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Volume 82

Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation

Yasutaka Sibuya

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Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation

Yasutaka Sibuya

American Mathematical Society
Providence, Rhode Island

複素領域における線型常微分方程式

解析接続の問題

Translated from the Japanese by Yasutaka Sibuya

2000 *Mathematics Subject Classification.* Primary 34–XX, 32D20, 32E10, 34E05, 30E15.

ABSTRACT. The main part of this book is an almost faithful translation of the book originally written in Japanese in 1976. In this book, focusing attention on intrinsic aspects as much as possible, we shall explain some problems of linear ordinary differential equations in complex domains. Since the original book was published, some new ideas have been introduced and rapidly developed. By adding five appendices to the translation, those new ideas are briefly explained in this book. Over 100 literatures are added to the original references.

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To My Mother

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Translator's Preface

Charles Darwin claims that the ape and the human being came from the same origin. The reader may or may not believe him. However, the author and the translator certainly came from the same origin. But they are not exactly the same person. They live in two different environments. Since this book was published, many research papers and books concerning similar subjects have appeared. Thus a new environment was created. So, the translator has tried to provide the reader a guided tour through this new environment by adding §§6.8–6.11 and five appendices. This is like sightseeing in the usual sense. If the reader wishes to discover more, he must go back there himself and look around more extensively. The translation is as faithful as the translator could achieve. Also, wherever necessary and suitable, he inserted remarks in [ ] to help the reader.

The main part of this translation was done at the University of Southern California in the academic year 1988/89 during the translator's sabbatical leave from the University of Minnesota. Also the translator was supported partially by a grant from the National Science Foundation. The translator wishes to express his appreciation to Professor William A. Harris, Jr. (U.S.C.), Mr. Hiroshi Mizuno (Kinokuniya), Ms. Mary C. Lane (A.M.S.), and Ms. Susan Connell (A.M.S.) for their kind help in the publication of this translation.

February, 1990, at Roseville — Yasutaka Sibuya

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Preface to the Japanese Edition

Research on differential equations is usually oriented toward obtaining explicit results, as it is motivated by applications. In this way many clever ideas have been found one after another. However, when a really difficult problem arises, we must dig out something intrinsic. As the study of topology, algebraic geometry, and functions of several complex variables has advanced, many methods which are useful in such fields have been introduced into the territory of differential equations. In this book, focusing attention on intrinsic aspects as much as possible, we shall explain some problems of linear ordinary differential equations in complex domains. The Riemann problem was solved quite naturally as the study of functions of several complex variables advanced. This was the most remarkable achievement since the beginning of this century. In this book, we shall first explain this problem. Also, we shall try to treat the generalized Riemann problem (or the Birkhoff problem) in a similar manner. Further, we shall explain the characterization of regular singular points due to W. Jurkat and D. A. Lutz. This result is recently attracting attention, because it is closely related to singularities of hypersurfaces.

The author of this book has never given a lecture of this kind. Professor Tosihusa Kimura of the University of Tokyo visited the University of Minnesota during the academic year 1971/72. He gave a series of lectures on algebraic differential equations, hypergeometric functions of several complex variables, and the Riemann problem. Also, during his stay, the two of us discussed various problems of mathematics, sometimes dangling fishing lines from a boat on a lake in Minnesota. Since then it has been four years. The author started writing this book as a term paper for the course given by Professor Kimura. The author tried his best. But this is the first draft without any extensive revisions. So he expects that the reader will find some pages hard to read and other pages containing mistakes. In particular, wherever the author's understanding of the material is shallow and uncertain, the reader will sense it immediately by detecting his somewhat vague way of writing. It is left to the reader to determine what grade Professor Kimura will give to this term paper. In the near future, the author will publish the contents of the last chapter [Chapter 6] as a research paper, because it contains new results [cf. Y. Sibuya [134]].

The author would like to express his appreciation to Professor Tosihusa Kimura (Univ. of Tokyo), Professor Seizo Ito (Univ. of Tokyo), Mr. Kenichi Uzuoka (Kinokuniya), and Mr. Kazumasa Yokota (Kinokuniya) for their kind help in the publication of this book. It should also be clearly noted that this work was partially supported by grant NSF-GP 38955 from the National Science Foundation.

August, 1976, at Roseville Yasutaka Sibuya

APPENDIX 1

The Hukuhara–Turrittin Theorem in Terms of $D_{\mathscr{K}}$-Modules

§A.1.0. Preliminaries

As in §6.8, we denote by $\mathscr{K}$ the quotient field of $\mathbf{C}[[z^{-1}]]$, i.e.

$$\mathscr{K} = \mathbf{C}[[z^{-1}]][z].$$

Let V be an n-dimensional vector space over $\mathscr{K}$. The differential operator $\delta = x\frac{d}{dx}$ defines a map : $\mathscr{K} \to \mathscr{K}$ which is additive and satisfies the Leibniz condition

$$\delta(a\, b) = \delta(a)\, b + a\, \delta(b) \quad \text{for } a,\ b \in \mathscr{K}. \tag{A.1.0.1}$$

A map $T : V \to V$ is called a linear differential operator if

(i) $T(u+v) = T(u) + T(v)$ for $u, v \in V$,
(ii) $T(a\,u) = \delta(a)\,u + a\,T(u)$ for $a \in \mathscr{K}$, $u \in V$.

Given a basis $\{u_1, \cdots, u_n\}$ for V over $\mathscr{K}$, there exists a matrix $A(z) \in gl(n, \mathscr{K})$ such that

$$[\, T(u_1), \cdots, T(u_n)\,] = [\, u_1, \cdots, u_n\,]\, A(z). \tag{A.1.0.2}$$

Observe that a vector $u = \sum_{1 \le j \le n} y_j\, u_j \in V$ $(y_j \in \mathscr{K})$ can be written in the form

$$u = [\, u_1, \cdots, u_n\,]\, y, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{bmatrix}. \tag{A.1.0.3}$$

Hence

$$\begin{aligned} T(u) &= [u_1, \cdots, u_n\,]\, \delta(y) + [\, T(u_1), \cdots, T(u_n)\,]\, y \\ &= [\, u_1, \cdots, u_n]\,(\, \delta(y) + A(z)\, y\,). \end{aligned} \tag{A.1.0.4}$$

It is easy to see that

$$\delta + A(z) : \mathscr{K}^n \to \mathscr{K}^n \tag{A.1.0.5}$$

is a linear differential operator : $\mathscr{K}^n \to \mathscr{K}^n$. (cf. (6.8.13)). We call $\delta + A(z)$ the *representation* of T with respect to the basis $\{ u_1, \cdots, u_n \}$.

For two bases $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_n\}$ for V over $\mathscr{K}$, set

$$\begin{cases} [\, T(u_1), \cdots, T(u_n) \,] = [\, u_1, \cdots, u_n \,] \, A(z), \\ [\, T(v_1), \cdots, T(v_n) \,] = [\, v_1, \cdots, v_n \,] \, B(z), \\ [\, v_1, \cdots, v_n \,] = [\, u_1, \cdots, u_n \,] \, P(z). \end{cases} \tag{A.1.0.6}$$

Then $P(z)B(z) = (\delta + A(z))\, P(z)$, or

$$\delta(P) + A\,P - P\,B = 0 \tag{A.1.0.7}$$

(cf. (6.8.15)).

A nonempty subset U of the vector space V is said to be *invariant* by T if $T(U) \subset U$. Suppose that V is the direct sum of two subspaces V_1 and V_2 which are invariant by T, i.e.,

$$V = V_1 + V_2, \qquad V_1 \cap V_2 = \{0\}, \qquad T(V_j) \subset V_j \qquad (j = 1, 2).$$

If $\dim_{\mathscr{K}} V_j = n_j$ $(j = 1, 2)$, we have $n_1 + n_2 = n$. A basis $\{u_{1,1}, \cdots, u_{1,n_1}\}$ for V_1 over $\mathscr{K}$ and a basis $\{u_{2,1}, \cdots, u_{2,n_2}\}$ for V_2 over $\mathscr{K}$ form a basis $\{u_{1,1}, \cdots, u_{1,n_1}, u_{2,1}, \cdots, u_{2,n_2}\}$ for V over $\mathscr{K}$. If we set

$$[T(u_{j,1}), \cdots, T(u_{j,n_j})] = [u_{j,1}, \cdots, u_{j,n_j}]\, A_j(z) \qquad (j = 1, 2),$$

then

$$[T(u_{1,1}), \cdots, T(u_{1,n_1}), T(u_{2,1}), \cdots, T(u_{2,n_2})]$$

$$= [u_{1,1}, \cdots, u_{1,n_1}, u_{2,1}, \cdots, u_{2,n_2}] \begin{bmatrix} A_1(z) & 0 \\ 0 & A_2(z) \end{bmatrix}.$$

From the observation above, we see that the conclusion of the Hukuhara-Turrittin theorem (cf. Theorem 6.8.1) means that V can be decomposed as a direct sum of suitable invariant subspaces of T. We can also explain this fact in terms of modules over a noncommutative ring.

§A.1.1. $D_{\mathscr{K}}$-modules

Let us consider a polynomial ring

$$\mathscr{K}[\sigma] = \left\{ \sum_{h=0}^{p} a_h\, \sigma^h \;;\; p \in \mathbf{N},\ a_h \in \mathscr{K} \right\}, \tag{A.1.1.1}$$

where N is the set of nonnegative integers. We define multiplication of σ and $a \in \mathscr{K}$ by

$$\sigma\, a - a\, \sigma = \delta(a). \tag{A.1.1.2}$$

Then $K[\sigma]$ becomes a noncommutative ring. We denote this ring by $D_{\mathscr{K}}$.

Let V be an n-dimensional vector space over $\mathscr{K}$ and a module over $D_{\mathscr{K}}$. In this case, we call V an n-dimensional $D_{\mathscr{K}}$-module. Given a basis $\{u_1, \cdots, u_n\}$ for V over $\mathscr{K}$, there exists a matrix $A(z) \in gl(n, \mathscr{K})$ such that

$$[\ \sigma u_1,\ \cdots,\ \sigma u_n\ ] = [\ u_1,\ \cdots,\ u_n\ ]\, A(z) \tag{A.1.1.3}$$

and hence

$$\sigma\ ([\ u_1,\ \cdots,\ u_n\ ]\, y) = [\ u_1,\ \cdots,\ u_n\ ]\,(\ \delta + A(z)\ )\, y, \tag{A.1.1.4}$$

where $y \in \mathscr{K}^n$.

