k^2

SMALL VALUES OF THE MODULUS

k^2	K	K'	K'/K	K/K'	k'^2
0,000001	1,57080	8,29405	5,28016	0,18939	0,999999
0,000002	1,57080	7,94748	5,05952	0,19765	0,999998
0,000003	1,57080	7,74475	4,93046	0,20282	0,999997
0,000004	1,57080	7,60091	4,83888	0,20666	0,999996
0,000005	1,57080	7,48934	4,76786	0,20974	0,999995
0,000006	1,57080	7,39818	4,70982	0,21232	0,999994
0,000007	1,57080	7,32111	4,66075	0,21456	0,999993
0,000008	1,57080	7,25434	4,61825	0,21653	0,999992
0,000009	1,57080	7,19545	4,58076	0,21830	0,999991
0,000010	1,57080	7,14277	4,54722	0,21991	0,999990
0,000100	1,57083	5,99159	3,81427	0,26217	0,999900
0,000200	1,57087	5,64512	3,59362	0,27827	0,999800
0,000300	1,57091	5,44249	3,46454	0,28864	0,999700
0,000400	1,57095	5,29875	3,37295	0,29648	0,999600
0,000500	1,57099	5,18727	3,30191	0,30286	0,999500
0,000600	1,57103	5,09620	3,24385	0,30828	0,999400
0,000700	1,57107	5,01921	3,19477	0,31301	0,999300
0,000800	1,57111	4,95253	3,15225	0,31723	0,999200
0,000900	1,57115	4,89373	3,11474	0,32105	0,999100
0,001000	1,57119	4,84113	3,08119	0,32455	0,999000
0,001100	1,57123	4,79356	3,05084	0,32778	0,998900
0,001200	1,57127	4,75014	3,02312	0,33078	0,998800
0,001300	1,57131	4,71020	2,99763	0,33360	0,998700
0,001400	1,57135	4,67322	2,97402	0,33624	0,998600
0,001500	1,57139	4,63880	2,95205	0,33875	0,998500
0,001600	1,57142	4,60661	2,93149	0,34112	0,998400
0,001700	1,57146	4,57638	2,91217	0,34339	0,998300
0,001800	1,57150	4,54788	2,89396	0,34555	0,998200
0,001900	1,57154	4,52092	2,87674	0,34762	0,998100
0,002000	1,57158	4,49535	2,86040	0,34960	0,998000
0,002100	1,57162	4,47103	2,84485	0,35151	0,997900
0,002200	1,57166	4,44784	2,83002	0,35335	0,997800
0,002300	1,57171	4,42569	2,81586	0,35513	0,997700
0,002400	1,57174	4,40448	2,80231	0,35685	0,997600
0,002500	1,57178	4,38414	2,78929	0,35851	0,997500
0,002600	1,57182	4,36461	2,77679	0,36013	0,997400
0,002700	1,57186	4,34581	2,76476	0,36170	0,997300
0,002800	1,57190	4,32769	2,75317	0,36322	0,997200
0,002900	1,57194	4,31022	2,74198	0,36470	0,997100
0,003000	1,57198	4,29334	2,73117	0,36614	0,997000

k'^2	K'	K	K/K'	K'/K	k^2

II. THE ELLIPTIC INTEGRAL OF THE FIRST KIND

$$F(\varphi, k) = \int_0^{\varphi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}, \quad k = \sin \alpha$$

φ°	α = 5°	α = 10°	α = 15°	α = 20°	α = 25°	α = 30°
1	0,01745	0,01745	0,01745	0,01745	0,01745	0,01745
2	0,03491	0,03491	0,03491	0,03491	0,03491	0,03491
3	0,05236	0,05236	0,05236	0,05236	0,05236	0,05237
4	0,06981	0,06981	0,06982	0,06982	0,06982	0,06983
5	0,08727	0,08727	0,08728	0,08728	0,08729	0,08729
6	0,1047	0,1047	0,1047	0,1047	0,1048	0,1048
7	0,1222	0,1222	0,1222	0,1222	0,1222	0,1223
8	0,1396	0,1396	0,1397	0,1397	0,1397	0,1397
9	0,1571	0,1571	0,1571	0,1572	0,1572	0,1572
10	0,1745	0,1746	0,1746	0,1746	0,1747	0,1748
11	0,1920	0,1920	0,1921	0,1921	0,1922	0,1923
12	0,2095	0,2095	0,2095	0,2096	0,2097	0,2098
13	0,2269	0,2270	0,2270	0,2271	0,2272	0,2274
14	0,2444	0,2444	0,2445	0,2446	0,2448	0,2450
15	0,2618	0,2619	0,2620	0,2622	0,2623	0,2625
16	0,2793	0,2794	0,2795	0,2797	0,2799	0,2802
17	0,2967	0,2968	0,2970	0,2972	0,2975	0,2978
18	0,3142	0,3143	0,3145	0,3148	0,3151	0,3154
19	0,3317	0,3318	0,3320	0,3323	0,3327	0,3331
20	0,3491	0,3493	0,3495	0,3499	0,3503	0,3508
21	0,3666	0,3668	0,3671	0,3675	0,3680	0,3686
22	0,3840	0,3843	0,3846	0,3851	0,3856	0,3863
23	0,4015	0,4017	0,4021	0,4027	0,4033	0,4041
24	0,4190	0,4192	0,4197	0,4203	0,4210	0,4219
25	0,4364	0,4367	0,4372	0,4379	0,4388	0,4397
26	0,4539	0,4542	0,4548	0,4556	0,4565	0,4576
27	0,4714	0,4717	0,4724	0,4732	0,4743	0,4755
28	0,4888	0,4893	0,4899	0,4909	0,4921	0,4935
29	0,5063	0,5068	0,5075	0,5086	0,5099	0,5114
30	0,5238	0,5243	0,5251	0,5263	0,5277	0,5294
31	0,5412	0,5418	0,5427	0,5440	0,5456	0,5475
32	0,5587	0,5593	0,5604	0,5618	0,5635	0,5656
33	0,5762	0,5769	0,5780	0,5795	0,5814	0,5837
34	0,5937	0,5944	0,5956	0,5973	0,5994	0,6018
35	0,6111	0,6119	0,6133	0,6151	0,6173	0,6200
36	0,6286	0,6295	0,6309	0,6329	0,6355	0,6383
37	0,6461	0,6470	0,6486	0,6507	0,6534	0,6566
38	0,6636	0,6646	0,6662	0,6685	0,6714	0,6749
39	0,6810	0,6821	0,6839	0,6864	0,6895	0,6932
40	0,6985	0,6997	0,7016	0,7043	0,7077	0,7117
41	0,7160	0,7173	0,7193	0,7222	0,7258	0,7301
42	0,7335	0,7348	0,7370	0,7401	0,7440	0,7486
43	0,7510	0,7524	0,7548	0,7581	0,7622	0,7671
44	0,7685	0,7700	0,7725	0,7760	0,7804	0,7857
45	0,7859	0,7876	0,7903	0,7940	0,7987	0,8044

