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Geometric Function Theory in Several Complex Variables

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Geometric Function Theory in Several Complex Variables

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Providence, Rhode Island

幾何学的関数論

KIKAGAKUTEKI KANSU-RON (Geometric Function Theory in Several Complex Variables)

by Junjiro Noguchi & Takushiro Ochiai

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ABSTRACT. This expanded edition in English gives a self-contained account of recent developments in geometric function theory in several complex variables and a fundamental treatise on the theory of positive currents, plurisubharmonic functions, and meromorphic mappings, culminating on the value-distribution theory for holomorphic curves.

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Foreword

This book is an expanded and largely rewritten version by J. Noguchi of the original Japanese edition by Noguchi with T. Ochiai ([89]). The purpose of this book is two fold. The first one is to give a self-contained and coherent account of recent developments in geometric function theory in several complex variables to those readers who have already learned the basics of complex function theory and the very elementary part of the theory of differential and complex manifolds. The second is to present, in a self-contained manner, sufficient fundamental accounts of the theory of positive currents and plurisubharmonic functions, and of the notion of meromorphic mappings, which are nowadays indispensable in the analytic and geometric theories of complex functions in several variables.

The elementary complex function theory in one variable consists, roughly speaking, of the following three themes:

I The contents from the definition of holomorphic functions to Cauchy's integral formula and the applications:

II Riemann's mapping theorem and the construction of Riemann surfaces via analytic continuation:

III The value distribution of meromorphic functions.

In the course, we have learned such theorems as Montel's theorem on normal families and Picard's little and big theorems. R. Nevanlinna evolved those theorems to the so-called Nevanlinna theory by establishing his first and second main theorems. The contents of III may be considered the basic part of the Nevanlinna theory.

In recent years, the contents of III have been generalized and extended to problems of the complex function theory in several variables and of the value distribution of holomorphic and meromorphic mappings between higher dimensional complex manifolds, and many interesting results have been obtained.

Problem of value distribution. *Let M and N be complex manifolds and $f: M \to N$ a holomorphic mapping (or more generally, meromorphic mapping). Then, investigate the properties of f and the image $f(M)$ in N.*

In this book we discuss the problem of the value distribution through geometric methods and treat the subject from the elementary level to the latest developments and topics.

Chapter I is devoted to the theory of Kobayashi pseudo-distances. S. Kobayashi defined a pseudo-distance d_M for an arbitrary complex manifold M, and called M a hyperbolic manifold if d_M is a distance. The definition is so natural that it immediately implies the decreasing principle $d_N(f(x), f(y)) \leq d_M(x, y)$ for all $x, y \in M$, which plays an essential role throughout his theory. A part of the problem of the value distribution can be reduced to those of Kobayashi pseudo-distances and be systematically discussed. Kobayashi wrote a comprehensive text book [54] and a survey paper [55]. We here define the Kobayashi pseudo-distance by making use of its infinitesimal form F_M due to Royden, and show that both coincide. Those results which have been obtained after the publication of Kobayashi's text book are described in detail. It might be helpful to read this chapter along with Chapters 4-6 of his book.

In Chapter II, we investigate the above problem of value distribution in the equidimensional case ($\dim M = \dim N$) for non-degenerate f; i.e., the differential df has maximal rank at a point. Similarly to the definition of F_M, we define the hyperbolic pseudo-volume form Ψ_M on M, which also satisfies the decreasing principle $f^*\Psi_N \leq \Psi_M$. We discuss the properties of Ψ_M and give applications.

Our next main object is to extend the Nevanlinna theory to the case of meromorphic mappings $f: \mathbf{C}^k \to M$, where M is compact. There is a long history in this subject beginning with R. Nevanlinna and represented by names such as H. Cartan, A. Bloch, H. and J. Weyl, L. Ahlfors, W. Stoll, and S. S. Chern. In the early 1970's, P. Griffiths and his co-authors gave a new insight in this subject and showed abundant connections to other fields such as differential geometry and algebraic geometry. The extended first main theorem is described in terms of holomorphic line bundles and Chern classes. As for the second main theorem, the problem, however, is more complicated. Nonetheless, P. Griffiths and his coauthors established a satisfactory second main theorem in the equidimensional case for non-degenerate f. We devote Chapter V to these works. The Poincaré-Lelong formula connecting plurisubharmonic functions with positive currents plays an important role there. Therefore we need the following items:

(i) Holomorphic line bundles and Chern classes,

(ii) Positive currents and plurisubharmonic functions,

(iii) Meromorphic mappings between complex manifolds,

(iv) Poincaré-Lelong formula.

We explain (i) at the beginning of Chapter II. Chapter III is devoted to (ii). The theory of positive currents was initiated by P. Lelong. Since the importance of (ii) has been lately increasing in various aspects of complex analysis and complex geometry, we made Chapter III worthwhile to be read independently and to be used as a reference.

We deal with (iii) in Chapter IV. We here discuss the notion of meromorphic mappings after R. Remmert. Elementary facts from the theory of complex analytic spaces are recalled without proofs (suitable references are given there). Then we give complete proofs to theorems on meromorphic mappings. Chapter IV should be used as a reference throughout the present book and may provide a standard reference on meromorphic mappings.

In Chapter VI we investigate the value distribution of holomorphic curves $f: \mathbf{C} \to M$. Except for certain special cases dealt by H. Cartan and L. Ahlfors, there is no known satisfactory second main theorem for $f: \mathbf{C} \to M$. On the other hand A. Bloch [9] stated a theorem that if M is projective algebraic and carries linearly independent holomorphic 1-forms more than the dimension of M, then the image $f(\mathbf{C})$ of f must be contained in a proper algebraic subset of M. His proof contained serious gaps and the statement was called Bloch's conjecture. We give a rigorous proof to this theorem. One notes that the theorem is connected to the problem of establishing the second main theorem for holomorphic curves (see Noguchi [77, 80, 82, 84, 85]). Here we use rather freely terminology from algebraic geometry, which is briefly explained in the second section. Readers without general basics of the algebraic geometry may find this chapter somewhat difficult, but we tried to make clear the essence of this theorem and our main idea.

In this English edition two appendices are added. In Appendix I, we explain holomorphic vector bundles and the determinant bundle. Then we calculate the Chern classes of canonical bundles of some complex submanifolds of the complex projective space $\mathbf{P}^m(\mathbf{C})$. We use these to discuss a number of examples.

Appendix II is devoted to the Weierstrass-Stoll canonical function on $\mathbf{C}^m$, which is a generalization of Weierstrass' canonical product. As an immediate consequence, we see that an effective analytic divisor D on $\mathbf{C}^m$ is algebraic if and only if the mass of $D \cap B(r)$ has polynomial growth in r, where $B(r)$ denotes a ball of $\mathbf{C}^m$ with radius r. We use this in Chapter V. Our construction is after P. Lelong and reveals a nice application of the theory of positive currents.

Since we started our research in these subjects, we have been inspired by Professor S. Kobayashi through his book, papers, lectures and many conversations. It is our pleasure to express deep gratitude to him.

We also thank Professor W. Stoll. By his invitation, J. Noguchi visited Notre Dame in 1984-1985 and gave a series of lectures on these subjects to graduate students, notes of which make up a part of this book.

Junjiro Noguchi
Takushiro Ochiai

October 1988

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Remarks and Notation

(i) In this book, theorems, propositions, lemmas and equations are consecutively numbered, so that for instance, (1.2.3) appears in Chapter I, § 2 and after (1.2.2).