Conversely, suppose that a differential operator $\delta + A(z) : \mathscr{K}^n \to \mathscr{K}^n$ and a vector space V over $\mathscr{K}$ are given. Then if, for a given basis $\{u_1, \cdots, u_n\}$ for V over $\mathscr{K}$, we define multiplication of σ and a vector in V by

$$\sigma\ ([\ u_1,\ \cdots,\ u_n\ ]\, y) = [\ u_1,\ \cdots,\ u_n\ ]\,(\ \delta + A(z)\ )\, y, \tag{A.1.1.5}$$

where $y \in \mathscr{K}^n$, the space V becomes an n-dimensional $D_{\mathscr{K}}$-module. For example, let $V = \mathscr{K}$ and $\lambda(z) \in \mathscr{K}$. If we set

$$\sigma\, a = (\delta + \lambda(z))\, a \quad \text{for } a \in \mathscr{K}, \tag{A.1.1.6}$$

then V becomes a one-dimensional $D_{\mathscr{K}}$-module. We denote this module by $E(\lambda)$.

Let V be an n-dimensional $D_{\mathscr{K}}$-module and $\lambda(z) \in \mathscr{K}$. Then the tensor product

$$E(\lambda) \bigotimes_{\mathscr{K}} V$$

becomes an n-dimensional $D_{\mathscr{K}}$-module if we set

$$\sigma(a \otimes u) = (\sigma a) \otimes u + a \otimes (\sigma u). \tag{A.1.1.7}$$

For a given basis $\{u_1, \cdots, u_n\}$ for V over $\mathscr{K}$, there exists a matrix $A(z) \in gl(n, \mathscr{K})$ such that (A.1.1.3) holds. Also there exists a nonzero vector e in $E(\lambda)$ such that $\sigma\, e = \lambda(z)\, e$. Then $\{e \otimes u_1, \cdots, e \otimes u_n\}$ is a basis for $E(\lambda) \otimes V$ over $\mathscr{K}$ and we have

$$\sigma\ ([\ e \otimes u_1,\ \cdots,\ e \otimes u_n\ ]\, y) = [\ e \otimes u_1,\ \cdots,\ e \otimes u_n\ ]\,(\ \delta + \lambda(z) I + A(z)\ )\, y, \tag{A.1.1.8}$$

where $y \in \mathscr{K}^n$ and I is the $n \times n$ identity matrix. In other words the differential operator

$$\delta + \lambda(z) I + A(z) : \mathscr{K}^n \to \mathscr{K}^n$$

represents the σ-multiplication with respect to the basis $\{e \otimes u_1, \cdots, e \otimes u_n\}$ for $E(\lambda) \otimes V$ over $\mathscr{K}$.

As above, let V be an n-dimensional $D_{\mathscr{K}}$-module. In order that a nonempty subset U of V be a $D_{\mathscr{K}}$-submodule of V, it is necessary and sufficient that U is a vector subspace of V over $\mathscr{K}$ and $\sigma\, U \subset U$. We can state the Hukuhara–Turrittin theorem as a decomposition of V into a direct sum of a finite number of irreducible $D_{\mathscr{K}}$-submodules of V. However, to do this, we need to extend the field $\mathscr{K}$ to

$$\mathscr{L} = \mathbf{C}[[\zeta^{-1}]][\zeta] \tag{A.1.1.9}$$

where $z = \zeta^s$ (cf. (6.8.2)). Since $s\,\delta = \zeta\frac{d}{d\zeta}$, there is a natural embedding : $D_{\mathscr{K}} \rightarrow D_{\mathscr{L}}$. Instead of the $D_{\mathscr{K}}$-module V, we shall look at the $D_{\mathscr{L}}$-module

$$\mathscr{L} \bigotimes_{\mathscr{K}} V.$$

§A.1.2. Fuchsian modules

So far we have given two definitions of regular singular points, i.e., one in §3.2 and the other in §5.2 (cf. Definition 5.2.1). The definition in §3.2 is algebraic, whereas the definition in §5.2 is analytic. The relation between these two definitions can be explained by Lemma 5.8.1 (case $\rho_0 = 0$). In this section, we shall explain the algebraic definition in terms of $D_{\mathscr{K}}$-modules.

DEFINITION A.1.2.1. Let V be an n-dimensional $D_{\mathscr{K}}$-module. A $\mathbf{C}[[z^{-1}]]$-submodule $\mathscr{M}$ of V is called a **lattice** if there exists a basis $\{m_1, \cdots, m_n\}$ for V over $\mathscr{K}$ that generates $\mathscr{M}$ over $\mathbf{C}[[z^{-1}]]$.

DEFINITION A.1.2.2. An n-dimensional $D_{\mathscr{K}}$-module V is called a Fuchsian module if there exists a lattice $\mathscr{M}$ such that

$$\sigma\,\mathscr{M} \in \mathscr{M}. \tag{A.1.2.1}$$

Suppose that V is a Fuchsian module and let $\mathscr{M}$ be a lattice satisfying condition (A.1.2.1). Since $\mathscr{M}$ is a lattice, there exists a basis $\{m_1, \cdots, m_n\}$ for V over $\mathscr{K}$ that generates $\mathscr{M}$ over $\mathbf{C}[[z^{-1}]]$. Condition (A.1.2.1) implies that

$$\sigma\,[\, m_1, \cdots, m_n \,] = [\, m_1, \cdots, m_n \,]\, A(z)$$

for some $n \times n$ matrix $A(z) \in gl(n, \mathbf{C}[[z^{-1}]])$. Therefore by utilizing the proof of Theorem 3.9.4 (due to M. Hukuhara), we can prove the theorem below.

THEOREM A.1.2.3. *An n-dimensional $D_{\mathscr{K}}$-module V is a Fuchsian module if and only if there exists a basis $\{u_1, \cdots, u_n\}$ for V over $\mathscr{K}$ and an $n \times n$ constant matrix $A_0 \in gl(n, \mathbf{C})$ such that*

$$\sigma\,[\, u_1, \cdots, u_n \,] = [\, u_1, \cdots, u_n \,]\, A_0. \tag{A.1.2.2}$$

The matrix A_0 can be chosen so that any two distinct eigenvalues of A_0 do not differ by an integer.

If the matrix A_0 is reduced to its Jordan canonical form, we obtain the following theorem.

THEOREM A.1.2.4. *A Fuchsian module V can be decomposed into a direct sum of a finite number of irreducible Fuchsian submodules as follows*:

$$V = \bigoplus_{j=1}^{q} \bigoplus_{\ell=1}^{m_j} E(\lambda_j) \bigotimes_{\mathscr{K}} N_{j\ell}, \tag{A.1.2.3}$$

where

(i) $\lambda_j \in \mathbf{C}$ *and* $\lambda_j - \lambda_k \notin \mathbf{Z}$ *if* $j \neq k$,
(ii) $N_{j\ell}$ *is a* $D_{\mathscr{K}}$*-module which admits a basis* $\{v_{j\ell 1}, \cdots, v_{j\ell n_{j\ell}}\}$ *such that*

$$\begin{cases} \sigma\,[\,v_{j\ell 1}, \cdots, v_{j\ell n_{j\ell}}\,] = [\,v_{j\ell 1}, \cdots, v_{j\ell n_{j\ell}}\,]J_{j\ell}, \\ J_{j\ell} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \end{cases} \tag{A.1.2.4}$$

REMARK A.1.2.5.

(1) The uniqueness of decomposition (A.1.2.3) can be proved by means of the Krull–Schmidt theorem (cf. N. Jacobson [72]).
(2) The ring $\mathbf{C}[[z^{-1}]]$ can be replaced by a complete discrete valuation ring whose residue field is algebraically closed of characteristic 0 (cf. I. J. Manin [104]).
(3) An n-dimensional $D_{\mathscr{K}}$-module V is a Fuchsian module if and only if every element $u \in V$ is contained in a finitely generated $\mathbf{C}[[z^{-1}]]$-submodule $\mathscr{M}(u)$ of V such that $\sigma\,\mathscr{M}(u) \in \mathscr{M}(u)$. This fact is closely related with Theorem 5.9.4 (due to D. Lutz). (See also Gérard–Levelt [16].)

§A.1.3. The general case

To state the Hukuhara–Turrittin theorem for the general case in terms of $D_{\mathscr{K}}$-modules, we must extend the field $\mathscr{K}$ to $\mathscr{L}$ by choosing a suitable positive integer s (cf. (6.8.2) and (A.1.1.9)).

THEOREM A.1.3.1. *Given an n-dimensional $D_{\mathscr{K}}$-module V, there exists a positive integer s such that the $D_{\mathscr{L}}$-module $\mathscr{L} \bigotimes_{\mathscr{K}} V$ can be decomposed into a direct sum of $D_{\mathscr{L}}$-submodules as follows:*

$$\mathscr{L} \bigotimes_{\mathscr{K}} V = \bigoplus_{j=1}^{p} \bigoplus_{\ell=1}^{m_j} E(\lambda_j) \bigotimes_{\mathscr{L}} N_{j\ell}, \qquad \text{(A.1.3.1)}$$

where

(i) $\lambda_j \in \mathbf{C}[\zeta]$ *and* $\lambda_j(\zeta) - \lambda_k(\zeta) \notin \mathbf{Z}$ *if* $j \neq k$,

(ii) $N_{j\ell}$ *is a* $D_{\mathscr{L}}$*-module which admits a basis* $\{v_{j\ell 1}, \cdots, v_{j\ell n_{j\ell}}\}$ *such that*

$$\begin{cases} \sigma\, [\, v_{j\ell 1}, \cdots, v_{j\ell n_{j\ell}}\,] = \dfrac{1}{s}\, [\, v_{j\ell 1}, \cdots, v_{j\ell n_{j\ell}}\,] J_{j\ell}, \\ \\ J_{j\ell} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \end{cases} \qquad \text{(A.1.3.2)}$$

REMARK A.1.3.2.

(1) The extension of $E(\lambda_j) \bigotimes_{\mathscr{L}} N_{j\ell}$ over any finite field extension $\mathscr{H}$ of $\mathscr{L}$ is an indecomposable $D_{\mathscr{H}}$-module.

(2) The uniqueness of decomposition (A.1.3.1) can be proved by means of the Krull–Schmidt theorem (cf. N. Jacobson [72]).