$F(\varphi, k)$, $k = \sin\alpha$

$\varphi°$	$\alpha = 5°$	$\alpha = 10°$	$\alpha = 15°$	$\alpha = 20°$	$\alpha = 25°$	$\alpha = 30°$
46	0,8034	0,8052	0,8080	0,8120	0,8170	0,8231
47	0,8209	0,8228	0,8258	0,8300	0,8354	0,8418
48	0,8384	0,8404	0,8436	0,8480	0,8537	0,8606
49	0,8559	0,8580	0,8614	0,8661	0,8721	0,8794
50	0,8734	0,8756	0,8792	0,8842	0,8905	0,8983
51	0,8909	0,8932	0,8970	0,9023	0,9090	0,9172
52	0,9084	0,9108	0,9148	0,9204	0,9275	0,9361
53	0,9259	0,9284	0,9326	0,9385	0,9460	0,9551
54	0,9434	0,9460	0,9505	0,9567	0,9646	0,9742
55	0,9609	0,9637	0,9683	0,9748	0,9832	0,9933
56	0,9784	0,9813	0,9862	0,9930	1,0018	1,0125
57	0,9959	0,9989	1,0041	1,0112	1,0204	1,0317
58	1,0134	1,0166	1,0219	1,0295	1,0391	1,0509
59	1,0309	1,0342	1,0398	1,0477	1,0578	1,0702
60	1,0484	1,0519	1,0577	1,0660	1,0766	1,0896
61	1,0659	1,0695	1,0757	1,0843	1,0953	1,1089
62	1,0834	1,0872	1,0936	1,1026	1,1141	1,1284
63	1,1009	1,1049	1,1115	1,1209	1,1330	1,1478
64	1,1184	1,1225	1,1295	1,1392	1,1518	1,1674
65	1,1359	1,1402	1,1474	1,1576	1,1707	1,1869
66	1,1534	1,1579	1,1654	1,1759	1,1896	1,2065
67	1,1709	1,1756	1,1833	1,1943	1,2085	1,2262
68	1,1884	1,1932	1,2013	1,2127	1,2275	1,2458
69	1,2059	1,2109	1,2193	1,2311	1,2465	1,2656
70	1,2235	1,2286	1,2373	1,2495	1,2655	1,2853
71	1,2410	1,2463	1,2553	1,2680	1,2845	1,3051
72	1,2585	1,2640	1,2733	1,2864	1,3036	1,3249
73	1,2760	1,2817	1,2913	1,3049	1,3226	1,3448
74	1,2935	1,2994	1,3093	1,3234	1,3417	1,3647
75	1,3110	1,3171	1,3273	1,3418	1,3608	1,3846
76	1,3285	1,3348	1,3454	1,3603	1,3800	1,4045
77	1,3461	1,3525	1,3634	1,3788	1,3991	1,4245
78	1,3636	1,3702	1,3814	1,3974	1,4183	1,4445
79	1,3811	1,3879	1,3995	1,4159	1,4374	1,4645
80	1,3986	1,4057	1,4175	1,4344	1,4566	1,4846
81	1,4161	1,4234	1,4356	1,4530	1,4758	1,5046
82	1,4336	1,4411	1,4536	1,4715	1,4950	1,5247
83	1,4512	1,4588	1,4717	1,4901	1,5143	1,5448
84	1,4687	1,4765	1,4897	1,5086	1,5335	1,5649
85	1,4862	1,4942	1,5078	1,5272	1,5527	1,5850
86	1,5037	1,5120	1,5259	1,5457	1,5720	1,6052
87	1,5212	1,5297	1,5439	1,5643	1,5912	1,6253
88	1,5388	1,5474	1,5620	1,5829	1,6105	1,6455
89	1,5563	1,5651	1,5801	1,6015	1,6297	1,6656
90	1,5738	1,5828	1,5981	1,6200	1,6490	1,6858