(ii) Throughout this book, manifolds are assumed to be connected unless otherwise mentioned. We follow the usual conventions in notation **N**, **Z**, **Q**, **R**, **C**, $\otimes$, $\wedge$, etc.; i.e., **N** denotes the set of natural numbers, **Z** the ring of integers, **Q** the field of rational numbers, **R** the field of real numbers, **C** the field of complex numbers, $\otimes$ the tensor product, and $\wedge$ the exterior product. For sets A and B, $A - B = \{x \in A \,;\, x \notin B\}$. For more symbols, see Symbols at the end of the book.

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APPENDIX I

Canonical Bundles of Complex Submanifolds of $P^m(C)$

1. Holomorphic Vector bundles

We need the general notion of holomorphic vector bundles. We call a triple $(\mathbf{E}, \pi, M)$ a **holomorphic vector bundle of rank** k if the following conditions are satisfied:

(1.1) (i) $\mathbf{E}$ is an $(m+k)$-dimensional complex manifold.

(ii) $\pi: \mathbf{E} \to M$ is a surjective holomorphic mapping.

(iii) For every $x \in M$, the fiber $\mathbf{E}_x = \pi^{-1}(x)$ is a complex vector space of complex dimension k.

(iv) For every $x \in M$, there exist an open neighborhood U of x and a biholomorphic mapping $\Phi: \mathbf{E}|U = \pi^{-1}(U) \to U \times \mathbf{C}^k$ such that $p \circ \Phi = \pi$ and $q \circ \Phi|_{\pi^{-1}(y)}: \pi^{-1}(y) \to \mathbf{C}^k$ are linear isomorphisms for all $y \in U$, where $p: U \times \mathbf{C} \to U$ and $q: U \times \mathbf{C} \to \mathbf{C}^k$ denote the natural projections.

The rank of $\mathbf{E}$ is denoted by rank $\mathbf{E}$. The mapping $\Phi: \mathbf{E}|U \to U \times \mathbf{C}^k$ is called a **local trivialization** of $\mathbf{E}$ over U, we consider $\mathbf{C}^k$ the column vector space of dimension k. Then we have k holomorphic sections on U

$$s_i(x) = \Phi^{-1}(x, {}^t(0, \ldots, \overset{i\text{-}th}{1}, \ldots, 0)), \quad 1 \le i \le k,$$

such that $\{s_i(x)\}_{i=1}^k$ is a base of $\mathbf{E}_x$ at every $x \in U$. Such $s = (s_1, \ldots, s_k)$ is called a **holomorphic local frame** of $\mathbf{E}$ over U. Conversely, if there is a holomorphic local frame over an open subset $U \subset M$, there is a local trivialization of $\mathbf{E}$ over U. Let $\{U_\lambda\}$ be an open covering of M such that there are local trivializations $\Phi_\lambda: \mathbf{E}|U \to U \times \mathbf{C}^k$. The pair $(\{U_\lambda\}, \{\Phi_\lambda\})$ is called a **local trivialization covering** of $\mathbf{E}$. Let $(x_\lambda, \xi_\lambda) \in U_\lambda \times \mathbf{C}^k$ and Let $(x_\mu, \xi_\mu) \in U_\mu \times \mathbf{C}^k$. Then they correspond to the same point of $\mathbf{E}$ if and only if

$$(1.2) \qquad x_\lambda = x_\mu = x \in U_\lambda \cap U_\mu, \quad \xi_\lambda = T_{\lambda\mu}(x)\xi_\mu,$$

where $T_{\lambda\mu}: U_\lambda \cap U_\mu \to GL(k, \mathbf{C})$ are holomorphic. The family $\{T_{\lambda\mu}\}$ is called the **system of holomorphic transitions** subordinated to the local trivialization covering. A complex submanifold $\mathbf{F} \subset \mathbf{E}$ is called a **holomorphic vector subbundle** if $\mathbf{F}$ is itself a holomorphic vector bundle of rank h $(0 \le h \le k)$ over M of which fiber structure is compatible with that of $\mathbf{E}$. Then we naturally have the quotient bundle $\mathbf{E}/\mathbf{F}$ which is a holomorphic vector bundle of rank $k - h$. We write the above facts as follows:

$$0 \to \mathbf{F} \to \mathbf{E} \to \mathbf{E}/\mathbf{F} \to 0.$$

For two holomorphic vector bundles $\mathbf{E}_1$ and $\mathbf{E}_2$ over M, the tensor product $\mathbf{E}_1 \otimes \mathbf{E}_2$ and the direct sum $\mathbf{E}_1 \oplus \mathbf{E}_2$ are naturally defined as holomorphic vector bundles over M. Furthermore, the exterior power bundle $\overset{l}{\wedge} \mathbf{E}$ $(1 \le l \le k)$ is defined. In special, $\overset{k}{\wedge} \mathbf{E}$ is called the **determinant bundle** of $\mathbf{E}$ and denoted by $\det \mathbf{E}$. In terms of (1.2), $\det \mathbf{E}$ is a holomorphic line bundle such that it is trivial over U_λ and the system of holomorphic transition functions is given by $\{\det T_{\lambda\mu}\}$. For instance, we have

$$(1.3) \qquad \det \mathbf{T}(M) = \mathbf{K}(M)^{-1}, \quad \det \mathbf{T}^*(M) = \mathbf{K}(M).$$

(1.4) Lemma. (i) *Let* $\mathbf{E}_i$ *$(i = 1, 2)$ be a holomorphic vector bundle of rank* k_i *over* M. *Then*

$$\det(\mathbf{E}_1 \otimes \mathbf{E}_2) = (\det \mathbf{E}_1)^{k_2} \otimes (\det \mathbf{E}_2)^{k_1}.$$

(ii) *Let* $\mathbf{E}$ *be a holomorphic vector bundle over* M *and* $\mathbf{F}$ *a holomorphic subbundle of* $\mathbf{E}$. *Then*

$$\det \mathbf{E} = (\det \mathbf{F}) \otimes (\det \mathbf{E}/\mathbf{F}).$$

Proof. (i) This immediately follows from the tensor algebra.

(ii) Let Let $x \in M$ be an arbitrary point. Then there is a holomorphic local frame $s = (s_1, \ldots, s_h)$ $(h = \operatorname{rank} \mathbf{F})$ on a neighborhood U of x. Taking U smaller if necessary, we may extend s to a holomorphic local frame $\tilde{s} = (s_1, \ldots, s_h, \ldots, s_k)$ $(k = \operatorname{rank} \mathbf{E})$ on U. Then $\hat{s} = (s_{h+1}, \ldots, s_k)$ induces a holomorphic local frame of $\mathbf{E}/\mathbf{F}$ over U. Taking a local trivialization covering as above, we easily see that the transition $T_{\lambda\mu}$ are of the type

$$\begin{bmatrix} A_{\lambda\mu} & B_{\lambda\mu} \\ O & C_{\lambda\mu} \end{bmatrix}$$

such that $\{A_{\lambda\mu}\}$ (resp. $\{C_{\lambda\mu}\}$) is a system of holomorphic transitions for $\mathbf{F}$ (resp. $\mathbf{E}/\mathbf{F}$). Therefore

$$\det(T_{\lambda\mu}) = \det(A_{\lambda\mu}) \cdot \det(C_{\lambda\mu}),$$

so that $\det \mathbf{E} = (\det \mathbf{F}) \otimes (\det \mathbf{E}/\mathbf{F})$. *Q.E.D.*