(3) The ring $\mathbf{C}[[z^{-1}]]$ can be replaced by a complete discrete valuation ring whose residue field is algebraically closed of characteristic 0 (cf. F. Baldassarri [42]).

§A.1.4. Cyclic vectors and Newton polygons

In the same way as we proved Lemma 5.7.1, we can prove the lemma below.

LEMMA A.1.4.1. *For a given n-dimensional $D_{\mathscr{K}}$-module V, there exists a vector $u_0 \in V$ such that*

$$u_0,\ \sigma\, u_0,\ \sigma^2\, u_0,\ \cdots,\ \sigma^{n-1}\, u_0 \qquad \text{(A.1.4.1)}$$

form a basis for V over $\mathscr{K}$.

As in §5.7, such a vector u_0 is called a *cyclic vector*.

Let u_0 be a cyclic vector. Then, for each vector $u \in V$, there exists a unique element $\sum_{0 \leq h \leq n-1} a_h \sigma^h \in D_{\mathscr{K}}$ such that

$$u = \left\{ \sum_{h=0}^{n-1} a_h \sigma^h \right\} u_0, \tag{A.1.4.2}$$

where $a_h \in \mathscr{K}$ $(h = 0, 1, \cdots, n-1)$. In particular there exist n elements $c_0, \cdots, c_{n-1} \in \mathscr{K}$ such that

$$\left\{ \sigma^n + \sum_{h=0}^{n-1} c_h \sigma^h \right\} u_0 = 0. \tag{A.1.4.3}$$

Set

$$\wp = \sigma^n + \sum_{h=0}^{n-1} c_h \sigma^h. \tag{A.1.4.4}$$

Utilizing a cyclic vector u_0, we can define a homomorphism ϕ of $D_{\mathscr{K}}$-modules : $D_{\mathscr{K}} \to V$ if we set

$$\phi(a) = a\, u_0 \quad \text{for } a \in D_{\mathscr{K}} \tag{A.1.4.5}$$

The homomorphism ϕ is surjective (cf. (A.1.4.2)) and

$$\text{Kernel}(\phi) = D_{\mathscr{K}}\, \wp.$$

(cf. (A.1.4.3) and (A.1.4.4)). Hence we have an isomorphism of $D_{\mathscr{K}}$-modules

$$\tilde{\phi} : V \to D_{\mathscr{K}} / D_{\mathscr{K}}\, \wp. \tag{A.1.4.6}$$

For example, the irreducible Fuchsian modules $E(\lambda_j) \bigotimes_{\mathscr{K}} N_{j\ell}$ given in Theorem A.1.2.4 are isomorphic to $D_{\mathscr{K}} / D_{\mathscr{K}}(\sigma - \lambda_j)^{n_{j\ell}}$ respectively.

For $a = \sum \alpha_m z^{-m} \in \mathscr{K}$, we set

$$\nu(a) = \inf\{ m \ ; \ \alpha_m \neq 0\}.$$

For a given element

$$P = \sum_{h=0}^{N} a_h \sigma^h \in D_{\mathscr{K}} \qquad (a_N \neq 0), \tag{A.1.4.7}$$

consider $N+1$ points $(h, \nu(a_h))$ $(h = 0, 1, \cdots, N)$ on an (X, Y)-plane. Set

$$\begin{cases} \mathscr{P}_h = \{ (X, Y) \ ; \ X \leq h, \ Y \geq \nu(a_h) \}, \\ \\ \mathscr{P} = \bigcup_{h=0}^{N} \mathscr{P}_h. \end{cases}$$

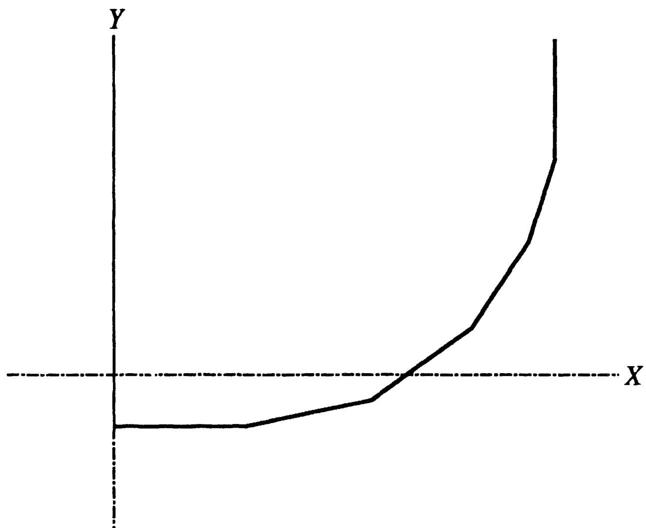

FIGURE 8

The boundary curve $\mathscr{C}$ of the smallest convex set containing $\mathscr{P}$ in the right half of the (X, Y)-plane is called the *Newton polygon* of the element P of $D_{\mathscr{K}}$. We denote by $\mathscr{N}(P)$ the Newton polygon of P. The polygon $\mathscr{N}(P)$ consists of a finite number of line segments with nonnegative slopes and two vertical rays (cf. Figure 8).

REMARK A.1.4.2. For a differential operator

$$\mathscr{L} = \sum_{h=0}^{N} a_h \delta^h,$$

where $a_h \in \mathscr{K}$ and $a_N \neq 0$, we can define the Newton polygon $\mathscr{N}(\mathscr{L})$ in exactly the same way as above. Let

$$0 \leq k_1 < k_2 < \cdots < k_p < +\infty \tag{A.1.4.8}$$

be all of the distinct slopes of sides of the Newton polygon $\mathscr{N}(\mathscr{L})$ and let

$$\{ (X, Y_j + k_j(X - X_j)) \ ; \ X_j \leq X \leq X_{j+1} \}$$

be the side of $\mathscr{N}(\mathscr{L})$ whose slope is k_j, where $Y_{j+1} = Y_j + k_j(X_{j+1} - X_j)$ (cf. Figure 9). Note that the k_j are rational numbers.
Then

$$X_1 = 0, \ X_{p+1} = N, \quad \text{and} \quad X_{j+1} - X_j \in \mathbf{Z}^+,$$

where $\mathbf{Z}^+$ is the set of all positive integers. Set

$$n_j = X_{j+1} - X_j \qquad (j = 1, 2, \cdots, p).$$

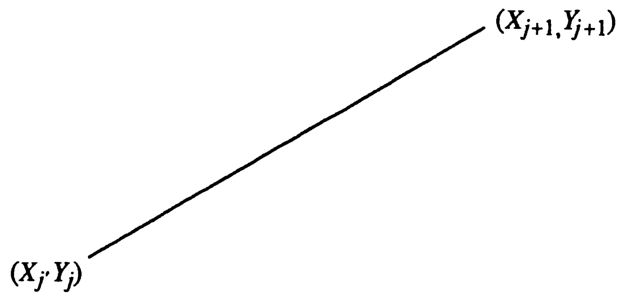

FIGURE 9

It is known that the differential equation

$$\mathscr{L}\, y = 0$$

admits n_j linearly independent formal solutions of the form

$$y_{j,h}(z) = z^{\mu_{j,h}} \exp[\lambda_{j,h} z^{k_j}\{1 + o(1)\}]\, f_{j,h}(z) \qquad (h = 1, \cdots, n_j), \quad (A.1.4.9)$$

where the $\mu_{j,h}$ are constants, the $\lambda_{j,h}$ are nonzero constants and $f_{j,h} \in \mathbf{C}[[z^{-\frac{1}{s}}]][z^{\frac{1}{s}}, \log z]$ for some positive integer s. Note that $\sum_{1\leq j\leq p} n_j = N$. These formal solutions are essentially the same as those given in §6.8 (cf. (6.8.9), (6.8.10), and (6.8.11)). We can construct formal solutions (A.1.4.9) systematically by utilizing the Newton polygon $\mathscr{N}(\mathscr{L})$ (cf. for example, W. Sternberg [145]).

§A.1.5. Theorem A.1.3.1 and Newton polygons

We can prove the Baldassarri theorem (cf. Theorem A.1.3.1) also by utilizing the Newton polygon $\mathscr{N}(\wp)$ if a $D_{\mathscr{K}}$-module V is given by

$$V = D_{\mathscr{K}} / D_{\mathscr{K}}\, \wp \qquad (A.1.5.1)$$

(cf. B. Malgrange [100], P. Robba [130] and Y. Sibuya [135]). The following observation will show a relation between a factorization of $\wp$ and a decomposition of module (A.1.5.1) into a direct sum of submodules.

OBSERVATION A.1.5.1. If $\wp = PQ$ for some P and Q in $D_{\mathscr{K}}$, we have an exact sequence

$$0 \to D_{\mathscr{K}}/D_{\mathscr{K}}\, P \to D_{\mathscr{K}}/D_{\mathscr{K}}\, \wp \to D_{\mathscr{K}}/D_{\mathscr{K}}\, Q \to 0 \qquad (A.1.5.2)$$

of homomorphisms of $D_{\mathscr{K}}$-modules. Exact sequence (A.1.5.2) splits if the Newton polygons $\mathscr{N}(P)$ and $\mathscr{N}(Q)$ do not have any common slopes. On the other hand, if all of the distinct slopes of the Newton polygon $\mathscr{N}(\wp)$ are given by (A.1.4.8), we have a factorization of $\wp$:

$$\wp = P_1 P_2 \cdots P_p$$

such that the Newton polygon $\mathcal{N}(P_j)$ has only one slope k_j. Hence

$$D_{\mathscr{K}}/D_{\mathscr{K}}\ \wp = \bigoplus_{j=1}^{p} D_{\mathscr{K}}/D_{\mathscr{K}}\ P_j. \tag{A.1.5.3}$$

Let us make more observations.