$F(\varphi, k)$, $k = \sin\alpha$

$\varphi°$	$\alpha = 35°$	$\alpha = 40°$	$\alpha = 45°$	$\alpha = 50°$	$\alpha = 55°$	$\alpha = 60°$
1	0,01745	0,01745	0,01745	0,01745	0,01745	0,01745
2	0,03491	0,03491	0,03491	0,03491	0,03491	0,03491
3	0,05237	0,05237	0,05237	0,05237	0,05238	0,05238
4	0,06983	0,06984	0,06984	0,06985	0,06985	0,06986
5	0,08730	0,08731	0,08732	0,08733	0,08734	0,08735
6	0,1048	0,1048	0,1048	0,1048	0,1049	0,1049
7	0,1223	0,1223	0,1223	0,1224	0,1224	0,1224
8	0,1398	0,1398	0,1399	0,1399	0,1399	0,1400
9	0,1573	0,1574	0,1574	0,1575	0,1575	0,1576
10	0,1748	0,1749	0,1750	0,1751	0,1751	0,1752
11	0,1924	0,1925	0,1926	0,1927	0,1928	0,1929
12	0,2099	0,2101	0,2102	0,2103	0,2105	0,2106
13	0,2275	0,2277	0,2279	0,2280	0,2282	0,2284
14	0,2451	0,2454	0,2456	0,2458	0,2460	0,2462
15	0,2628	0,2630	0,2633	0,2636	0,2638	0,2641
16	0,2804	0,2808	0,2811	0,2814	0,2817	0,2820
17	0,2981	0,2985	0,2989	0,2993	0,2997	0,3000
18	0,3159	0,3163	0,3168	0,3172	0,3177	0,3181
19	0,3336	0,3341	0,3347	0,3352	0,3357	0,3362
20	0,3514	0,3520	0,3526	0,3533	0,3539	0,3545
21	0,3692	0,3699	0,3706	0,3714	0,3721	0,3728
22	0,3871	0,3879	0,3887	0,3896	0,3904	0,3912
23	0,4049	0,4059	0,4068	0,4078	0,4088	0,4097
24	0,4229	0,4239	0,4250	0,4261	0,4272	0,4283
25	0,4408	0,4420	0,4433	0,4446	0,4458	0,4470
26	0,4589	0,4602	0,4616	0,4630	0,4645	0,4658
27	0,4769	0,4784	0,4800	0,4816	0,4832	0,4847
28	0,4950	0,4967	0,4985	0,5003	0,5021	0,5038
29	0,5132	0,5150	0,5170	0,5190	0,5210	0,5229
30	0,5313	0,5334	0,5356	0,5379	0,5401	0,5422
31	0,5496	0,5519	0,5543	0,5568	0,5593	0,5617
32	0,5679	0,5704	0,5731	0,5759	0,5786	0,5812
33	0,5862	0,5890	0,5920	0,5950	0,5980	0,6010
34	0,6046	0,6077	0,6109	0,6143	0,6176	0,6208
35	0,6231	0,6264	0,6300	0,6336	0,6373	0,6409
36	0,6416	0,6452	0,6491	0,6531	0,6572	0,6610
37	0,6602	0,6641	0,6684	0,6727	0,6771	0,6814
38	0,6788	0,6831	0,6877	0,6925	0,6973	0,7020
39	0,6975	0,7021	0,7071	0,7123	0,7176	0,7227
40	0,7162	0,7213	0,7267	0,7323	0,7380	0,7436
41	0,7350	0,7405	0,7463	0,7524	0,7586	0,7647
42	0,7539	0,7598	0,7661	0,7727	0,7794	0,7860
43	0,7728	0,7791	0,7859	0,7931	0,8004	0,8075
44	0,7918	0,7986	0,8059	0,8136	0,8215	0,8293
45	0,8109	0,8182	0,8260	0,8343	0,8428	0,8512

$F(\varphi, k)$, $k = \sin\alpha$

$\varphi°$	$\alpha = 35°$	$\alpha = 40°$	$\alpha = 45°$	$\alpha = 50°$	$\alpha = 55°$	$\alpha = 60°$
46	0,8300	0,8378	0,8462	0,8552	0,8643	0,8734
47	0,8492	0,8575	0,8666	0,8761	0,8860	0,8959
48	0,8685	0,8773	0,8870	0,8973	0,9079	0,9185
49	0,8878	0,8973	0,9076	0,9186	0,9300	0,9415
50	0,9072	0,9173	0,9283	0,9401	0,9523	0,9647
51	0,9267	0,9374	0,9491	0,9617	0,9748	0,9881
52	0,9462	0,9576	0,9701	0,9835	0,9976	1,0119
53	0,9658	0,9778	0,9912	1,0055	1,0206	1,0359
54	0,9855	0,9982	1,0124	1,0277	1,0437	1,0602
55	1,0052	1,0187	1,0337	1,0500	1,0672	1,0848
56	1,0250	1,0393	1,0552	1,0725	1,0908	1,1097
57	1,0449	1,0600	1,0768	1,0952	1,1147	1,1349
58	1,0648	1,0807	1,0985	1,1180	1,1389	1,1605
59	1,0848	1,1016	1,1204	1,1411	1,1633	1,1864
60	1,1049	1,1226	1,1424	1,1643	1,1879	1,2126
61	1,1250	1,1436	1,1646	1,1877	1,2128	1,2392
62	1,1453	1,1648	1,1869	1,2113	1,2379	1,2661
63	1,1655	1,1860	1,2093	1,2351	1,2633	1,2933
64	1,1859	1,2074	1,2318	1,2591	1,2890	1,3209
65	1,2063	1,2288	1,2545	1,2833	1,3149	1,3489
66	1,2267	1,2503	1,2773	1,3076	1,3411	1,3773
67	1,2472	1,2719	1,3002	1,3321	1,3675	1,4060
68	1,2678	1,2936	1,3233	1,3568	1,3942	1,4351
69	1,2885	1,3154	1,3464	1,3817	1,4212	1,4646
70	1,3092	1,3372	1,3697	1,4068	1,4484	1,4944
71	1,3299	1,3592	1,3931	1,4320	1,4759	1,5246
72	1,3507	1,3812	1,4167	1,4574	1,5036	1,5552
73	1,3716	1,4033	1,4403	1,4830	1,5316	1,5862
74	1,3924	1,4254	1,4640	1,5087	1,5597	1,6175
75	1,4134	1,4477	1,4879	1,5346	1,5882	1,6492
76	1,4344	1,4700	1,5118	1,5606	1,6168	1,6812
77	1,4554	1,4923	1,5359	1,5867	1,6457	1,7136
78	1,4765	1,5147	1,5600	1,6130	1,6748	1,7463
79	1,4976	1,5372	1,5842	1,6394	1,7040	1,7792
80	1,5187	1,5597	1,6085	1,6660	1,7335	1,8125
81	1,5399	1,5823	1,6328	1,6926	1,7631	1,8461
82	1,5611	1,6049	1,6573	1,7194	1,7929	1,8799
83	1,5823	1,6276	1,6817	1,7462	1,8228	1,9140
84	1,6035	1,6502	1,7063	1,7731	1,8528	1,9482
85	1,6248	1,6730	1,7308	1,8001	1,8830	1,9826
86	1,6461	1,6957	1,7554	1,8271	1,9132	2,0172
87	1,6673	1,7184	1,7801	1,8542	1,9435	2,0519
88	1,6886	1,7412	1,8047	1,8813	1,9739	2,0867
89	1,7099	1,7640	1,8294	1,9084	2,0043	2,1216
90	1,7313	1,7868	1,8541	1,9356	2,0347	2,1565