2. Non-Singular Complete Intersections of $\mathbf{P}^m(\mathbf{C})$ and the Canonical Bundles

In general, let D be an analytic hypersurface of a complex manifold M. Let $F = 0$ be a defining equation of D in a neighborhood of a point $x \in D$ (cf. Chapter IV, §2). It follows from Theorem (4.2.3) and the implicit function theorem that

$$D \cap U \text{ is non-singular} \iff dF \neq O \text{ on } D \cap U.$$

Let $[z^0; \cdots; z^m]$ be a homogeneous coordinate system of $\mathbf{P}^m(\mathbf{C})$. Then we set $U_i = \{z^i \neq 0\}$ and $z_i^j = z^j/z^i$, $j \neq i$. Then (z_i^j) are the affine coordinate of U_i. Let P be a homogeneous polynomial in $(z^0, \ldots, z^m)$ of degree d without multiple factor and D an analytic hypersurface defined by

$$D = \{[z^0; \cdots; z^m]; P(z^0, \ldots, z^m) = 0\}.$$

Put

$$P_i(z_i^j) = P(z_i^0, \ldots, \overset{i\text{-th}}{1}, \ldots, z^m) \text{ on } U_i.$$

Then for $i \neq k$

$$P_i(x) = (z_i^k)^d P_k(x). \tag{2.1}$$

Assume that D is non-singular. Then it follows that

$$dP_i \neq O \quad \text{on } D \cap U_i. \tag{2.2}$$

Let $\iota_D: D \to \mathbf{P}^m(\mathbf{C})$ be the natural inclusion mapping. Then we obtain from (2.1)

$$(\iota_D | U_i)^* dP_i = (z_i^k)^d (\iota_D | U_k)^* dP_k \text{ on } U_i \cap U_k \cap D. \tag{2.3}$$

Let $\mathbf{H}_0$ be the hyperplane bundle over $\mathbf{P}^m(\mathbf{C})$. Then $\{(\iota_D | U_i)^* dP_i\}$ defines a global holomorphic section σ_D of the holomorphic vector bundle $\mathbf{H}_0^d \otimes \mathbf{T}^*(\mathbf{P}^m(\mathbf{C}))|D$, which does not vanish anywhere in D. We identify the trivial line bundle $\mathbf{1}_D$ with a holomorphic vector subbundle of $\mathbf{H}_0^d \otimes \mathbf{T}^*(\mathbf{P}^m(\mathbf{C}))|D$ through

$$(x, a) \in D \times \mathbf{C} = \mathbf{1}_D \to a\sigma_D(x) \in \mathbf{H}_0^d \otimes \mathbf{T}^*(\mathbf{P}^m(\mathbf{C}))|D.$$

Then the quotient bundle $\mathbf{H}_0^d \otimes \mathbf{T}^*(\mathbf{P}^m(\mathbf{C}))|D/\mathbf{1}_D$ is isomorphic to $(\mathbf{H}_0|D)^d \otimes \mathbf{T}^*(D)$. Therefore we have the following.

(2.4) Lemma. *Let the notation be as above. Then*

$$0 \to \mathbf{1}_D \to \mathbf{H}_0^d \otimes \mathbf{T}^*(\mathbf{P}^m(\mathbf{C}))|D \to (\mathbf{H}_0|D)^d \mathbf{T}^*(D) \to 0.$$

(2.5) Lemma. *Let D be a non-singular hypersurface of degree d of $\mathbf{P}^m(\mathbf{C})$. Then*

$$\det \mathbf{T}^*(D) = (\mathbf{H}_0|D)^{d-m-1}.$$

Proof. This follows from Lemma (2.4), Lemma (1.4), (1.3) and Example (2.1.26). *Q.E.D.*

Let D_ν, $\nu = 1, \dots, l$ be non-singular analytic hypersurfaces of degree d_ν of $\mathbf{P}^m(\mathbf{C})$. Let P_ν be homogeneous polynomials without multiple factor such that $D_\nu = \{P_\nu = 0\}$. Assume that $\sum D_\nu$ has only simple normal crossings. Put $M = \bigcap_{\nu=1}^{l} D_i$. Then M is a complex submanifold of dimension $m - l$ and called a **non-singular complete intersection of degree $d = \sum d_\nu$**.

(2.6) Theorem. *Let $M \subset \mathbf{P}^m(\mathbf{C})$ be a non-singular complete intersection of degree d. Then*

$$\mathbf{K}(M) = (\mathbf{H}_0|M)^{d-m-1}.$$

Proof. Let D_ν, $1 \le \nu \le l$ be non-singular analytic hypersurfaces of degree d_ν such that $M = \bigcap_{\nu=1}^{l} D_i$ as above. By Lemma (2.5)

$$\det \mathbf{T}^*(D_1) = (\mathbf{H}_0|D_1)^{d_1-m-1}. \tag{2.7}$$

Applying the same arguments as above to the non-singular analytic hypersurface $D_1 \cap D_2$ of D_1 and making use of (2.7), we have

$$\det \mathbf{T}^*(D_1 \cap D_2) = (\mathbf{H}_0|D_1 \cap D_2)^{d_1+d_2-m-1}.$$

Inductively, we get

$$\det \mathbf{T}^*(D_1 \cap \cdots \cap D_l) = (\mathbf{H}_0|D_1 \cap \cdots \cap D_l)^{d_1+\cdots+d_l-m-1},$$

so that $\det \mathbf{T}^*(M) = (\mathbf{H}_0|M)^{d-m-1}$. *Q.E.D.*

APPENDIX II

Weierstrass-Stoll Canonical Functions

1. Review of Potential Theory on $\mathbf{R}^m$

We recall several fundamental facts from the real potential theory on the euclidean space $\mathbf{R}^m$ ($m \geq 2$). Cf. [32], Chapter 2 for this section. Let $x = (x^1, \ldots, x^m)$ be the standard coordinate system of $\mathbf{R}^m$ and set

$$\|x\| = \left\{ \sum_{i=1}^{m} |x^i|^2 \right\}^{1/2}, \quad B(r) = \{ \|x\| < r \}.$$

The **Laplacian** Δ is defined by

$$\Delta = \frac{\partial^2}{\partial (x^1)^2} + \cdots + \frac{\partial^2}{\partial (x^m)^2}.$$

Let U be a domain of $\mathbf{R}^m$. Then the Laplacian Δ operates on the distribution space $D(U)'$. We use the same notation and terminologies as in Chapter III, §1. A locally integrable function $u: U \to [-\infty, \infty)$ is called a **subharmonic function** if the following are satisfied:

(1.1) (i) u is upper semicontinuous.

(ii) $\Delta[u]$ is a positive Radon measure.

Remark. For the definition of subharmonic functions on U, it is equivalent to take the same definition as Definition (3.3.1), where (iii) has to be replaced with spherical mean integrals.