OBSERVATION A.1.5.2. We shall look at a $D_{\mathscr{K}}$-module V given by

$$V = D_{\mathscr{K}}/D_{\mathscr{K}}\ P, \tag{A.1.5.4}$$

where P is given by (A.1.4.7), assuming that the Newton polygon $\mathcal{N}(P)$ has only one slope k. Module (A.1.5.4) is a Fuchsian module if and only if $k = 0$. If $k > 0$, write

$$k = \frac{t}{s}, \tag{A.1.5.5}$$

where t and s are two relatively prime positive integers. Then s divides N and we can write P in the form

$$P = z^r \left\{ \sum_{\ell=0}^{\frac{N}{s}} \alpha_\ell \, (z^{-t}\sigma^s)^\ell + \sum_{\ell > kh} \alpha_{\ell h}\, z^{-\ell}\sigma^h \right\},$$

where $\alpha_\ell \in \mathbf{C}$, $\alpha_{\ell h} \in \mathbf{C}$, $r \in \mathbf{Z}$ and

$$\alpha_0 \neq 0, \qquad \alpha_{\frac{N}{s}} \neq 0.$$

The polynomial

$$\phi_P(\xi) = \sum_{\ell=0}^{\frac{N}{s}} \alpha_\ell\, \xi^\ell$$

is called the *determining polynomial* of P. Let

$$\phi_P(\xi) = \gamma \prod_{j=1}^{q} (\xi + \mu_j)^{m_j},$$

where the μ_j and γ are nonzero numbers in $\mathbf{C}$ and $\mu_j \neq \mu_h \quad (j \neq h)$.

The result below is a Hensel-type lemma (cf. P. Robba [130]).

OBSERVATION A.1.5.3. We have a factorization

$$P = P_1 \cdots P_q,$$

where, for each j, the Newton polygon $\mathcal{N}(P_j)$ has only one slope k and the determining polynomial of P_j is given by

$$\phi_{P_j}(\xi) = \gamma_j\, (\xi + \mu_j)^{m_j} \qquad (j = 1, \cdots, q),$$

where the γ_j are nonzero numbers in $\mathbf{C}$. Furthermore module (A.1.5.4) can be decomposed as

$$V = \bigoplus_{j=1}^{q} D_{\mathscr{K}}/D_{\mathscr{K}}\, P_j. \tag{A.1.5.6}$$

This decomposition of V is similar to the block-diagonalization which was explained in §5.5 (cf. Lemma 5.5.2). However in order to apply Lemma 5.5.2 to V, we must extend the field $\mathscr{K}$ to $\mathscr{L} = \mathbf{C}[[z^{-\frac{1}{s}}]][z^{\frac{1}{s}}]$. Decomposition (A.1.5.6) does not require such a field extension.

OBSERVATION A.1.5.4. Let us consider now the case when the Newton polygon $\mathscr{N}(P)$ has only one slope $k > 0$ and the determining polynomial of P is $\phi_P(\xi) = c(\xi + \mu)^m$, where c and μ are nonzero numbers in $\mathbf{C}$ and $m \in \mathbf{Z}^+$. Let us write $k = \frac{t}{s}$, where t and s are two relatively prime positive integers. We extend the field $\mathscr{K}$ to $\mathscr{L} = \mathbf{C}[[\zeta^{-1}]][\zeta]$, where $z = \zeta^s$. Then the Newton polygon $\mathscr{N}(P)$ of P as an element of $D_{\mathscr{L}}$ has only one slope t and the determining polynomial of P is $\tilde{\phi}_P(\eta) = c((\frac{\eta}{s})^s + \mu)^m$. Hence if $s > 1$, Observation A.1.5.3 applies. If $s = 1$ (i.e., $k = t$), we have

$$W = E(\mu z^t) \bigotimes_{\mathscr{K}} V \simeq D_{\mathscr{K}}/D_{\mathscr{K}} Q, \tag{A.1.5.7}$$

where

$$Q = \sum_{h=0}^{N} a_h\, (\sigma - \mu z^t)^h$$

and $\simeq$ means that the both sides of (A.1.5.7) are isomorphic to each other. Note that P is given by (A.1.4.7) and the one-dimensional $D_{\mathscr{K}}$-module $E(\lambda)$ was defined in §A.1.1. It can be proved that all the slopes of the Newton polygon $\mathscr{N}(Q)$ are strictly smaller than t. In this manner, we shall arrive at the same conclusion as Theorem A.1.3.1.

REMARK A.1.5.5. In some environments, a differential equation is not given *explicitly* as $\frac{dW}{dz} = A(z)W$, but a $D_{\mathscr{K}}$-module is given. In such a case computation of the matrix $A(z)$ is difficult. Even if $A(z)$ could be computed, the analytic expression of $A(z)$ might be hideously complicated. In any case, the method based on the theory of $D_{\mathscr{K}}$-modules is more intrinsic. Through such an approach, we find many helpful ideas in the theory of commutative algebra (cf. M. F. Atiyah and I. G. MacDonald [36]). To study formal structure of a completely integrable Pfaffian system at singularities, the best approach is to look at the problem as a study of D_R-modules, where $R = \mathbf{C}[[z_1, \cdots, z_m]]$ for example (cf. H. Charrière [52], H. Charrière and R. Gérard [53], and A. R. P. van den Essen and A. H. M. Levelt [56]). [The translator finds that book [120] of F. Pham is helpful to understand the theory of differential equations at singularities in terms of D_R-modules, where R is a differential ring.]

APPENDIX 2

Gevrey Asymptotics

§A.2.0. Preliminaries

In §5.1 we introduced a concept of asymptotic expansions. We defined there a map

$$A_{sp} : \mathscr{A}(R_0, a, b) \rightarrow \mathbf{C}[[z^{-1}]] \tag{A.2.0.1}$$

as follows:

$$A_{sp}(f) = p \tag{A.2.0.2}$$

means that

(i) $f \in \mathscr{O}(R_0, a, b)$,
(ii) $p = \sum_{m \geq 0} a_m z^{-m} \in \mathbf{C}[[z^{-1}]]$,
(iii) an inequality

$$\left| f(z) - \sum_{m=0}^{N-1} a_m z^{-m} \right| \leq K_{N,R,\alpha,\beta} \, |z|^{-N} \tag{A.2.0.3}$$

holds on $\mathscr{D}(R, \alpha, \beta)$ for every positive integer N and every (R, α, β) satisfying the inequalities $0 < R_0 < R$ and $a < \alpha < \beta < b$, where $K_{N,R,\alpha,\beta}$ is a nonnegative number determined by (N, R, α, β). Also,

$$\begin{cases} \mathscr{D}(R_0, a, b) = \{ z \; ; \; |z| > R_0, \; a < \arg z < b \}, \\ \\ \mathscr{O}(R_0, a, b) = \mathscr{O}(\mathscr{D}(R_0, a, b)). \end{cases}$$

The domain of definition of A_{sp} is denoted by $\mathscr{A}(R_0, a, b)$. This definition of asymptotic expansions was originally given by H. Poincaré [122].

As we indicated in (4) of §5.1 , A_{sp} is surjective but not injective. Let us denote the kernel of A_{sp} by $\mathscr{A}_{+\infty,0}(R_0, a, b)$, i.e.,

$$\mathscr{A}_{+\infty,0}(R_0, a, b) = \{ f \in \mathscr{A}(R_0, a, b) \; ; \; A_{sp}(f) = 0 \}. \tag{A.2.0.4}$$

If we examine various cases of asymptotic solutions of differential equations, we notice that $\mathbf{C}[[z^{-1}]]$ and $\mathscr{A}_{+\infty,0}(R_0, a, b)$ are too large. The following theorem due to E.Maillet suggests a suitable restriction of $\mathbf{C}[[z^{-1}]]$.

THEOREM A.2.0.1 (E. Maillet [94]). *Let $F(x, y_0, y_1, \cdots, y_n)$ be a nonzero polynomial in $y_0, y_1, \cdots, y_n$ with coefficients belonging to $\mathbf{C}\{x\}$, and let $f = \sum_{m \geq 0} a_m x^m \in \mathbf{C}[[x]]$ be a formal solution of the differential equation*:

$$F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}\right) = 0. \tag{A.2.0.5}$$

Then there exist nonnegative numbers K, p, and A such that

$$|a_m| \leq K(m!)^p A^m \qquad (m \geq 0). \tag{A.2.0.6}$$

N. Watson [148] and F. Nevanlinna [111] developed a theory of asymptotic series under restriction (A.2.0.6). J.-P. Ramis reorganized and developed further such a theory systematically (cf. J.-P.Ramis [123, 124]). Condition (A.2.0.6) is also related with *ultradistributions* with support at the origin (cf. H. Komatsu [81, 82, 83]). The initial motivation for the Ramis theory was the duality between the space of such ultradistributions and the space of formal power series satisfying condition (A.2.0.6) (cf. J.-P. Ramis [123, 125]). Thus he called his theory of asympotic series the **Gevrey asymptotics.**

The Gevrey asymptotics are based on the two definitions below.

DEFINITION A.2.0.2. Let s be a nonnegative number. A formal power series $f = \sum_{m \geq 0} a_m z^{-m} \in \mathbf{C}[[z^{-1}]]$ is said to be of **Gevrey order** s if there exist two nonnegative numbers C and A such that

$$|a_m| \leq C(m!)^s A^m \quad \text{for } m \in \mathbf{N}. \tag{A.2.0.7}$$

We denote by $\mathbf{C}[[z^{-1}]]_s$ the set of all power series of Gevrey order s.

DEFINITION A.2.0.3. A function f of z is said to admit an asymptotic expansion of **Gevrey order** s $(s \geq 0)$ as $z \to \infty$ on a sectorial domain $\mathscr{D}(R_0, a, b)$ if

(i) $f \in \mathscr{A}(R_0, a, b)$,
(ii) $A_{sp}(f) = \sum_{m \geq 0} a_m z^{-m} \in \mathbf{C}[[z^{-1}]]_s$,
(iii) an inequality

$$\left| f(z) - \sum_{m=0}^{N-1} a_m z^{-m} \right| \leq K_{R,\alpha,\beta} \, (N!)^s \, (B_{R,\alpha,\beta})^N \, |z|^{-N} \tag{A.2.0.8}$$

holds on $\mathscr{D}(R, \alpha, \beta)$ for every positive integer N and every (R, α, β) satisfying the inequalities $0 < R_0 < R$ and $a < \alpha < \beta < b$, where $K_{R,\alpha,\beta}$ and $B_{R,\alpha,\beta}$ are nonnegative numbers determined by (R, α, β). We denote by $\mathscr{A}_s(R_0, a, b)$ the set of all functions admitting asymptotic expansions of Gevrey order s as $z \to \infty$ on the sectorial domain $\mathscr{D}(R_0, a, b)$.

As we shall explain in §A.2.2, the kernel of the map $A_{sp} : \mathscr{A}_s(R_0, a, b) \to \mathbf{C}[[z^{-1}]]_s$ is characterized by the theorem below.