$F(\varphi, k)$, $k = \sin\alpha$

$\varphi°$	$\alpha = 65°$	$\alpha = 70°$	$\alpha = 75°$	$\alpha = 80°$	$\alpha = 85°$	$\alpha = 90°$
1	0,01745	0,01745	0,01745	0,01745	0,01745	0,01745
2	0,03491	0,03491	0,03491	0,03491	0,03491	0,03491
3	0,05238	0,05238	0,05238	0,05238	0,06238	0,05238
4	0,06986	0,06986	0,06987	0,06987	0,06987	0,06987
5	0,08736	0,08736	0,08737	0,08737	0,08738	0,08738
6	0,1049	0,1049	0,1049	0,1049	0,1049	0,1049
7	0,1224	0,1224	0,1225	0,1225	0,1225	0,1225
8	0,1400	0,1400	0,1401	0,1401	0,1401	0,1401
9	0,1576	0,1577	0,1577	0,1577	0,1577	0,1577
10	0,1753	0,1753	0,1754	0,1754	0,1754	0,1754
11	0,1930	0,1930	0,1931	0,1931	0,1932	0,1932
12	0,2107	0,2108	0,2109	0,2109	0,2110	0,2110
13	0,2285	0,2286	0,2287	0,2288	0,2289	0,2289
14	0,2464	0,2465	0,2466	0,2467	0,2468	0,2468
15	0,2643	0,2645	0,2646	0,2648	0,2648	0,2648
16	0,2823	0,2825	0,2827	0,2828	0,2829	0,2830
17	0,3003	0,3006	0,3009	0,3010	0,3011	0,3012
18	0,3185	0,3188	0,3191	0,3193	0,3194	0,3195
19	0,3367	0,3371	0,3374	0,3377	0,3378	0,3379
20	0,3550	0,3555	0,3559	0,3562	0,3563	0,3564
21	0,3734	0,3740	0,3744	0,3747	0,3749	0,3750
22	0,3919	0,3926	0,3931	0,3935	0,3937	0,3938
23	0,4105	0,4113	0,4119	0,4123	0,4126	0,4127
24	0,4293	0,4301	0,4308	0,4313	0,4316	0,4317
25	0,4481	0,4490	0,4498	0,4504	0,4508	0,4509
26	0,4670	0,4681	0,4690	0,4697	0,4701	0,4702
27	0,4861	0,4874	0,4884	0,4891	0,4896	0,4897
28	0,5053	0,5067	0,5079	0,5087	0,5092	0,5094
29	0,5247	0,5262	0,5275	0,5285	0,5291	0,5293
30	0,5442	0,5459	0,5474	0,5484	0,5491	0,5493
31	0,5639	0,5658	0,5674	0,5686	0,5693	0,5696
32	0,5837	0,5858	0,5876	0,5889	0,5898	0,5900
33	0,6037	0,6060	0,6080	0,6095	0,6104	0,6107
34	0,6238	0,6265	0,6287	0,6303	0,6313	0,6317
35	0,6442	0,6471	0,6495	0,6513	0,6525	0,6528
36	0,6647	0,6679	0,6706	0,6726	0,6739	0,6743
37	0,6854	0,6890	0,6919	0,6941	0,6955	0,6960
38	0,7063	0,7102	0,7135	0,7159	0,7175	0,7180
39	0,7275	0,7318	0,7353	0,7380	0,7397	0,7403
40	0,7488	0,7535	0,7575	0,7604	0,7623	0,7629
41	0,7704	0,7756	0,7799	0,7831	0,7852	0,7859
42	0,7922	0,7979	0,8026	0,8062	0,8084	0,8092
43	0,8143	0,8205	0,8256	0,8295	0,8320	0,8328
44	0,8367	0,8433	0,8490	0,8533	0,8560	0,8569
45	0,8593	0,8665	0,8727	0,8774	0,8804	0,8814

$F(\varphi, k)$, $k = \sin\alpha$

$\varphi°$	$\alpha = 65°$	$\alpha = 70°$	$\alpha = 75°$	$\alpha = 80°$	$\alpha = 85°$	$\alpha = 90°$
46	0,8821	0,8901	0,8968	0,9019	0,9052	0,9063
47	0,9053	0,9139	0,9212	0,9269	0,9304	0,9316
48	0,9288	0,9381	0,9461	0,9523	0,9561	0,9575
49	0,9525	0,9627	0,9714	0,9781	0,9824	0,9838
50	0,9766	0,9876	0,9971	1,0044	1,0091	1,0107
51	1,0010	1,0130	1,0233	1,0313	1,0364	1,0381
52	1,0258	1,0387	1,0500	1,0587	1,0643	1,0662
53	1,0509	1,0649	1,0771	1,0867	1,0927	1,0948
54	1,0764	1,0916	1,1048	1,1152	1,1219	1,1242
55	1,1022	1,1187	1,1331	1,1444	1,1517	1,1542
56	1,1285	1,1462	1,1619	1,1743	1,1823	1,1851
57	1,1551	1,1743	1,1914	1,2049	1,2136	1,2167
58	1,1822	1,2030	1,2215	1,2362	1,2458	1,2492
59	1,2097	1,2321	1,2522	1,2684	1,2789	1,2826
60	1,2376	1,2619	1,2837	1,3014	1,3129	1,3170
61	1,2660	1,2922	1,3159	1,3352	1,3480	1,3524
62	1,2949	1,3231	1,3490	1,3701	1,3841	1,3890
63	1,3243	1,3547	1,3828	1,4059	1,4214	1,4268
64	1,3541	1,3870	1,4175	1,4429	1,4599	1,4659
65	1,3844	1,4199	1,4532	1,4810	1,4998	1,5065
66	1,4153	1,4536	1,4898	1,5203	1,5411	1,5485
67	1,4467	1,4880	1,5274	1,5610	1,5840	1,5923
68	1,4786	1,5232	1,5661	1,6030	1,6287	1,6379
69	1,5111	1,5591	1,6059	1,6466	1,6752	1,6856
70	1,5441	1,5959	1,6468	1,6918	1,7237	1,7354
71	1,5777	1,6335	1,6891	1,7388	1,7745	1,7877
72	1,6118	1,6720	1,7326	1,7876	1,8277	1,8427
73	1,6465	1,7113	1,7774	1,8384	1,8837	1,9008
74	1,6818	1,7516	1,8237	1,8915	1,9427	1,9623
75	1,7176	1,7927	1,8715	1,9468	2,0050	2,0276
76	1,7540	1,8347	1,9207	2,0047	2,0711	2,0973
77	1,7909	1,8777	1,9716	2,0653	2,1414	2,1721
78	1,8284	1,9215	2,0240	2,1288	2,2164	2,2528
79	1,8664	1,9663	2,0781	2,1954	2,2969	2,3404
80	1,9048	2,0119	2,1339	2,2653	2,3837	2,4362
81	1,9438	2,0584	2,1913	2,3387	2,4775	2,5421
82	1,9831	2,1057	2,2504	2,4157	2,5795	2,6603
83	2,0229	2,1537	2,3110	2,4965	2,6911	2,7942
84	2,0630	2,2024	2,3731	2,5811	2,8136	2,9487
85	2,1035	2,2518	2,4366	2,6694	2,9487	3,1313
86	2,1442	2,3017	2,5013	2,7612	3,0978	3,3547
87	2,1852	2,3520	2,5670	2,8561	3,2620	3,6425
88	2,2263	2,4027	2,6336	2,9537	3,4412	4,0481
89	2,2675	2,4535	2,7007	3,0530	3,6328	4,7413
90	2,3088	2,5046	2,7681	3,1534	3,8317	∞