Subharmonic functions satisfy properties similar to those of subharmonic functions on $\mathbf{C} = \mathbf{R}^2$. Let u: $U \to [-\infty, \infty)$ be a subharmonic function on U. Then

$$u(a) \leq \frac{1}{r^m |B(1)|} \int_{a + B(r)} u dx^1 \cdots dx^m,$$

provided that $a + B(r) \subset U$, where $|B(1)| = \int_{B(1)} dx^1 \cdots dx^m$. The smoothing u_ε defined as in Chapter III, §1 is a C^∞ subharmonic function on U_ε and satisfies

$$u_\varepsilon \downarrow u \quad \text{as } \varepsilon \downarrow 0. \tag{1.2}$$

Moreover, if $-u$ is also subharmonic, then u is called a **harmonic** function on U. A harmonic function u on U are C^∞ by (1.2), and $\Delta u \equiv 0$. Let $S(r) \subset \mathbf{R}^m$ denote the sphere of radius $r > 0$ with center origin and dS_r the rotation invariant measure on $S(r)$ induced from the Euclidean metric, normalized as

$$\int_{S(r)} dS_r = 1.$$

Let v be an integrable function on $S(r)$ with respect to dS_r. Then the integral

$$u(x) = \int_{S(r)} v(y) \frac{r^2 - \|x\|^2}{\|y - x\|^m} r^{m-2} dS_r(y), \quad \|x\| < r$$

is called the **Poisson integral**. The function u is harmonic in the ball $B(r) = \{\|x\| < r\}$ with boundary value v. For a harmonic function u in a neighborhood of $\overline{B(r)}$ we have

$$u(x) = \int_{S(r)} u(y) \frac{r^2 - \|x\|^2}{\|y - x\|^m} r^{m-2} dS_r(y), \quad \|x\| < r \tag{1.3}$$

(1.4) Lemma. *Let u be a harmonic function on $\mathbf{R}^m$. Assume that there are a positive increasing sequence $r_\nu \uparrow \infty$ ($\nu \uparrow \infty$) and constants $C > 0$, $d \geq 0$ such that*

$$\sup\{|u(x)|; x \in B(r_\nu)\} \leq C r_\nu^d, \quad \nu = 1, 2, \ldots$$

Then u is a polynomial of degree $\leq d$ in $x^1, \ldots, x^m$.

Proof. Taking the complexification $(z^j) = (x^j + iy^j)$ of (x^j), we put

$$\hat{u}_r(z) = \int_{S(r)} u(w) \frac{r^2 - \sum_{j=1}^{m} (z^j)^2}{\left(\sum_{j=1}^{m} (w^j - z^j)^2 \right)^{\frac{m}{2}}} r^{m-2} dS_r(w) \tag{1.5}$$

for $\|z\| < r/2$. Then $\hat{u}_r$ is holomorphic in $\{\|z\| < r/2\}$ and $\hat{u}_r | \{y^1 = \cdots = y^m = 0\} = u$. Hence $\hat{u}_r = \hat{u}_{r'}$ for $r < r'$, so that they define a holomorphic function $\hat{u}$ on $\mathbf{C}^m$ such that $\hat{u} | \{y^1 = \cdots = y^m = 0\} = u$. In (1.5) we put $r = r_\nu$ and $\|z\| < r_\nu/4$. Then there is a constant $C_1 > 0$ such that

$$|\hat{u}(z)| \leq C_1 \left(\frac{r_\nu}{4} \right)^d, \quad \nu = 1, 2 \ldots$$

Since $\hat{u}$ is a holomorphic function on $\mathbf{C}^m$, $\hat{u}$ is a polynomial of degree $\leq d$ in $z^1, \ldots, z^m$, and hence so is u in $x^1, \ldots, x^m$. *Q.E.D.*

We define the potential kernel function $P(a, x)$ on $\mathbf{R}^m$ by

$$(1.6)\qquad P(a, x) = \frac{1}{\|a-x\|^{m-2}} \quad \text{for } m \geq 3,$$
$$P(a, x) = -\log\|a-x\| \quad \text{for } m = 2.$$

In the case of $m \geq 3$ (resp. $m = 2$) $P(a, x)$ is called the **Newton kernel function** (resp. **logarithmic kernel function**). The potential kernel function $P(a, x)$ with a fixed is harmonic in $\mathbf{R}^m - \{a\}$ and $-P(a, x)$ is subharmonic in $\mathbf{R}^m$. It is a classical fact that

$$\Delta[P(a, x)] = -\frac{1}{(m-2)|S(1)|}\delta_a,$$

where $|S(1)|$ denotes the area of the unit sphere $S(1) \subset \mathbf{R}^m$ with respect to the Euclidean metric and δ_a the Dirac measure at a. This leads to the following:

(1.7) Proposition. *Let σ be a positive Radon measure on $B(R)$ and $0 < r < R$ and set*

$$(1.8)\qquad U_r(x, \sigma) = -\frac{1}{(m-2)|S(1)|}\int_{y \in B(r)} P(y, x)\,d\sigma(y).$$

Then $U_r(x, \sigma)$ is subharmonic in $\mathbf{R}^m$ and satisfies

$$\Delta[U_r(\cdot, \sigma)] = \sigma \quad \text{in } B(r).$$

Here we call the above integral $U(x, \sigma)$ the **potential** of the Radon measure σ.

2. Local Potentials of Positive Closed (1, 1)-Currents and Modified Kernel Function

Let $z^j = x^{2j-1} + ix^{2j}$, $1 \leq j \leq m$, be the complex coordinates of $\mathbf{C}^m$ with real variables x^k, $1 \leq k \leq 2m$. By this we identify $\mathbf{C}^m = \mathbf{R}^{2m}$. Note that

$$\Delta = \sum_{j=1}^{m}\left(\frac{\partial^2}{(\partial x^{2j-1})^2} + \frac{\partial^2}{(\partial x^{2j})^2}\right) = 4\sum_{j=1}^{m}\left(\frac{\partial^2}{\partial z^j \partial \bar{z}^j}\right).$$

We use the same notation as in Chapter III, §3. For a locally integrable function u on a domain of $\mathbf{C}^m$ we have

$$(2.1)\qquad dd^c[u] \wedge \alpha^{m-1} = \frac{1}{4m}\Delta[u]\cdot\alpha^m.$$

Hence, if u is plurisubharmonic, then u is subharmonic. Let

$$T = \sum_{j,k}\frac{i}{2\pi}T_{j\bar{k}}\,dz^j \wedge d\bar{z}^k$$

be a positive current of type (1, 1) on $B(R)$. Define the **potential** of T by

$$(2.2)\qquad U_r(z, T) = -\frac{1}{m-1}\int_{y\in B(r)} P(y, z)T(y)\wedge\alpha^{m-1}(y), \quad 0 < r < R.$$

Then $U_r(z, T)$ is subharmonic in $\mathbf{C}^m$ and satisfies

$$(2.3)\qquad \Delta[U_r(z, T)] = 4\sum_{j=1}^{m} T_{j\bar{j}} \text{ on } B(r).$$