THEOREM A.2.0.4. *For $s > 0$, set*

$$\mathscr{A}_{s,0}(R_0, a, b) = \mathscr{A}_s(R_0, a, b) \cap \mathscr{A}_{+\infty,0}(R_0, a, b). \qquad \text{(A.2.0.9)}$$

Then $f \in \mathscr{A}_{s,0}(R_0, a, b)$ if and only if

(i) $f \in \mathscr{O}(R_0, a, b)$,

(ii) *an inequality*

$$|f(z)| \leq \mu(R, \alpha, \beta) \exp\left\{ -\lambda(R, \alpha, \beta)\ |z|^{\frac{1}{s}} \right\} \qquad \text{(A.2.0.10)}$$

holds on $\mathscr{D}(R, \alpha, \beta)$ for every (R, α, β) satisfying the inequalities $0 < R_0 < R$ and $a < \alpha < \beta < b$, where $\mu(R, \alpha, \beta)$ is a non-negative number and $\lambda(R, \alpha, \beta)$ is a positive number, and they are determined by (R, α, β).

As in §6.1, if N sectorial domains

$$\mathscr{S}_\ell = \mathscr{D}(R, a_\ell, b_\ell) \qquad (\ell = 1, \cdots, N) \qquad \text{(A.2.0.11)}$$

satisfy the condition

$$\bigcup_{\ell=1}^{N} \mathscr{S}_\ell = \{ z\ ;\ |z| > R \}, \qquad \text{(A.2.0.12)}$$

we call $\{ \mathscr{S}_1, \cdots, \mathscr{S}_N \}$ **a covering at** $z = \infty$. Furthermore we call $\{ \mathscr{S}_1, \cdots, \mathscr{S}_N \}$ a **good covering** at $z = \infty$ if

(i) $a_\ell < a_{\ell+1} \quad (\ell = 1, \cdots, N) \quad$ where $\quad a_{N+1} = a_1 + 2\pi$,

(ii) $b_\ell - a_\ell < \pi \quad (\ell = 1, \cdots, N)$,

(iii) $\mathscr{S}_\ell \cap \mathscr{S}_{\ell+1} \neq \emptyset \quad (\ell = 1, \cdots, N)$ and $\mathscr{S}_\ell \cap \mathscr{S}_k = \emptyset$ otherwise if $\ell \neq k$, where $\mathscr{S}_{N+1} = \mathscr{S}_1$.

In order to improve various results of the Poincaré asymptotics by means of the Gevrey asymptotics, the following lemma is fundamental.

LEMMA A.2.0.5. *Assume that a covering $\{ \mathscr{S}_1, \cdots, \mathscr{S}_N \}$ at $z = \infty$ is good and that N functions $\phi_1(z), \cdots, \phi_N(z)$ satisfy the conditions*

(1) $\phi_\ell(z) \in \mathscr{O}(R, a_\ell, b_\ell)$,

(2) *$\phi_\ell(z)$ is bounded on $\mathscr{D}(R, a_\ell, b_\ell)$,*

(3) *we have*

$$|\phi_\ell(z) - \phi_{\ell+1}(z)| \leq \gamma \exp\left\{ -\lambda\ |z|^k \right\} \qquad \text{on} \quad \mathscr{S}_\ell \cap \mathscr{S}_{\ell+1}, \qquad \text{(A.2.0.13)}$$

where $\gamma \geq 0$, $\lambda > 0$, and $k > 0$ are suitable numbers.

Then

$$\phi_\ell \in \mathscr{A}_{\frac{1}{k}}(R, a_\ell, b_\ell) \qquad (\ell = 1, \cdots, N). \tag{A.2.0.14}$$

[The author of this book also found a similar result (cf. Theorem A.2.5.2) through his study of the Matkowsky condition for the Ackerberg–O'Malley resonance (cf. Ackerberg–O'Malley [35], Y. Sibuya [136], B. J. Matkowsky [107], N. Kopell [84, 85] and C.-H. Lin [87]).]

In this appendix, we shall explain some basic properties of $\mathbf{C}[[z^{-1}]]_s$ and $\mathscr{A}_s(R_0, a, b)$, the existence and uniqueness of Gevrey asymptotic solutions, applications of Lemma A.2.0.5, and some results similar to Lemma A.2.0.5.

§A.2.1. Properties of $\mathbf{C}[[z^{-1}]]_s$

In this section, we shall explain some basic properties of $\mathbf{C}[[z^{-1}]]_s$

PROPERTY A.2.1.1. For $s \geq 0$, $\mathbf{C}[[z^{-1}]]_s$ is a subalgebra of $\mathbf{C}[[z^{-1}]]$ as a commutative differential algebra over $\mathbf{C}$. In particular, $\mathbf{C}[[z^{-1}]]_0 = \mathbf{C}\{z^{-1}\}$.

Actually we can prove that

(i) $f + g \in \mathbf{C}[[z^{-1}]]_s$ if f and $g \in \mathbf{C}[[z^{-1}]]_s$,
(ii) $fg \in \mathbf{C}[[z^{-1}]]_s$ if f and $g \in \mathbf{C}[[z^{-1}]]_s$,
(iii) $cf \in \mathbf{C}[[z^{-1}]]_s$ if $c \in \mathbf{C}$ and $f \in \mathbf{C}[[z^{-1}]]_s$,
(iv) $\frac{df}{dz} \in \mathbf{C}[[z^{-1}]]_s$ if $f \in \mathbf{C}[[z^{-1}]]_s$,
(v) $\int_\infty^z f dz \in \mathbf{C}[[z^{-1}]]_s$ if $f \in z^{-2}\mathbf{C}[[z^{-1}]]_s$.

It is easy to prove (i), (iii), (iv), and (v). Property (ii) follows from the lemma below (cf. G. N. Watson [148]).

LEMMA A.2.1.2. *For $s > 0$ and any integer $h \geq 2$, we have*

$$\sum_{p_1+\cdots+p_h=n,\ p_j \in \mathbf{N}} (p_1!)^s (p_2!)^s \cdots (p_h!)^s \leq (\gamma(s))^{h-1}(n!)^s \qquad (n \in \mathbf{N}), \tag{A.2.1.1}$$

where $\mathbf{N}$ is the set of all nonnegative integers and $\gamma(s)$ is a positive number depending only on s.

PROPERTY A.2.1.3. Let $s_1, \cdots, s_n$ and $\sigma_1, \cdots, \sigma_n$ be nonnegative real numbers. Assume also that a formal power series in n variables $y_1, \cdots, y_n$,

$$F(y_1, \cdots, y_n) = \sum_{(p_1, \cdots, p_n) \in \mathbf{N}^n} \alpha_{p_1 \cdots p_n} y_1^{p_1} \cdots y_n^{p_n},$$

with coefficients $\alpha_{p_1 \cdots p_n}$ in $\mathbf{C}$ satisfy the conditions

$$|\alpha_{p_1 \cdots p_n}| \leq C(p_1!)^{s_1} \cdots (p_n!)^{s_n} A_1^{p_1} \cdots A_n^{p_n} \qquad ((p_1, \cdots, p_n) \in \mathbf{N}^n)$$

for some nonnegative numbers $C, A_1, \cdots, A_n$ and that

$$f_j = \sum_{m\geq 1} z^{-m} \in \mathbf{C}[[z^{-1}]]_{\sigma_j} \qquad (j = 1, \cdots, n).$$

Set

$$s = \max\{s_1, \cdots, s_n, \sigma_1, \cdots, \sigma_n\}. \tag{A.2.1.2}$$

Then

$$F(f_1, \cdots, f_n) \in \mathbf{C}[[z^{-1}]]_s. \tag{A.2.1.3}$$

In particular, if $f = \sum_{m\geq 0} b_m z^{-m} \in \mathbf{C}[[z^{-1}]]_s$ $(s \geq 0)$ and if $b_0 \neq 0$, then

$$\frac{1}{f} \in \mathbf{C}[[z^{-1}]]_s \tag{A.2.1.4}$$

and

$$f^{\frac{1}{p}} \in \mathbf{C}[[z^{-1}]]_s \tag{A.2.1.5}$$

for every positive integer p.

This is a good exercise (cf. M. Gevrey [59]).

EXAMPLE A.2.1.4. The power series $f = \sum_{m\geq 0} m! z^{-m}$ belongs to $\mathbf{C}[[z^{-1}]]_1$. Hence $\frac{1}{f} \in \mathbf{C}[[z^{-1}]]_1$ and $f^{\frac{1}{p}} \in \mathbf{C}[[z^{-1}]]_1$ for every positive integer p. The power series f is a formal solution of the linear differential equation

$$\frac{d}{dz}\left[z\left(\frac{d}{dz} + 1\right)\right] z^{-1}y = 0.$$

However, $\frac{1}{f}$ and $f^{\frac{1}{p}}$ $(p \geq 2)$ do not satisfy any linear ordinary differential equations of the form

$$\sum_{h=0}^{N} a_h(z) \frac{d^h y}{dz^h} = 0 \qquad (a_h \in \mathbf{C}\{z^{-1}\})$$

(cf. W. A. Harris–Y. Sibuya [64, 65] , S. Sperber [144] and M. Singer [143]).

For two real numbers $s \geq 0$ and $A > 0$ and for a formal power series $f = \sum_{m\geq 0} a_m z^{-m} \in \mathbf{C}[[z^{-1}]]$ we set

$$||f||_{s,A} = \sum_{m\geq 0} \frac{|a_m|}{(m!)^s A^m}. \tag{A.2.1.6}$$

Note that

$$||f||_{s_1,A} \leq ||f||_{s_2,A} \qquad \text{if} \qquad s_2 \leq s_1. \tag{A.2.1.7}$$

In order that $f \in \mathbf{C}[[z^{-1}]]_s$, it is necessary and sufficient that $||f||_{s,A} < +\infty$ for some $A > 0$. Let us denote by $E(s, A)$ the set of all $f \in \mathbf{C}[[z^{-1}]]$ such that $||f||_{s,A} < +\infty$, i.e.,

$$E(s, A) = \{f \in \mathbf{C}[[z^{-1}]];\ ||f||_{s,A} < +\infty\}. \tag{A.2.1.8}$$

It is easy to show that $E(s, A)$ is a Banach space with the norm $||f||_{s,A}$.