III. THE ELLIPTIC INTEGRAL OF THE SECOND KIND

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 t}\, dt, \quad k = \sin \alpha$$

φ°	$\alpha = 5^\circ$	$\alpha = 10^\circ$	$\alpha = 15^\circ$	$\alpha = 20^\circ$	$\alpha = 25^\circ$	$\alpha = 30^\circ$
1	0,0175	0,0175	0,0175	0,0175	0,0175	0,0175
2	0,0349	0,0349	0,0349	0,0349	0,0349	0,0349
3	0,0524	0,0524	0,0524	0,0524	0,0524	0,0524
4	0,0698	0,0698	0,0698	0,0698	0,0698	0,0698
5	0,0873	0,0873	0,0873	0,0873	0,0873	0,0872
6	0,1047	0,1047	0,1047	0,1047	0,1047	0,1047
7	0,1222	0,1222	0,1222	0,1221	0,1221	0,1221
8	0,1396	0,1396	0,1396	0,1396	0,1396	0,1395
9	0,1571	0,1571	0,1570	0,1570	0,1570	0,1569
10	0,1745	0,1745	0,1745	0,1744	0,1744	0,1743
11	0,1920	0,1920	0,1919	0,1919	0,1918	0,1917
12	0,2094	0,2094	0,2093	0,2093	0,2092	0,2091
13	0,2269	0,2268	0,2268	0,2267	0,2266	0,2264
14	0,2443	0,2443	0,2442	0,2441	0,2439	0,2437
15	0,2618	0,2617	0,2616	0,2615	0,2613	0,2611
16	0,2792	0,2791	0,2790	0,2788	0,2786	0,2784
17	0,2967	0,2966	0,2964	0,2962	0,2959	0,2956
18	0,3141	0,3140	0,3138	0,3136	0,3133	0,3129
19	0,3316	0,3314	0,3312	0,3309	0,3305	0,3301
20	0,3490	0,3489	0,3486	0,3483	0,3478	0,3473
21	0,3665	0,3663	0,3660	0,3656	0,3651	0,3645
22	0,3839	0,3837	0,3834	0,3829	0,3823	0,3817
23	0,4014	0,4011	0,4007	0,4002	0,3996	0,3988
24	0,4188	0,4185	0,4181	0,4175	0,4168	0,4159
25	0,4362	0,4359	0,4354	0,4348	0,4339	0,4330
26	0,4537	0,4533	0,4528	0,4520	0,4511	0,4500
27	0,4711	0,4707	0,4701	0,4693	0,4682	0,4670
28	0,4886	0,4881	0,4875	0,4865	0,4854	0,4840
29	0,5060	0,5055	0,5048	0,5037	0,5025	0,5010
30	0,5234	0,5229	0,5221	0,5209	0,5195	0,5179
31	0,5409	0,5403	0,5394	0,5381	0,5366	0,5348
32	0,5583	0,5577	0,5567	0,5553	0,5536	0,5516
33	0,5757	0,5751	0,5740	0,5725	0,5706	0,5684
34	0,5932	0,5924	0,5912	0,5896	0,5876	0,5852
35	0,6106	0,6098	0,6085	0,6067	0,6045	0,6019
36	0,6280	0,6272	0,6258	0,6238	0,6214	0,6186
37	0,6455	0,6445	0,6430	0,6409	0,6383	0,6353
38	0,6629	0,6619	0,6602	0,6580	0,6552	0,6519
39	0,6803	0,6792	0,6775	0,6750	0,6720	0,6685
40	0,6977	0,6966	0,6947	0,6921	0,6888	0,6851
41	0,7152	0,7139	0,7119	0,7091	0,7056	0,7016
42	0,7326	0,7313	0,7291	0,7261	0,7224	0,7180
43	0,7500	0,7486	0,7463	0,7431	0,7391	0,7345
44	0,7674	0,7659	0,7634	0,7600	0,7558	0,7509
45	0,7849	0,7832	0,7806	0,7770	0,7725	0,7672