(2.4) Lemma. *Let T be a closed positive current of type* (1, 1) *on $B(R)$ and $0 < r < R$. Then there is a subharmonic function u on $B(r)$ such that*

$$dd^c[u] = T \quad \text{on } B(r).$$

Proof. Suppose first that T is C^∞ but not necessarily positive. Let $r < r' < R$. By Poincaré's lemma (Lemma (3.2.30)) there is a real 1-form v on $B(r')$ such that $dv = T$ on $B(r')$. Decompose $v = v' + v''$, where v' (resp. v'') is a 1-form of type (1, 0) (resp. (0, 1)). Since v is real and T is of type (1, 1), $v' = \overline{v''}$ and $\partial v' = \bar{\partial} v'' = 0$. By Dolbeault's lemma (cf. Hörmander [48], Chapter II, §3), there is a C^∞-function u'' on $B(r)$ with $\bar{\partial} u'' = v''$. Putting $u = 2\pi i(\overline{u''} - u'')$, we have

$$dd^c u = T.$$

In the general case, we take $U_{r'}(z, T)$ defined by (2.2) and put

$$S = T - dd^c U_{r'}(\cdot, T).$$

Put $S = \sum_{i,j} \frac{i}{2\pi} S_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Then (2.1) and (2.3) imply that

$$(2.5)\qquad \sum_{j=1}^{m} S_{j\bar{j}} = 0 \text{ in } B(r').$$

Moreover, we put

$$S_{i\bar{j}k\bar{l}} = \frac{\partial^2}{\partial z^k \partial \bar{z}^l} S_{i\bar{j}}.$$

The d-closedness of S implies that $S_{i\bar{j}k\bar{l}}$ is invariant under index exchanges of i and j, and of k and l. Using (2.5), we have

$$\Delta S_{i\bar{j}} = 4\sum_{k=1}^{m} S_{i\bar{j}k\bar{k}} = 4\left(\sum_{k=1}^{m} S_{k\bar{k}}\right)_{i\bar{j}} = 0 \text{ in } B(r').$$

Therefore $S_{i\bar{j}}$ are harmonic in the sense of distribution and hence C^∞. Then the result of the first half implies our assertion. *Q.E.D.*

We are going to define a modified kernel function derived from $P(a, x)$. The potential kernel function $P(a, x)$ for $m \geq 2$ is quite different to that for $m = 1$. Since our main concern is in the case of $m \geq 2$, we restrict ourselves to the case of $m \geq 2$. Now we expand $P(a, z)$:

$$P(a, z) = \frac{1}{\|a - z\|^{2m-2}}$$
$$= P_0(a, z) + P_1(a, z) + \cdots + P_\lambda(a, z) + \cdots,$$

where $P_0(a, z) = \dfrac{1}{\|a\|^{2m-2}}$ and $P_\lambda(a, z)$ are harmonic homogeneous polynomials of degree λ in z^j and $\bar{z}^j$, $j = 1, 2, \ldots, m$. For $q \in \mathbf{Z}^+$ we set

$$e(q; a, z) = -P(a, z) + P_0(a, z) + \cdots + P_q(a, z) \tag{2.6}$$
$$= -P_{q+1}(a, z) - P_{q+2}(a, z) - \cdots.$$

This $e(q; a, z)$ is the modified kernel function, which will be used to construct Weierstrass-Stoll canonical functions in the next section. Put $t = \|z\|/\|a\| < 1$ and let θ be the angle formed by the vectors a and z. Then

$$P(a, z) = \|a\|^{2-2m}(1 - 2t\cos\theta + t^2)^{1-m} \tag{2.7}$$
$$= \|a\|^{2-2m}\left[1 + B_1(\cos\theta)t + \cdots + B_\lambda(\cos\theta)t^\lambda + \cdots\right].$$

On the other hand, we have

$$P(a, z) \leq \|a\|^{2-2m}(1 - t)^{2-2m} = \|a\|^{2-2m}\sum_{\lambda=0}^{\infty} b_\lambda t^\lambda.$$

Here note that $P_\lambda(a, z) = \|a\|^{2-2m}B_\lambda(\cos\theta)t^\lambda$. Hence

$$|B_\lambda(\cos\theta)| \leq b_\lambda, \tag{2.8}$$

$$e(q; a, z) = -\|a\|^{2-2m}\sum_{\lambda=q+1}^{\infty} B_\lambda(\cos\theta)t^\lambda. \tag{2.9}$$

Now we consider general $t = \|z\|/\|a\|$. We fix $0 < \tau < 1$. Suppose first that $t \leq \tau$. Then it follows from (2.8) and (2.9) that

$$|e(q; a, z)| \leq \|a\|^{2-2m}t^{q+1}\sum_{\lambda=q+1}^{\infty} b_\lambda \tau^{\lambda-q-1} \tag{2.10}$$
$$= C_1(q, \tau)\|a\|^{2-2m}t^{q+1},$$

where $C_1(q, \tau) = \sum_{\lambda=q+1}^{\infty} b_\lambda \tau^{\lambda-q-1}$. We next suppose that $t \geq \tau$. It follows from (2.6)~(2.7) that

$$\begin{aligned} e(q;a,z) &\le \|a\|^{2-2m}(1+B_1(\cos\theta)t+\cdots+B_q(\cos\theta)t^q) \\ &\le \|a\|^{2-2m}(1+b_1t+\cdots+b_qt^q) \\ &\le \|a\|^{2-2m}t^q(t^{-q}+b_1t^{-q+1}+\cdots+b_q) \\ &\le \|a\|^{2-2m}t^q(\tau^{-q}+b_1\tau^{-q+1}+\cdots+b_q). \end{aligned}$$

Putting $C_2(q,\tau)=(\tau^{-q}+b_1\tau^{-q+1}+\cdots+b_q)$, we have

$$e(q;a,z)\le C_2(q,\tau)\|a\|^{2-2m}t^q. \tag{2.11}$$

Thus (2.11) and (2.10) imply the following.

(2.12) Lemma. *Let the notation be as above. Then*

$$e(q;a,z)\le C(q,\tau)\|a\|^{2-2m}\frac{\|z\|^{q+1}}{\|a\|^q(\|z\|+\|a\|)}$$

for any z, $a\in\mathbf{C}^m$ with $a\neq O$, and

$$|e(q;a,z)|\le C(q,\tau)\|a\|^{2-2m}\frac{\|z\|^{q+1}}{\|a\|^q(\|z\|+\|a\|)}$$

for $\|z\|\le\tau\|a\|$, where $C(q,\tau)=\max\{(1+\tau)C_1(q,\tau),\,(1+1/\tau)C_2(q,\tau)\}$.

3. Weierstrass-Stoll Canonical Functions

Let $T=\sum_{j,k}\frac{i}{2\pi}T_{j\bar{k}}\,dz^j\wedge d\bar{z}^k$ be a closed positive current of type (1, 1) on $\mathbf{C}^m$. Consider the equation

$$dd^c[U]=T. \tag{3.1}$$

If U is a solution of (3.1), then

$$\Delta[U]=4\sum_{j=1}^m T_{j\bar{j}}.$$

The potential $U_r(z,T)$ defined by (2.2) is subharmonic in $\mathbf{C}^m$ and satisfies

$$\Delta[U_r(z,T)]=4\sum_{j=1}^m T_{j\bar{j}}\ \text{ in } B(r).$$

Hence, if $U_r(z,T)$ converges as $r\to\infty$, then the limit is a candidate for a solution of (3.1). In general, it is not convergent. Therefore we will use the modified kernel function $e(q;a,z)$ defined by (2.6), of which difference to $-P(a,z)$ is only a harmonic polynomial of degree q.

We first investigate the condition for the convergence. Let $q\in\mathbf{Z}^+$. Then

$$\int_1^r t^{-q} dn(t, T) = \left[t^{-q} n(t, T) \right]_1^r + q \int_1^r \frac{n(t, T)}{t^{q+1}} dt.$$

This implies the following lemma.