PROPERTY A.2.1.5. If f and $g \in E(s, A)$, then $fg \in E(s, A)$ and

$$||fg||_{s,A} \le ||f||_{s,A}||g||_{s,A}. \tag{A.2.1.9}$$

Also if $f \in E(s, A)$, then $f \in E(s, B)$ for all $B \ge A$ and

$$\begin{cases} ||f||_{s,B} \ \le \ ||f||_{s,A}, \\ \\ ||f||_{s,B} - |f(0)| \le \dfrac{A}{B}\ \{||f||_{s,A} - |f(0)|\ \}, \end{cases} \tag{A.2.1.10}$$

for $B \ge A$. Furthermore, if $f \in E(s, A)$ and if q is an integer not less than $\frac{1}{s}$, then

$$\left|\left| z^{1-q}\frac{df}{dz}\right|\right|_{s,A} \ \le \ \frac{1}{A^q}\ ||f||_{s,A}. \tag{A.2.1.11}$$

PROPERTY A.2.1.6. Consider a formal power series

$$\vec{F}(z, \vec{y}, \vec{u}) = \sum_{|\vec{p}|+|\vec{q}|\ge 0} \vec{f}_{\vec{p},\vec{q}}\ \vec{y}^{\vec{p}}\ \vec{u}^{\vec{q}}, \tag{A.2.1.12}$$

where $\vec{f}_{\vec{p},\vec{q}} \in \mathbf{C}[[z^{-1}]]^n$, $\vec{p} = (p_1, \cdots, p_n) \in \mathbf{N}^n$, $\vec{q} = (q_1, \cdots, q_m) \in \mathbf{N}^m$, $|\vec{p}| = \sum_{1\le h\le n} p_h$, $|\vec{q}| = \sum_{1\le h\le m} q_h$, $\vec{y}^{\vec{p}} = y_1^{p_1}\cdots y_n^{p_n}$, $\vec{u}^{\vec{q}} = u_1^{q_1}\cdots u_m^{q_m}$ and

$$\vec{y} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}, \qquad \vec{u} = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_m \end{bmatrix}.$$

Let us denote by $\vec{F}_{\vec{y}}(z, \vec{y}, \vec{u})$ the Jacobian matrix of $\vec{F}$ with respect to $\vec{y}$. Then we can prove an **implicit function theorem**.

THEOREM A.2.1.7. *Assume that power series* (A.2.1.12) *satisfies the following conditions*:

(1) $\vec{f}_{\vec{0},\vec{0}} \in z^{-1}\mathbf{C}[[z^{-1}]]^n$,
(2) $\vec{F}_{\vec{y}}(z, \vec{0}, \vec{0}) - \Phi \in z^{-1}gl(n, \mathbf{C}[[z^{-1}]])$ *for some invertible matrix* $\Phi \in GL(n, \mathbf{C})$,
(3) $\vec{f}_{\vec{p},\vec{q}} \in E(s, A)^n$ *for some* $s \ge 0$ *and* $A > 0$,

(4) $\sum_{|\vec{p}|\geq 0, |\vec{q}|\geq 0} ||\vec{f}_{\vec{p},\vec{q}}||_{s,A}\ \vec{y}^{\vec{p}}\ \vec{u}^{\vec{q}}$ *is a convergent power series in* $\vec{y}$ *and* $\vec{u}$ *, where*

$$||\vec{f}||_{s,A} = \max_{1\leq j\leq n} ||f_j||_{s,A}$$

for

$$\vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ f_n \end{bmatrix} \in E(s,A)^n.$$

Then, there exists a unique power series in z^{-1} *and* $\vec{u}$

$$\vec{\phi}(z,\vec{u}) = \sum_{|\vec{q}|\geq 0} \vec{\phi}_{\vec{q}}\ \vec{u}^{\vec{q}} \tag{A.2.1.13}$$

such that

(i)

$$\begin{cases} \vec{\phi}_{\vec{q}} \in \mathbf{C}[[z^{-1}]]^n, \qquad \vec{\phi}_{\vec{0}} \in z^{-1}\mathbf{C}[[z^{-1}]]^n, \\ \vec{F}(z,\vec{\phi}(z,\vec{u}),\vec{u}) = 0, \end{cases}$$

(ii) $\vec{\phi}_{\vec{q}} \in E(s,B)^n$ *for some* $B > 0$*, and*

(iii) *the power series* $\sum_{|\vec{q}|\geq 0} ||\vec{\phi}_{\vec{q}}||_{s,B}\ \vec{u}^{\vec{q}}$ *is a convergent power series in* $\vec{u}$.

This is also a good exercise. To verify this result, it suffices to construct a suitable majorizing series.

PROPERTY A.2.1.8. Let us look at a differential equation

$$\vec{y} = \vec{g}_0(z) + z^{1-q}\,\Phi(z)\,\frac{d\vec{y}}{dz} + \sum_{|\vec{p}|\geq 2} \vec{g}_{\vec{p}}(z)\ \vec{y}^{\vec{p}}, \tag{A.2.1.14}$$

where

$$\vec{g}_0 \in z^{-1}\ E(s,A)^n, \qquad \Phi \in gl(n,E(s,A)), \qquad \vec{g}_{\vec{p}} \in E(s,A)^n$$

for some $s \geq 0$ and $A > 0$ and $\sum_{|\vec{p}|\geq 2} ||\vec{g}_{\vec{p}}||_{s,A}\ \vec{y}^{\vec{p}}$ is a convergent power series in $\vec{y}$.

Utilizing (A.2.1.7), (A.2.1.9), (A.2.1.10), and (A.2.1.11), we can prove Theorem A.2.1.9.

THEOREM A.2.1.9. *Differential equation* (A.2.1.14) *admits a unique solution* $\vec{\phi}(z) \in z^{-1}\ E(\sigma, B)^n$, *where*

$$\sigma = \max\left\{ s,\ \frac{1}{q} \right\} \tag{A.2.1.15}$$

and B is a sufficiently large positive number.

Note that, if B is sufficiently large, the right-hand side of differential equation (A.2.1.14) defines a contraction map : $E(\sigma, B)^n \to E(\sigma, B)^n$.

Theorem A.2.1.9 is a result better than the Maillet theorem (cf. Theorem A.2.0.1), although it applies only to differential equation (A.2.1.14). In §A.2.4, we shall explain by means of Lemma A.2.0.5 a generalization of Theorem A.2.1.9 that applies to algebraic differential equation (A.2.0.5) .

§A.2.2. Properties of $\mathscr{A}_s(R_0, a, b)$

In this section, we shall explain some basic properties of $\mathscr{A}_s(R_0, a, b)$.

PROPERTY A.2.2.1. In §5.1, we indicated in (1) and (2) that $\mathscr{A}(R_0, a, b)$ is a commutative differential algebra over $\mathbf{C}$ and that the map

$$A_{sp} : \mathscr{A}(R_0, a, b) \to \mathbf{C}[[z^{-1}]] \tag{A.2.2.1}$$

is a homomorphism of commutative differential algebras over $\mathbf{C}$. To verify a similar result for $\mathscr{A}_s(R_0, a, b)$ $(s \geq 0)$ the following characterization of $\mathscr{A}_s(R_0, a, b)$ is helpful.

LEMMA A.2.2.2. *Let s be a nonnegative number. Then a function* $f(z)$ *belongs to* $\mathscr{A}_s(R_0, a, b)$ *if and only if*

(i) $f \in \mathscr{O}(R_0, a, b)$ *and*

(ii) *for every* (R, α, β) *satisfying the condition*

$$0 < R_0 < R, \qquad a < \alpha < \beta < b, \tag{A.2.2.2}$$

there exist two nonnegative numbers $C_{R,\alpha,\beta}$ *and* $A_{R,\alpha,\beta}$ *such that*

$$\left| \frac{d^p f\left(\frac{1}{x}\right)}{dx^p} \right| \leq C_{R,\alpha,\beta}\ (p!)^{s+1}\ \left(A_{R,\alpha,\beta}\right)^p \tag{A.2.2.3}$$

for every $p \in \mathbf{N}$ *and* $z = \frac{1}{x} \in \mathscr{D}(R, \alpha, \beta)$.

By virtue of Lemma A.2.2.2, we can show for $s \geq 0$ that $\mathscr{A}_s(R_0, a, b)$ is a subalgebra of $\mathscr{A}(R_0, a, b)$ as a commutative differential algebra over $\mathbf{C}$ and that $\mathscr{A}_0(R_0, a, b)$ is the set of all functions in $\mathscr{O}(R_0, a, b)$ that are holomorphic at $z = \infty$. The map

$$A_{sp} : \mathscr{A}_s(R_0, a, b) \to \mathbf{C}[[z^{-1}]]_s \tag{A.2.2.4}$$

is a restriction of map (A.2.2.1) and hence is a homomorphism of commutative differential algebras over **C**. We can also show that

$$\int_{\infty}^{z} f(\zeta)d\zeta \in \mathscr{A}_s(R_0, a, b) \qquad \text{if } f \in z^{-2}\mathscr{A}_s(R_0, a, b).$$

PROPERTY A.2.2.3. Consider a map

$$\frac{1}{f} : \mathscr{D}(R_1, a_1, b_1) \rightarrow \mathscr{D}(R_2, a_2, b_2)$$

such that

$$\begin{cases} f \in \mathscr{A}_{s_1}(R_1, a_1, b_1), \\ f \notin \mathscr{A}_{s_1,0}(R_1, a_1, b_1), \\ A_{sp}(f) \in z^{-1}\mathbf{C}[[z^{-1}]]_{s_1}. \end{cases}$$

Then, if $g \in \mathscr{A}_{s_2}(R_2, a_2, b_2)$, we have

$$\begin{cases} g \circ \left(\frac{1}{f}\right) \in \mathscr{A}_{\max(s_1,s_2)}(R_1, a_1, b_1), \\ A_{sp}\left(g \circ \left(\frac{1}{f}\right)\right) = (A_{sp}(g)) \circ \left(\frac{1}{A_{sp}(f)}\right). \end{cases}$$

This can be proved basically by using Lemma A.2.2.2.