$E(\varphi, k)$, $k = \sin\alpha$

$\varphi°$	$\alpha = 5°$	$\alpha = 10°$	$\alpha = 15°$	$\alpha = 20°$	$\alpha = 25°$	$\alpha = 30°$
46	0,8023	0,8006	0,7978	0,7939	0,7891	0,7835
47	0,8197	0,8179	0,8149	0,8108	0,8057	0,7998
48	0,8371	0,8352	0,8320	0,8277	0,8223	0,8160
49	0,8545	0,8525	0,8491	0,8446	0,8389	0,8322
50	0,8719	0,8698	0,8663	0,8614	0,8554	0,8483
51	0,8894	0,8871	0,8834	0,8783	0,8719	0,8644
52	0,9068	0,9044	0,9005	0,8951	0,8884	0,8805
53	0,9242	0,9217	0,9175	0,9119	0,9048	0,8965
54	0,9416	0,9390	0,9346	0,9287	0,9212	0,9125
55	0,9590	0,9562	0,9517	0,9454	0,9376	0,9284
56	0,9764	0,9735	0,9687	0,9622	0,9540	0,9443
57	0,9938	0,9908	0,9858	0,9789	0,9703	0,9602
58	1,0112	1,0080	1,0028	0,9956	0,9866	0,9760
59	1,0286	1,0253	1,0198	1,0123	1,0029	0,9918
60	1,0460	1,0426	1,0368	1,0290	1,0192	1,0076
61	1,0634	1,0598	1,0538	1,0456	1,0354	1,0233
62	1,0808	1,0771	1,0708	1,0623	1,0516	1,0390
63	1,0982	1,0943	1,0878	1,0789	1,0678	1,0546
64	1,1156	1,1115	1,1048	1,0955	1,0839	1,0702
65	1,1330	1,1288	1,1218	1,1121	1,1001	1,0858
66	1,1504	1,1460	1,1387	1,1287	1,1162	1,1013
67	1,1678	1,1632	1,1557	1,1453	1,1323	1,1168
68	1,1852	1,1805	1,1726	1,1619	1,1483	1,1323
69	1,2026	1,1977	1,1896	1,1784	1,1644	1,1478
70	1,2200	1,2149	1,2065	1,1949	1,1804	1,1632
71	1,2374	1,2321	1,2234	1,2115	1,1964	1,1786
72	1,2548	1,2494	1,2403	1,2280	1,2124	1,1939
73	1,2722	1,2666	1,2573	1,2445	1,2284	1,2093
74	1,2896	1,2838	1,2742	1,2609	1,2443	1,2246
75	1,3070	1,3010	1,2911	1,2774	1,2603	1,2399
76	1,3244	1,3182	1,3080	1,2939	1,2762	1,2552
77	1,3418	1,3354	1,3249	1,3104	1,2921	1,2704
78	1,3592	1,3526	1,3417	1,3268	1,3080	1,2857
79	1,3765	1,3698	1,3586	1,3433	1,3239	1,3009
80	1,3939	1,3870	1,3755	1,3597	1,3398	1,3161
81	1,4113	1,4042	1,3924	1,3761	1,3556	1,3312
82	1,4287	1,4214	1,4093	1,3925	1,3715	1,3464
83	1,4461	1,4386	1,4261	1,4090	1,3873	1,3616
84	1,4635	1,4558	1,4430	1,4254	1,4032	1,3767
85	1,4809	1,4729	1,4599	1,4418	1,4190	1,3919
86	1,4983	1,4901	1,4767	1,4582	1,4348	1,4070
87	1,5157	1,5073	1,4936	1,4746	1,4507	1,4221
88	1,5330	1,5245	1,5104	1,4910	1,4665	1,4372
89	1,5504	1,5417	1,5273	1,5074	1,4823	1,4524
90	1,5678	1,5589	1,5442	1,5238	1,4981	1,4675

$E(\varphi, k)$, $k = \sin\alpha$

$\varphi°$	$\alpha = 35°$	$\alpha = 40°$	$\alpha = 45°$	$\alpha = 50°$	$\alpha = 55°$	$\alpha = 60°$
1	0,0175	0,0175	0,0175	0,0175	0,0175	0,0175
2	0,0349	0,0349	0,0349	0,0349	0,0349	0,0349
3	0,0524	0,0524	0,0524	0,0524	0,0523	0,0523
4	0,0698	0,0698	0,0698	0,0698	0,0698	0,0698
5	0,0872	0,0872	0,0872	0,0872	0,0872	0,0872
6	0,1047	0,1046	0,1046	0,1046	0,1046	0,1046
7	0,1221	0,1221	0,1220	0,1220	0,1220	0,1220
8	0,1395	0,1394	0,1394	0,1394	0,1393	0,1393
9	0,1569	0,1568	0,1568	0,1567	0,1567	0,1566
10	0,1743	0,1742	0,1741	0,1740	0,1739	0,1739
11	0,1916	0,1915	0,1914	0,1913	0,1912	0,1911
12	0,2089	0,2088	0,2087	0,2086	0,2084	0,2083
13	0,2263	0,2261	0,2259	0,2258	0,2256	0,2254
14	0,2436	0,2434	0,2431	0,2429	0,2427	0,2425
15	0,2608	0,2606	0,2603	0,2601	0,2598	0,2596
16	0,2781	0,2778	0,2775	0,2771	0,2768	0,2766
17	0,2953	0,2949	0,2946	0,2942	0,2938	0,2935
18	0,3125	0,3121	0,3116	0,3112	0,3107	0,3103
19	0,3297	0,3291	0,3286	0,3281	0,3276	0,3271
20	0,3468	0,3462	0,3456	0,3450	0,3444	0,3438
21	0,3639	0,3632	0,3625	0,3618	0,3611	0,3604
22	0,3809	0,3802	0,3793	0,3785	0,3777	0,3770
23	0,3980	0,3971	0,3961	0,3952	0,3943	0,3935
24	0,4150	0,4139	0,4129	0,4118	0,4108	0,4098
25	0,4319	0,4308	0,4296	0,4284	0,4272	0,4261
26	0,4488	0,4475	0,4462	0,4449	0,4436	0,4423
27	0,4657	0,4643	0,4628	0,4613	0,4598	0,4584
28	0,4825	0,4809	0,4793	0,4776	0,4760	0,4744
29	0,4993	0,4975	0,4957	0,4938	0,4920	0,4903
30	0,5161	0,5141	0,5121	0,5100	0,5080	0,5061
31	0,5328	0,5306	0,5283	0,5261	0,5239	0,5218
32	0,5494	0,5470	0,5446	0,5421	0,5396	0,5373
33	0,5660	0,5634	0,5607	0,5580	0,5553	0,5528
34	0,5826	0,5797	0,5768	0,5738	0,5709	0,5681
35	0,5991	0,5960	0,5928	0,5895	0,5863	0,5833
36	0,6155	0,6122	0,6087	0,6052	0,6017	0,5984
37	0,6319	0,6283	0,6245	0,6207	0,6169	0,6134
38	0,6483	0,6444	0,6403	0,6361	0,6321	0,6282
39	0,6646	0,6604	0,6559	0,6515	0,6471	0,6429
40	0,6808	0,6763	0,6715	0,6667	0,6620	0,6575
41	0,6970	0,6921	0,6870	0,6819	0,6768	0,6719
42	0,7132	0,7079	0,7025	0,6969	0,6914	0,6862
43	0,7293	0,7237	0,7178	0,7118	0,7059	0,7003
44	0,7453	0,7393	0,7330	0,7267	0,7204	0,7144
45	0,7613	0,7549	0,7482	0,7414	0,7347	0,7282