(3.2) Lemma. *Let $r_\nu \uparrow \infty$, $\nu = 1, 2, \ldots$ be a positive increasing sequence. Then*

$$\lim_{\nu \to \infty} \int_1^{r_\nu} t^{-q} dn(t, T) < \infty \tag{3.3}$$

if and only if

$$\lim_{\nu \to \infty} \frac{n(r_\nu, T)}{r_\nu^q} < \infty \text{ and } \lim_{\nu \to \infty} \int_1^{r_\nu} \frac{n(t, T)}{t^{q+1}} dt < \infty.$$

We define the **order** ρ_T of T by

$$\rho_T = \overline{\lim_{r \to \infty}} \frac{\log N(r, T)}{\log r} = \overline{\lim_{r \to \infty}} \frac{\log n(r, T)}{\log r}. \tag{3.4}$$

Assume that there is a positive increasing sequence $r_\nu \uparrow \infty$ such that

$$\lim_{\nu \to \infty} \int_1^{r_\nu} t^{-q} dn(t, T) < \infty.$$

We may use any $r_n \uparrow \infty$ if $\overline{\lim}_{r \to \infty} \int_1^r t^{-q} dn(t, T) < \infty$. For instance, we may put $q = 0$ if $n(r, T) = O(1)$; if $\rho_T < \infty$, then we may put

$$q = [\rho_T] + 1, \tag{3.5}$$

where $[\cdot]$ stands for the Gauss' symbol. We define the **canonical potential** $U(q; z, T)$ of T due to Lelong [63] by

$$U(q; z, T) = \lim_{\nu \to \infty} \frac{1}{m-1} \int_{a \in B(r_\nu)} e(q; a, z) T \wedge \alpha^{m-1}. \tag{3.6}$$

We check the convergence. For simplicity, we assume that $O \notin \operatorname{supp} T$. Let $\|z\| = r < r_\nu$. It follows from Lemma (2.12) that

$$\begin{aligned} &\frac{1}{m-1} \int_{a \in B(r_\nu)} e(q; a, z) T \wedge \alpha^{m-1} \\ &\le \frac{1}{m-1} \int_{a \in B(r_\nu)} C(q, \tau) \|a\|^{2-2m} \frac{r^{q+1}}{\|a\|^q (r + \|a\|)} T \wedge \alpha^{m-1} \\ &= \frac{C(q, \tau)}{m-1} \int_0^{r_\nu} t^{2-2m-q} \frac{r^{q+1}}{r+t} d(T \wedge \alpha^{m-1}(B(t))) \end{aligned} \tag{3.7}$$

Continued

$$= \frac{C(q,\tau)}{m-1}\left\{\left[\frac{r^{q+1}}{r+t}t^{-q}n(t,T)\right]_0^{r_\nu}\right.$$

$$\left. + r^{q+1}\int_0^{r_\nu}\left[(q+2m-2)\frac{t^{1-2m-q}}{r+t}+\frac{t^{2-2m-q}}{(r+t)^2}\right]T\wedge\alpha^{m-1}(B(t))dt\right\}$$

$$= \frac{C(q,\tau)}{m-1}\left\{\frac{r^{q+1}}{r+r_\nu}\frac{n(r_\nu,T)}{r_\nu^q}\right.$$

$$\left. + r^{q+1}\int_0^{r_\nu}\frac{(q+2m-1)t+(q+2m-2)r}{(r+t)^2t^{q+1}}n(t,T)dt\right\}.$$

By Lemma (3.2)

$$\frac{r^{q+1}}{r+r_\nu}\frac{n(r_\nu,T)}{r_\nu^q}\to 0\ (\nu\to\infty). \tag{3.8}$$

Moreover we have

$$\frac{C(q,\tau)}{m-1}r^{q+1}\int_0^{r_\nu}\frac{(q+2m-1)t+(q+2m-2)r}{(r+t)^2t^{q+1}}n(t,T)dt \tag{3.9}$$

$$\le C'(q,\tau)r^{q+1}\int_0^{r_\nu}\frac{n(t,T)}{(r+t)t^{q+1}}dt,$$

where $C'(q,\tau)=C(q,\tau)(q+2m-1)/(m-1)$. Lemma (3.2) implies that the last integral in (3.9) converges as $\nu\to\infty$. Without loss of generality, we may assume that $T\wedge\alpha^{m-1}(S(r_\nu))=0$, $\nu=1,2,\ldots$ (cf. the proof of Theorem (3.2.31)). Making use of Lemma (2.12) in the same way as above, we infer that the sequence

$$\int_{a\in B(r_\nu)}|e(q;a,z)|T\wedge\alpha^{m-1},\quad \nu=1,2,\ldots$$

is a Cauchy sequence which is uniform for z belonging to a fixed compact subset. Therefore the right hand of (3.6) converges uniformly on compact subsets and $U(q;z,T)$ is a subharmonic function on $\mathbf{C}^m$. Note that

$$\int_0^{r_\nu}\frac{n(t,T)}{(r+t)t^{q+1}}dt=\int_0^{r}\frac{n(t,T)}{(r+t)t^{q+1}}dt+\int_r^{r_\nu}\frac{n(t,T)}{(r+t)t^{q+1}}dt \tag{3.10}$$

$$\le\frac{1}{r}\int_0^{r}\frac{n(t,T)}{t^{q+1}}dt+\int_r^{r_\nu}\frac{n(t,T)}{t^{q+2}}dt.$$

Put

$$C(q) = \inf\{C'(q, \tau);\ 0 < \tau < 1\}.$$

Then (3.6)~(3.10) implies the following.

(3.11) Lemma. *The canonical potential $U(q; z, T)$ is a subharmonic function on $\mathbf{C}^m$ and satisfies*

$$\Delta[U(q; z, T)] = 4\sum_{j=1}^{m} T_{j\bar{j}},$$

$$U(q; z, T) \le C(q) r^q \left\{ \int_0^r \frac{n(t, T)}{t^{q+1}} dt + r \lim_{\nu\to\infty} \int_r^{r_\nu} \frac{n(t, T)}{t^{q+2}} dt \right\}. \tag{3.12}$$

Put

$$M(r) = \sup\{U(q; z, T);\ z \in B(r)\}.$$

(3.13) Corollary. $\dfrac{1}{r^{2m}} \displaystyle\int_{z\in B(r)} |U(q; z, T)| \alpha^m \le 2M(r).$

Proof. Put $U^{\pm}(q; z, T) = \max\{0, \pm U(q; z, T)\}$. Then

$$U(q; z, T) = U^+(q; z, T) - U^-(q; z, T).$$

Since $U(q; z, T)$ is subharmonic and $U(q; O, T) = 0$,

$$r^{-2m} \int_{B(r)} \{U^+(q; z, T) - U^-(q; z, T)\} \alpha^m \ge 0,$$

so that

$$r^{-2m} \int_{B(r)} U^-(q; z, T) \alpha^m \le r^{-2m} \int_{B(r)} U^+(q; z, T) \alpha^m \le M(r).$$

Since $|U(q; z, T)| = U^+(q; z, T) + U^-(q; z, T)$, we have the desired estimate. *Q.E.D.*