PROPERTY A.2.2.4. We define a norm for functions in $\mathscr{O}(R_0, a, b)$ by

$$||f||_{s,A,V} = \sup_{\frac{1}{x} \in V} \sum_{p=0}^{+\infty} \frac{1}{(p!)^{s+1}A^p} \left| \frac{d^p f\left(\frac{1}{x}\right)}{dx^p} \right| \qquad (f \in \mathscr{O}(R_0, a, b)), \tag{A.2.2.5}$$

where $s \geq 0$, $A > 0$ and $V = \mathscr{D}(R_0, a, b)$. Then Lemma A.2.2.2 implies that $f \in \mathscr{O}(R_0, a, b)$ belongs to $\mathscr{A}_s(R_0, a, b)$ if and only if

$$||f||_{s,A_W,W} < +\infty \tag{A.2.2.6}$$

for every $W = \mathscr{D}(R, \alpha, \beta)$ satisfying condition (A.2.2.2) and some positive constant A_W.

PROPERTY A.2.2.5. Set

$$E(s, A, V) = \{ f \in \mathscr{O}(R_0, a, b) ; ||f||_{s,A,V} < +\infty \}. \tag{A.2.2.7}$$

Then $E(s, A, V)$ is a Banach space with the norm $||f||_{s,A,V}$. The following theorem is a good exercise.

THEOREM A.2.2.6. *Let*

$$F(z, \vec{y}) = \sum_{|\vec{p}| \geq 0} f_{\vec{p}}(z)\, \vec{y}^{\vec{p}} \tag{A.2.2.8}$$

be a power series in $\vec{y} = (y_1, \cdots, y_n)$ *with coefficients* $f_{\vec{p}} \in \mathscr{A}_s(R_0, a, b)$. *Assume that, for every* $W = \mathscr{D}(R, \alpha, \beta)$ *satisfying condition* (A.2.2.2), *there exist two positive numbers* A_W *and* r_W *such that the power series*

$$\sum_{|\vec{p}|\geq 0} ||f_{\vec{p}}(z)||_{s,A_W,W}\ \vec{y}^{\vec{p}}$$

is uniformly and absolutely convergent for $|y_j| \leq r_W$ $(j = 1, \cdots, n)$. *Let* $\vec{g} = (g_1, \cdots, g_n) \in \mathscr{A}_s(R_0, a, b)^n$ *be such that*

$$\begin{cases} A_{sp}(g_j) \in z^{-1}\mathbf{C}[[z^{-1}]]_s, \\ ||g_j||_{s,A_W,W} \leq r_W \end{cases} \tag{A.2.2.9}$$

for $j = 1, \cdots, n$. *Then*

$$\begin{cases} F(z, \vec{g}) \in \mathscr{A}_s(R_0, a, b), \\ A_{sp}(F(x, \vec{g})) = \displaystyle\sum_{|\vec{p}|\geq 0} (A_{sp}(f_{\vec{p}}))(A_{sp}(\vec{g}))^{\vec{p}}. \end{cases} \tag{A.2.2.10}$$

To prove Theorem A.2.2.6, we need the lemma below.

LEMMA A.2.2.7.

(i) *If* $f \in E(s, A, V)$, *then* $f \in E(s, B, V)$ *for all* $B \geq A$ *and*

$$\begin{cases} \sup_{z\in V} |f(z)| \leq ||f||_{s,B,V} \leq ||f||_{s,A,V}, \\ ||f||_{s,B,V} \leq \sup_{z\in V} |f(z)| + \dfrac{A}{B}\ ||f||_{s,A,V} \end{cases} \tag{A.2.2.11}$$

for $B \geq A$.

(ii) *If* f *and* g *belong to* $E(s, A, V)$, *then* fg *also belongs to* $E(s, A, V)$ *and*

$$||fg||_{s,A,V} \leq ||f||_{s,A,V}\ ||g||_{s,A,V}. \tag{A.2.2.12}$$

REMARK A.2.2.8. In general, the condition: $||g_j||_{s,A_W,W} \leq r_W$ might not be satisfied by a given $\vec{g} \in \mathscr{A}_s(R_0, a, b)^n$. In such a case, we can find a positive-valued continuous function $\rho(\theta)$ on the interval $a < \theta < b$ such that $F(z, \vec{g}) \in \mathscr{A}_s(\rho(\theta), a, b)$, where

$$\mathscr{D}(\rho(\theta), a, b) = \{z\ ;\ a < \arg z < b,\ |z| > \rho(\arg z)\}. \tag{A.2.2.13}$$

For example, if $f \in \mathscr{A}_s(R_0, a, b)$ satisfies the condition $A_{sp}(f) \notin z^{-1}\mathbf{C}[[z^{-1}]]_s$, then there exists a positive-valued continuous function $\rho(\theta) > R_0$ on the interval $a < \theta < b$ such that $\frac{1}{f} \in \mathscr{A}_s(\rho(\theta), a, b)$. We can also prove an implicit function theorem similar to Theorem A.2.1.7.

THEOREM A.2.2.9. *Consider a power series*

$$\vec{F}(z,\vec{y},\vec{u}) = \vec{f}_0(z) + \Phi(z)\vec{y} + \Psi(z)\vec{u} + \sum_{|\vec{p}|+|\vec{q}|\geq 2} \vec{f}_{\vec{p},\vec{q}}\ \vec{y}^{\vec{p}}\ \vec{u}^{\vec{q}},$$

where $\vec{f}_0 \in \mathscr{A}_s(R_0,a,b)^n$, $\vec{f}_{\vec{p},\vec{q}} \in \mathscr{A}_s(R_0,a,b)^n$, $\Phi \in gl(n,\mathscr{A}_s(R_0,a,b))$, Ψ *is an* $n \times m$ *matrix with entries in* $\mathscr{A}_s(R_0,a,b)$, *and*

$$\vec{y} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}, \qquad \vec{u} = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_m \end{bmatrix}.$$

Assume that the following conditions are satisfied:

(1) $A_{sp}(\vec{f}_0) \in z^{-1}\mathbf{C}[[z^{-1}]]_s{}^n$,

(2) $A_{sp}(\Phi) - \Phi_0 \in z^{-1}gl(n,\mathscr{A}_s(R_0,a,b))$ *for some invertible matrix* $\Phi_0 \in GL(n,\mathbf{C})$,

(3) *for every* $W = \mathscr{D}(R,\alpha,\beta)$ *satisfying the condition:* $R_0 < R$, $a < \alpha < \beta < b$ *there exists a positive number* A_W *such that the power series*

$$\sum_{|\vec{p}|+|\vec{q}|\geq 2} ||\vec{f}_{\vec{p},\vec{q}}||_{s,A_W,W}\ \vec{y}^{\vec{p}}\ \vec{u}^{\vec{q}},$$

is convergent in $\vec{y}$ *and* $\vec{u}$.

Then, there exists a unique power series in $\vec{u}$

$$\vec{\phi}(z,\vec{u}) = \sum_{|\vec{q}|\geq 0} \vec{\phi}_{\vec{q}}\ \vec{u}^{\vec{q}}$$

such that

(i)

$$\begin{cases} \vec{\phi}_{\vec{q}} \in \mathscr{A}_s(\rho(\theta),a,b)^n, \\ A_{sp}(\vec{\phi}_{\vec{q}}) \in\ z^{-1}\mathbf{C}[[z^{-1}]]^n \qquad if \qquad |\vec{q}| = 0, \end{cases}$$

for a suitable positive-valued continuous function $\rho(\theta) > R_0$ *on the interval* $a < \theta < b$,

(ii) *if* $W = \mathscr{D}(R,\alpha,\beta)$ *satisfies the condition*

$$a < \alpha < \beta < b, \qquad \max_{\alpha\leq\theta\leq\beta} \rho(\theta) < R$$

there exist two positive numbers B_W *and* r_W *such that the series*

$$\sum_{|\vec{q}|\geq 0} ||\vec{\phi}_{\vec{q}}||_{s,B_W,W}\ (r_W)^{|\vec{q}|}$$

is convergent,

(iii) $\vec{F}(z, \vec{\phi}(z, \vec{u}), \vec{u}) = 0$ *for* $z \in W$ *and* $\|u\| \leq r_W$ *if* W *satisfies the condition given in* (ii).

PROPERTY A.2.2.10. Let us investigate the surjectivity of homomorphism (A.2.2.4). The homomorphism A_{sp} is not always surjective. For example, since $\mathbf{C}[[z^{-1}]]_0 = \mathbf{C}\{z^{-1}\}$, but $\mathcal{A}_0(R_0, a, b)$ is the set of all functions in $\mathcal{O}(R_0, a, b)$ that are holomorphic at $z = \infty$, homomorphism (A.2.2.4) is not surjective if $s = 0$. As a matter of fact, we can prove the following theorem.

THEOREM A.2.2.11. *Homomorphism* (A.2.2.4) *is surjective if and only if*

$$b - a \leq s\,\pi. \tag{A.2.2.14}$$

To prove the surjectivity of A_{sp}, we utilize the *incomplete Leroy transforms* $\ell_{r,k}$ *of order* k of a convergent power series $\phi \in \mathbf{C}\{z^{-1}\}$ which are defined by

$$\ell_{r,k}(\phi) = k\ z^k \int_0^r \phi(\frac{1}{t}) \exp\left\{-(tz)^k\right\}\ t^{k-1}\ dt, \tag{A.2.2.15}$$

where r and k are two positive numbers. Note that $\phi\left(\frac{1}{t}\right) \in \mathbf{C}\{t\}$. Hence, if r is smaller than the radius of convergence of $\phi(\frac{1}{t})$ and if $|\arg z| \leq \frac{\pi}{2k}$, integral (A.2.2.15) is well defined. The "if" part of Theorem A.2.2.11 is a corollary of the lemma below.

LEMMA A.2.2.12. *Given a convergent power series* $\phi = \sum_{m \geq 0} c_m\ z^{-m} \in \mathbf{C}\{z^{-1}\}$, *we have*

(i) $\ell_{r,k}(\phi) \in \mathcal{A}_{\frac{1}{k}}(R, -\frac{\pi}{2k}, \frac{\pi}{2k})$ *for any positive numbers* k *and* R,

(ii) $A_{sp}(\ell_{r,k}(\phi)) = \sum_{m \geq 0} \Gamma(1 + \frac{m}{k})\ c_m\ z^{-m}$.

Here we assume that r *is smaller than the radius of convergence of* $\phi(\frac{1}{t})$.

We shall come back to the "only if" part of Theorem A.2.2.11 in Appendix 4.