$E(\varphi, k)$, $k = \sin\alpha$

$\varphi°$	$\alpha = 35°$	$\alpha = 40°$	$\alpha = 45°$	$\alpha = 50°$	$\alpha = 55°$	$\alpha = 60°$
46	0,7772	0,7704	0,7633	0,7560	0,7488	0,7420
47	0,7931	0,7858	0,7782	0,7705	0,7629	0,7555
48	0,8089	0,8012	0,7931	0,7849	0,7768	0,7690
49	0,8247	0,8165	0,8079	0,7992	0,7905	0,7823
50	0,8404	0,8317	0,8227	0,8134	0,8042	0,7954
51	0,8560	0,8469	0,8373	0,8275	0,8177	0,8004
52	0,8716	0,8620	0,8518	0,8414	0,8311	0,8212
53	0,8872	0,8770	0,8663	0,8553	0,8444	0,8339
54	0,9026	0,8919	0,8806	0,8690	0,8575	0,8464
55	0,9181	0,9068	0,8949	0,8827	0,8705	0,8588
56	0,9335	0,9216	0,9091	0,8962	0,8834	0,8710
57	0,9488	0,9363	0,9232	0,9097	0,8961	0,8831
58	0,9641	0,9510	0,9372	0,9230	0,9088	0,8950
59	0,9793	0,9656	0,9511	0,9362	0,9213	0,9068
60	0,9945	0,9801	0,9650	0,9493	0,9336	0,9184
61	1,0096	0,9946	0,9787	0,9623	0,9459	0,9299
62	1,0247	1,0090	0,9924	0,9752	0,9580	0,9412
63	1,0397	1,0233	1,0060	0,9880	0,9700	0,9524
64	1,0547	1,0376	1,0195	1,0007	0,9818	0,9634
65	1,0696	1,0518	1,0329	1,0133	0,9936	0,9743
66	1,0845	1,0660	1,0463	1,0259	1,0052	0,9850
67	1,0993	1,0801	1,0596	1,0383	1,0167	0,9956
68	1,1141	1,0941	1,0728	1,0506	1,0282	1,0061
69	1,1289	1,1081	1,0859	1,0628	1,0395	1,0164
70	1,1436	1,1221	1,0990	1,0750	1,0506	1,0266
71	1,1583	1,1359	1,1120	1,0871	1,0617	1,0367
72	1,1729	1,1498	1,1250	1,0991	1,0727	1,0467
73	1,1875	1,1636	1,1379	1,1110	1,0836	1,0565
74	1,2021	1,1773	1,1507	1,1228	1,0944	1,0662
75	1,2167	1,1910	1,1635	1,1346	1,1051	1,0759
76	1,2312	1,2047	1,1762	1,1463	1,1158	1,0854
77	1,2457	1,2183	1,1889	1,1580	1,1263	1,0948
78	1,2601	1,2319	1,2015	1,1695	1,1368	1,1041
79	1,2746	1,2454	1,2141	1,1811	1,1472	1,1133
80	1,2890	1,2590	1,2266	1,1926	1,1576	1,1225
81	1,3034	1,2725	1,2391	1,2040	1,1678	1,1316
82	1,3177	1,2859	1,2516	1,2154	1,1781	1,1406
83	1,3321	1,2994	1,2640	1,2267	1,1883	1,1495
84	1,3464	1,3128	1,2765	1,2381	1,1984	1,1584
85	1,3608	1,3262	1,2889	1,2493	1,2085	1,1673
86	1,3751	1,3396	1,3012	1,2606	1,2186	1,1761
87	1,3894	1,3530	1,3136	1,2719	1,2286	1,1848
88	1,4037	1,3664	1,3260	1,2831	1,2387	1,1936
89	1,4180	1,3798	1,3383	1,2943	1,2487	1,2023
90	1,4323	1,3931	1,3506	1,3055	1,2587	1,2111