(3.14) Theorem. *Let T be a closed positive current of type* (1, 1) *on* $\mathbf{C}^m$ *such that* $O \notin \operatorname{supp} T$. *Assume that there is a sequence $r_\nu \uparrow \infty$ ($\nu \uparrow \infty$) such that*

$$\lim_{\nu\to\infty} \int_1^{r_\nu} t^{-q} dn(t, T) < \infty.$$

Then the canonical potential $U(q; z, T)$ satisfies the estimate (3.12) *and a solution of* (3.1):

$$dd^c[U(q; z, T)] = T.$$

Proof. There remains to show that $dd^c[U(q;z,T)]=T$. We may assume that $B(r_1)\cap\operatorname{supp}T=\varnothing$. By Lemma (2.4) there are plurisubharmonic functions u_ν on $B(r_\nu)$ such that $u_1\equiv 0$ and $dd^c[u_\nu]=T|B(r_\nu)$. Put

$$H_\nu(z)=U(q,z,T)-u_\nu(z),\quad z\in B(r_\nu).$$

Then $\Delta[H_\nu]=0$, so that H_ν are C^∞ harmonic functions on $B(r_\nu)$. Put

$$H_{\nu i\bar{j}}=\frac{\partial^2 H_\nu}{\partial z^i\partial\bar{z}^j}.$$

Then

$$H_{(\nu+1)i\bar{j}}-H_{\nu i\bar{j}}=0\ \text{ on } B(r_\nu).$$

Therefore $H_{\nu i\bar{j}}$ define harmonic functions $H_{i\bar{j}}$ on $\mathbf{C}^m$. It follows from (2.6) that

(3.15) all partial derivatives of order $\leq q-2$ of $H_{i\bar{j}}$ vanish at O.

Let χ and χ_ε be the convolution kernels defined in Chapter III, §1, (c). Then

$$\chi_\varepsilon(z)=\frac{1}{\varepsilon^{2m}}\chi\left(\frac{z}{\varepsilon}\right).$$

Put

$$C=\max\left\{\frac{\partial^2\chi}{\partial z^i\partial\bar{z}^j}(z);\ z\in\mathbf{C}^m,\ 1\leq i,\ j\leq m\right\}.$$

Then

$$\left|\frac{\partial^2\chi_\varepsilon}{\partial z^i\partial\bar{z}^j}(z)\right|\leq C\varepsilon^{-2m-2}. \tag{3.16}$$

Since $H_{i\bar{j}}$ is harmonic, $H_{i\bar{j}\varepsilon}=H_{i\bar{j}}*\chi_\varepsilon=H_{i\bar{j}}$. By definition

$$H_{i\bar{j}}=\frac{\partial^2[U(q;z,T)]}{\partial z^i\partial\bar{z}^j}*\chi_\varepsilon-\frac{\partial^2[u_\nu]}{\partial z^i\partial\bar{z}^j}*\chi_\varepsilon.$$

Put

$$S_1=\frac{\partial^2[U(q;z,T)]}{\partial z^i\partial\bar{z}^j}*\chi_\varepsilon,$$

$$S_2=\frac{\partial^2[u_\nu]}{\partial z^i\partial\bar{z}^j}*\chi_\varepsilon=T_{i\bar{j}}*\chi_\varepsilon.$$

It follows from (3.16) that

$$|S_1(z)|\leq C\varepsilon^{-2m-2}\int_{w\in B(\varepsilon)}|U(q;z+w,T)|\alpha^m(w).$$

Put $r = \|z\|$ and $\varepsilon = r$. Then by Corollary (3.13)

$$(3.17)\qquad |S_1(z)| \le Cr^{-2m-2}\int_{w\in B(2r)} |U(q; z, T)|\alpha^m$$
$$\le Cr^{-2m-2}(2r)^{2m}2M(2r) \le 2^{2m+1}Cr^{-2}M(2r).$$

Since $\|T_{i\bar{j}}\| \le \frac{1}{\sqrt{2}}\sum T_{i\bar{i}}$ as measures,

$$(3.18)\qquad |S_2(z)| = \left|\int_{w\in B(z;\,\varepsilon)} T_{ij}\chi_\varepsilon(w-z)\alpha^m(w)\right|$$
$$\le \int_{w\in B(z;\,\varepsilon)} \|T_{i\bar{j}}\|\chi_\varepsilon(w-z)\alpha^m(w)$$
$$\le \frac{m}{\sqrt{2}}\int_{w\in B(z;\,\varepsilon)} \chi_\varepsilon(w-z)T\wedge\alpha^{m-1}(w)$$
$$\le \frac{m}{\sqrt{2}}\varepsilon^{-2m}(\max\chi)\int_{w\in B(z;\,\varepsilon)} T\wedge\alpha^{m-1}$$
$$\le \frac{m}{\sqrt{2}}r^{-2m}(\max\chi)\int_{B(2r)} T\wedge\alpha^{m-1}$$
$$= 2^{2m-5/2}m(\max\chi)r^{-2}n(2r, T).$$

We have by (3.17) and (3.18)

$$(3.19)\qquad |H_{i\bar{j}}(z)| \le C'r^{-2}\{M(2r) + n(2r, T)\}$$

for $\|z\| \le r$, where C' is a positive constant independent of r and T. We infer from the assumption, Lemma (3.2), (3.19) and Lemma (3.11) that

$$|H_{i\bar{j}}(z)| \le O(r_\nu^{q-2}) \le O\left(\left(\frac{r_\nu}{2}\right)^{q-2}\right), \quad \|z\| \le \frac{r_\nu}{2}.$$

If $0 \le q \le 1$, then $H_{i\bar{j}} \equiv 0$. In the case of $q \ge 2$, we see by Lemma (1.4) that $H_{i\bar{j}}$ are polynomials of degree $\le q-2$. Then it follows from (3.15) that $H_{i\bar{j}} = 0$. Therefore $dd^c[U(q; z, T)] = T$. *Q.E.D.*

(3.20) Theorem (Stoll). *Let D be an effective divisor on $\mathbf{C}^m$ such that $O \notin \operatorname{supp} D$. Assume that there are $q \in \mathbf{Z}^+$ and a positive increasing sequence $r_\nu \uparrow \infty$ $(\nu \to \infty)$ such that*

$$\lim_{\nu\to\infty}\int_1^{r_\nu} t^{-q}dn(t, D) < \infty.$$

Then there exists a unique holomorphic function F satisfying the conditions:

(i) $(F) = D$.

(ii) $F(O) = 1$ *and all partial derivatives of order* $\leq q$ *vanish at* O.

(iii) *There is a positive constant* $A(q)$ *depending only on* q *such that*

$$\log|F(z)| \leq A(q)\|z\|^q \left\{ \int_0^{\|z\|} \frac{n(t, T)}{t^{q+1}} dt + \|z\| \lim_{\nu\to\infty} \int_{\|z\|}^{r_\nu} \frac{n(t, T)}{t^{q+2}} dt \right\}.$$

Proof. Take a locally finite open covering $\{W_\lambda\}_{\lambda=1}^{\infty}$ of $\mathbf{C}^m$ so that W_λ are balls and there are holomorphic functions F_λ on W_λ satisfying $(F_\lambda) = D|W_\lambda$. By Theorem (3.14) we have the canonical potential $U(q; z, D)$ of D satisfying

$$dd^c[U(q; z, D)] = D.$$

Put

$$h_\lambda = \frac{1}{2} U(q; z, D) - \log|F_\lambda|.$$

Then $dd^c h_\lambda = 0$; i.e., h_λ are pluriharmonic. Therefore there are holomorphic functions g_λ such that the real parts Re g_λ of g_λ coincide with h_λ. Hence we get

$$\frac{1}{2} U(q; z, D) = \mathrm{Re}(g_\lambda + \log F_\lambda)$$

When $W_\lambda \cap W_\mu \neq \varnothing$,

$$g_\lambda + \log F_\lambda = g_\mu + \log F_\mu + i a_{\lambda\mu},$$

where $a_{\lambda\mu} \in \mathbf{R}$ satisfying the cocycle condition

$$a_{\lambda\mu} = -a_{\mu\lambda}, \quad a_{\lambda\mu} + a_{\mu\nu} + a_{\nu\lambda} = 0.$$

Hence we have a Cech cohomology class $(a_{\lambda\mu}) \in H^1(\mathbf{C}^m, \mathbf{R})$. Since $H^1(\mathbf{C}^m, \mathbf{R}) = \{0\}$, there are constants $b_\lambda \in \mathbf{R}$ such that

$$a_{\lambda\mu} = b_\mu - b_\lambda.$$

Then we have

$$g_\lambda + ib_\lambda + \log F_\lambda = g_\mu + ib_\mu + \log F_\mu \ \text{ on } W_\lambda \cap W_\mu \neq \varnothing.$$

Then we define a holomorphic function F on $\mathbf{C}^m$ by

$$F = F_\lambda \exp(g_\lambda + ib_\lambda) \ \text{ on } W_\lambda.$$

By Theorem (3.14) this F satisfies the required properties.

Let G be a holomorphic function on $\mathbf{C}^m$ satisfying (i), (ii) and (iii). Then $A = F/G$ is a nowhere vanishing holomorphic function on $\mathbf{C}^m$. By (5.2.25), Corollary (5.2.30), condition (iii) and Lemma (3.2), we see that

$$T(r_\nu, A^{\pm 1}) \le T(r_\nu, F) + T(r_\nu, G) + O(1) = O(r_\nu^q).$$

Thus Theorem (5.3.13) implies that $\log M(r_\nu, A^{\pm 1}) = O(r_\nu^q)$. Taking $\log A(z)$ so that $\log A(O) = \log 1 = 0$, we get

$$\sup\{|\mathrm{Re}\log A(z)|;\ z \in B(r_\nu)\} = O(r_\nu^q).$$

Since $\mathrm{Re}\log A(z)$ is harmonic in $\mathbf{C}^m$, Lemma (1.4) implies that $\mathrm{Re}\log A(z)$ is a polynomial of degree $\le q$ in z^i and $\bar{z}^i$. Then condition (ii) implies that $\mathrm{Re}\log A(z) = 0$ and hence $A(z) \equiv 1$. *Q.E.D.*

The unique holomorphic function F for a given divisor D in Theorem (3.20) is called the **Weierstrass-Stoll canonical function** of D.

We give a criterion of the algebraicity of a divisor on $\mathbf{C}^m$.

(3.21) Lemma. *Let P be a polynomial on $\mathbf{C}^m$. Then*

$$\text{degree of } P = p \iff N(r, (P)) = (p + o(1))\log r.$$

Proof. Remark that $N(r, (P))$ is a convex increasing function in $\log r$. Hence it suffices to show that for any $p \in \mathbf{R}$ with $p \ge 0$

$$\text{degree of } P \le p \iff N(r, (P)) \le (p + o(1))\log r.$$

Suppose that the degree of $P \le p$. Then by Proposition (5.3.14)

$$N(r, (P)) \le T(r, P) \le (p + o(1))\log r.$$

Conversely, suppose that $N(r, (P)) \le (p + o(1))\log r$. Let p' be a degree of P. Then Theorem (5.1.15) implies that

$$N(r, (P)) = \int_{\Gamma(r)} \log|P|\eta - \int_{\Gamma(1)} \log|P|\eta. \tag{3.22}$$

Let $\rho: \mathbf{C}^m - \{O\} \to \mathbf{P}^{m-1}(\mathbf{C})$ be the Hopf fibering and ω_0 the Fubini-Study Kähler form. Then

$$\eta = d^c \log\|z\| \wedge \rho^*\omega^{m-1}. \tag{3.23}$$

For every complex line $l \in \mathbf{P}^{m-1}(\mathbf{C})$ through O, we take a point $a_l \in l \cap \Gamma(1)$. Then it follows from (3.22) and (3.23) that

$$\begin{aligned} N(r, (P)) &= \int_{l \in \mathbf{P}^{m-1}(\mathbf{C})} \left\{ \log|P(re^{i\theta}a_l)|d\theta - \log|P(e^{i\theta}a_l)|d\theta \right\} \omega_0^{m-1} \\ &= \int_{l \in \mathbf{P}^{m-1}(\mathbf{C})} N(r, (P|l))\omega_0^{m-1} = \int_1^r \frac{dt}{t} \int_{l \in \mathbf{P}^{m-1}(\mathbf{C})} n(r, (P|l))\omega_0^{m-1}. \end{aligned} \tag{3.24}$$

Since P is a polynomial of degree p', $n(r, (P|l)) \uparrow p'$ as $r \uparrow \infty$. Therefore we obtain

$$N(r, (P)) = (p' + o(1))\log r \leq (p + o(1))\log r,$$

so that $p' \leq p$. *Q.E.D.*

(3.25) Theorem (Stoll). *Let D be an effective divisor on $\mathbf{C}^m$. Then D is an algebraic divisor defined by a polynomial of degree p if and only if*

$$N(r, D) = (p + o(1))\log r.$$

Proof. By Lemma (3.21) it suffices to show that if $N(r, D) = O(\log r)$, then D is algebraic. Suppose that $N(r, D) = O(\log r)$. Then

$$n(r, D) = O(1). \tag{3.26}$$

We may assume that $O \notin \operatorname{supp} D$. Let $q = 0$ and take any positive increasing sequence $r_\nu \uparrow \infty$ in Theorem (3.20). Then we have the Weierstrass-Stoll canonical function F satisfying

$$\log |F(z)| \leq A(0) \left\{ \int_0^r \frac{n(t, D)}{t} dt + r \int_r^\infty \frac{n(t, D)}{t^2} dt \right\}$$

for $z \in B(r)$. It follows from this and (3.26) that

$$T(r, F) = m(r, F) \leq O(\log r).$$

By Proposition (5.3.14) F is a polynomial. *Q.E.D.*

Notes

In the case of $m = 1$, the Weierstrass canonical product of a given effective divisor on $\mathbf{C}$, of which order is finite, is well known (cf., e.g., Hayman [46]). Stoll [110] proved its higher dimensional version, Theorem (3.20). The proof given here is due to Lelong [63]. By the uniqueness, the function constructed here must coincide with that of Stoll. Let $l \in \mathbf{P}^{m-1}(\mathbf{C})$ denote a complex line of $\mathbf{C}^m$ through O. Then the restriction $F|l$ of the Weierstrass-Stoll canonical function of an effective divisor D on $\mathbf{C}^m$ are the Weierstrass canonical product of the intersections $D \cdot l$ for all $l \in \mathbf{P}^{m-1}(\mathbf{C})$. This is trivial by Stoll's construction. Stoll [114] is a nice survey on his method. There is an application for meromorphic mappings $f: \mathbf{C}^m \to \mathbf{P}^N(\mathbf{C})$ of finite order due to Noguchi [75]. Mok-Siu-Yau [70] applied the method of the canonical potential of Theorem (3.14) to the characterization problem of $\mathbf{C}^m$.

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