$E(\varphi, k)$. $k = \sin\alpha$

$\varphi°$	$\alpha = 65°$	$\alpha = 70°$	$\alpha = 75°$	$\alpha = 80°$	$\alpha = 85°$	$\alpha = 90°$
1	0,0175	0,0175	0,0175	0,0175	0,0175	0,0175
2	0,0349	0,0349	0,0349	0,0349	0,0349	0,0349
3	0,0523	0,0523	0,0523	0,0523	0,0523	0,0523
4	0,0698	0,0698	0,0698	0,0698	0,0698	0,0698
5	0,0872	0,0872	0,0872	0,0872	0,0872	0,0872
6	0,1046	0,1046	0,1045	0,1045	0,1045	0,1045
7	0,1219	0,1219	0,1219	0,1219	0,1219	0,1219
8	0,1393	0,1392	0,1392	0,1392	0,1392	0,1392
9	0,1566	0,1565	0,1565	0,1565	0,1564	0,1564
10	0,1738	0,1738	0,1737	0,1737	0,1737	0,1737
11	0,1910	0,1910	0,1909	0,1908	0,1908	0,1908
12	0,2082	0,2081	0,2080	0,2080	0,2079	0,2079
13	0,2253	0,2252	0,2251	0,2250	0,2250	0,2250
14	0,2424	0,2422	0,2421	0,2420	0,2419	0,2419
15	0,2594	0,2592	0,2590	0,2589	0,2588	0,2588
16	0,2763	0,2761	0,2759	0,2758	0,2757	0,2756
17	0,2932	0,2929	0,2927	0,2925	0,2924	0,2924
18	0,3100	0,3096	0,3094	0,3092	0,3091	0,3090
19	0,3267	0,3263	0,3260	0,3258	0,3256	0,3256
20	0,3433	0,3429	0,3425	0,3422	0,3421	0,3420
21	0,3599	0,3593	0,3589	0,3586	0,3584	0,3584
22	0,3763	0,3757	0,3753	0,3749	0,3747	0,3746
23	0,3927	0,3920	0,3915	0,3911	0,3908	0,3907
24	0,4090	0,4082	0,4076	0,4071	0,4068	0,4067
25	0,4251	0,4243	0,4236	0,4230	0,4227	0,4226
26	0,4412	0,4402	0,4394	0,4389	0,4385	0,4384
27	0,4572	0,4561	0,4552	0,4545	0,4541	0,4540
28	0,4730	0,4718	0,4708	0,4701	0,4696	0,4695
29	0,4888	0,4874	0,4863	0,4855	0,4850	0,4848
30	0,5044	0,5029	0,5017	0,5007	0,5002	0,5000
31	0,5199	0,5182	0,5169	0,5159	0,5153	0,5150
32	0,5352	0,5334	0,5319	0,5308	0,5302	0,5299
33	0,5505	0,5485	0,5468	0,5456	0,5449	0,5446
34	0,5656	0,5634	0,5616	0,5603	0,5595	0,5592
35	0,5806	0,5782	0,5762	0,5748	0,5739	0,5736
36	0,5954	0,5928	0,5907	0,5891	0,5881	0,5878
37	0,6101	0,6073	0,6050	0,6032	0,6022	0,6018
38	0,6247	0,6216	0,6191	0,6172	0,6161	0,6157
39	0,6391	0,6357	0,6330	0,6310	0,6297	0,6293
40	0,6533	0,6497	0,6468	0,6446	0,6432	0,6428
41	0,6675	0,6636	0,6604	0,6580	0,6566	0,6561
42	0,6814	0,6772	0,6738	0,6712	0,6697	0,6691
43	0,6952	0,6907	0,6870	0,6843	0,6826	0,6820
44	0,7088	0,7040	0,7001	0,6971	0,6953	0,6947
45	0,7223	0,7172	0,7129	0,7097	0,7078	0,7071

$E(\varphi, k)$, $k = \sin\alpha$

$\varphi°$	$\alpha = 65°$	$\alpha = 70°$	$\alpha = 75°$	$\alpha = 80°$	$\alpha = 85°$	$\alpha = 90°$
46	0,7356	0,7301	0,7255	0,7222	0,7201	0,7193
47	0,7488	0,7429	0,7380	0,7344	0,7321	0,7314
48	0,7618	0,7555	0,7503	0,7464	0,7440	0,7431
49	0,7746	0,7679	0,7623	0,7582	0,7556	0,7547
50	0,7872	0,7801	0,7741	0,7697	0,7670	0,7660
51	0,7997	0,7921	0,7858	0,7811	0,7781	0,7772
52	0,8120	0,8039	0,7972	0,7922	0,7891	0,7880
53	0,8242	0,8155	0,8084	0,8031	0,7998	0,7986
54	0,8361	0,8270	0,8194	0,8137	0,8102	0,8090
55	0,8479	0,8382	0,8302	0,8242	0,8204	0,8192
56	0,8595	0,8493	0,8408	0,8344	0,8304	0,8290
57	0,8709	0,8601	0,8511	0,8443	0,8401	0,8387
58	0,8822	0,8707	0,8612	0,8540	0,8496	0,8481
59	0,8933	0,8812	0,8711	0,8635	0,8588	0,8572
60	0,9042	0,8914	0,8808	0,8728	0,8677	0,8660
61	0,9149	0,9015	0,8903	0,8818	0,8764	0,8746
62	0,9254	0,9113	0,8995	0,8905	0,8849	0,8830
63	0,9358	0,9210	0,9085	0,8990	0,8930	0,8910
64	0,9460	0,9304	0,9173	0,9072	0,9009	0,8988
65	0,9561	0,9397	0,9258	0,9152	0,9086	0,9063
66	0,9659	0,9487	0,9341	0,9230	0,9160	0,9136
67	0,9756	0,9576	0,9422	0,9305	0,9231	0,9205
68	0,9852	0,9662	0,9501	0,9377	0,9299	0,9272
69	0,9946	0,9747	0,9578	0,9447	0,9364	0,9336
70	1,0038	0,9830	0,9652	0,9514	0,9427	0,9397
71	1,0129	0,9911	0,9724	0,9579	0,9487	0,9455
72	1,0218	0,9990	0,9794	0,9642	0,9544	0,9511
73	1,0306	1,0067	0,9862	0,9702	0,9599	0,9563
74	1,0392	1,0143	0,9928	0,9759	0,9650	0,9613
75	1,0477	1,0217	0,9992	0,9814	0,9699	0,9659
76	1,0561	1,0290	1,0053	0,9867	0,9745	0,9703
77	1,0643	1,0361	1,0113	0,9917	0,9789	0,9744
78	1,0725	1,0430	1,0171	0,9965	0,9829	0,9782
79	1,0805	1,0498	1,0228	1,0011	0,9867	0,9816
80	1,0884	1,0565	1,0282	1,0054	0,9902	0,9848
81	1,0962	1,0630	1,0335	1,0096	0,9935	0,9877
82	1,1040	1,0695	1,0387	1,0135	0,9965	0,9903
83	1,1116	1,0758	1,0437	1,0173	0,9992	0,9926
84	1,1192	1,0821	1,0486	1,0209	1,0017	0,9945
85	1,1267	1,0883	1,0534	1,0244	1,0039	0,9962
86	1,1342	1,0944	1,0581	1,0277	1,0060	0,9976
87	1,1417	1,1004	1,0628	1,0309	1,0078	0,9986
88	1,1491	1,1064	1,0674	1,0340	1,0095	0,9994
89	1,1565	1,1124	1,0719	1,0371	1,0111	0,9999
90	1,1638	1,1184	1,0764	1,0401	1,0127	1,0000

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