PROPERTY A.2.2.13. Let us investigate the injectivity of homomorphism (A.2.2.4). The kernel $\mathcal{A}_{s,0}(R_0, a, b)$ of A_{sp} consists of all functions f in $\mathcal{O}(R_0, a, b)$ such that

$$|f(z)| \leq K_{R,\alpha,\beta}(N!)^s (B_{R,\alpha,\beta})^N |z|^{-N} \tag{A.2.2.16}$$

holds on $\mathcal{D}(R, \alpha, \beta)$ for every positive integer N and every (R, α, β) satisfying condition (A.2.2.2), where $K_{R,\alpha,\beta}$ and $B_{R,\alpha,\beta}$ are nonnegative numbers determined by (R, α, β). (cf. Definition A.2.0.3). The characterization of $\mathscr{A}_{s,0}(R_0, a, b)$ given by (A.2.0.10) of Lemma A.2.0.4 follows from the characterization given by (A.2.2.16.) (cf. N. Watson [148]).

We can derive the following theorem from Lemma A.2.0.4.

THEOREM A.2.2.14. *Homomorphism* (A.2.2.4) *is injective if and only if*

$$b - a > s\,\pi. \tag{A.2.2.17}$$

The "if" part of this theorem is a corollary of the lemma below.

LEMMA A.2.2.15 (N. Watson [148]). *Suppose that a function f is holomorphic on $\mathscr{D}(R_0, a, b)$ and satisfies the condition*

$$|f(z)| \leq \mu \exp\{-\lambda\, |z|^k\} \tag{A.2.2.18}$$

for $z \in \mathscr{D}(R_0, a, b)$, where μ is a nonnegative number and λ is a positive number. In this case, if a, b, and $s = \frac{1}{k}$ satisfy condition (A.2.2.17), *then $f(z) = 0$ identically on $\mathscr{D}(R_0, a, b)$.*

The "only if" part of Theorem A.2.2.14 is evident.

§A.2.3. Gevrey asymptotic solutions

For a given linear differential operator

$$P = \sum_{h=0}^{n} a_h(z)\, \delta^h, \tag{A.2.3.1}$$

where $a_h \in \mathscr{A}_s(\rho(\theta), a, b)$ and $\delta = z\frac{d}{dz}$, we write

$$A_{sp}(P) = \sum_{h=0}^{n} A_{sp}(a_h)\, \delta^h. \tag{A.2.3.2}$$

Here $\rho(\theta)$ is a positive-valued continuous function on the interval $a < \theta < b$. The operator P is a **C**-linear map : $\mathscr{A}_s(\rho(\theta), a, b) \to \mathscr{A}_s(\rho(\theta), a, b)$ and the operator $A_{sp}(P)$ is a **C**-linear map : $\mathbf{C}[[z^{-1}]]_s \to \mathbf{C}[[z^{-1}]]_s$. We define the Newton polygon $\mathscr{N}(P)$ of the operator P by

$$\mathscr{N}(P) = \mathscr{N}(A_{sp}(P)), \tag{A.2.3.3}$$

where $\mathscr{N}(A_{sp}(P))$ is the Newton polygon of $A_{sp}(P)$ (cf. Remark A.1.4.2 of §A.1.4). If a given operator P satisfies the condition $A_{sp}(a_n) \neq 0$, there exists a nonnegative integer $M(P)$ such that $\mathscr{N}(P + z^{-M}R) = \mathscr{N}(P)$ for any integer M not less than $M(P)$ and any linear differential operator $R = \sum_{0 \leq h \leq n} b_h(z)\, \delta^h \quad (b_h \in \mathscr{A}_s(\rho(\theta), a, b))$.

In this section we shall explain the existence and uniqueness of Gevrey asymptotic solutions of a differential equation

$$P[y] = z^{-M}\, G(z, y, \delta y, \cdots, \delta^{n-1} y), \tag{A.2.3.4}$$

where

$$\begin{cases} P \;=\; \sum_{h=0}^{n} a_h(z)\,\delta^h, \\ G(z,y_0,y_1,\cdots,y_{n-1}) = G_0(z) + \sum_{|\vec{p}|\geq 1} G_{\vec{p}}(z)\,\vec{y}^{\vec{p}} \end{cases} \tag{A.2.3.5}$$

and $\vec{p} = (p_0,\cdots,p_{n-1}) \in \mathbf{N}^n$, $\quad |\vec{p}| = \sum_{0\leq h\leq n-1} p_h$, $\vec{y} = (y_0,y_1,\cdots,y_{n-1})$.

ASSUMPTION A.2.3.1.

(i) $a_h \in \mathscr{A}_s(\rho(\theta),a,b)$, $G_0 \in \mathscr{A}_s(\rho(\theta),a,b)$, *and* $G_{\vec{p}} \in \mathscr{A}_s(\rho(\theta),a,b)$,

(ii) $A_{sp}(a_n) \neq 0$,

(iii) *for any* $W = \mathscr{D}(R,\alpha,\beta)$ *satisfying the condition*

$$a < \alpha < \beta < b, \qquad \max_{\alpha\leq\theta\leq\beta} \rho(\theta) \;<\; R, \tag{A.2.3.6}$$

there exist two positive numbers A_W *and* r_W *such that the power series*

$$\sum_{|\vec{p}|\geq 1} ||G_{\vec{p}}||_{s,A_W,W}\;\vec{y}^{\vec{p}}$$

is convergent uniformly and absolutely for $|y_j| < r_W$ $(j = 0,1,\cdots,n)$,

(iv) *the integer* M *is not less than* $M(P)$ (*i.e.*, $M \geq M(P)$),

(v) *the differential equation*

$$A_{sp}(P)[y] = z^{-M}\left\{A_{sp}(G_0) + \sum_{|\vec{p}|\geq 1} A_{sp}(G_{\vec{p}})\; y^{p_0}\,(\delta y)^{p_1}\cdots(\delta^{n-1}y)^{p_{n-1}}\right\} \tag{A.2.3.7}$$

admits a solution

$$f(z) = \sum_{1\leq m<+\infty} c_m\, z^{-m} \;\in\; z^{-1}\mathbf{C}[[z^{-1}]]_s. \tag{A.2.3.8}$$

Under this assumption we can prove the following theorem.

THEOREM A.2.3.2. *For any given direction* $\arg z = \theta$ *such that* $a < \theta < b$ *there exist a sectorial domain* $W = \mathscr{D}(R,\alpha,\beta)$ *and a function* $\phi(z) \in \mathscr{A}_s(R,\alpha,\beta)$ *such that*

(i) W *satisfies condition* (A.2.3.6),

(ii) $\alpha < \theta < \beta$,

(iii) $A_{sp}(\phi) = f$,

(iv) *the function* ϕ *satisfies differential equation* (A.2.3.4) *on* W.

SKETCH OF PROOF. Since the case $s = 0$ is a well-known case, we assume that $s > 0$. Set

$$\psi = \sum_{m \geq 1} \frac{c_m}{\Gamma(1+sm)} (e^{i\theta} z)^{-m} \quad \text{and} \quad \phi_0(e^{i\theta} z) = \ell_{r,\frac{1}{s}}(\psi).$$

Then

$$\psi \in \mathbf{C}\{z^{-1}\}, \qquad \phi_0 \in \mathscr{A}_s(R, \theta - \frac{s\pi}{2}, \theta + \frac{s\pi}{2}), \qquad A_{sp}(\phi_0) = f \quad \text{(A.2.3.9)}$$

for any positive number R (cf. Property A.2.2.10 of §A.2.2). Here we choose r smaller than the radius of convergence of ψ.

Change the unknown y by

$$y = u + \phi_0(z). \quad \text{(A.2.3.10)}$$

Then differential equation (A.2.3.4) becomes

$$P[u] = z^{-M} G(z, u + \phi_0, \delta[u + \phi_0], \cdots, \delta^{n-1}[u + \phi_0]) - P[\phi_0].$$

Set

$$\begin{cases} F_0(z) & = z^{-M} G(z, \phi_0, \delta[\phi_0], \cdots, \delta^{n-1}[\phi_0]) - P[\phi_0], \\ F(z, \vec{u}) & = G(z, u_0 + \phi_0, u_1 + \delta[\phi_0], \cdots, u_n + \delta^{n-1}[\phi_0]) \\ & \quad - G(z, \phi_0, \delta[\phi_0], \cdots, \delta^{n-1}[\phi_0]) \end{cases} \quad \text{(A.2.3.11)}$$

Then differential equation (A.2.3.4) can be written in the form

$$P[u] = F_0(z) + z^{-M} F(z, u, \delta u, \cdots, \delta^{n-1} u). \quad \text{(A.2.3.12)}$$

Note that

$$\begin{aligned} & A_{sp}(F_0) \\ & = z^{-M} \left\{ A_{sp}(G_0) + \sum_{|\vec{p}| \geq 1} A_{sp}(G_{\vec{p}}) \, (A_{sp}(\phi_0))^{p_0} \cdots (\delta^{n-1}[A_{sp}(\phi_0)])^{p_{n-1}} \right\} \\ & \quad - A_{sp}(P)[A_{sp}(\phi_0)] \\ & = z^{-M} \left\{ A_{sp}(G_0) + \sum_{|\vec{p}| \geq 1} A_{sp}(G_{\vec{p}}) \, f^{p_0} \cdots (\delta^{n-1}[f])^{p_{n-1}} \right\} \\ & \quad - A_{sp}(P)[f] \\ & = 0. \end{aligned}$$

Hence $F_0(z)$ belongs to $\mathscr{A}_{s,0}(R, a, b)$ for every $W = \mathscr{D}(R, \alpha, \beta)$ satisfying conditions (A.2.3.6) and

$$\theta - \frac{s\pi}{2} < \alpha < \beta < \theta + \frac{s\pi}{2}. \quad \text{(A.2.3.13)}$$

This is an application of Theorem A.2.2.6. Therefore, there exist a non-negative number $\mu(R,\alpha,\beta)$ and a positive number $\lambda(R,\alpha,\beta)$ such that

$$|F_0(z)| \leq \mu(R,\alpha,\beta)\exp\{-\lambda(R,\alpha,\beta)\,|z|^{\frac{1}{s}}\} \tag{A.2.3.14}$$

on a sectorial domain $W = \mathscr{D}(R,\alpha,\beta)$ if W satisfies conditions (A.2.3.6) and (A.2.3.13) (cf. Theorem A.2.0.4).

Note also that $F(z,\vec{u})$ is holomorphic and bounded in $(z,\vec{u})$ on any domain

$$z \in W = \mathscr{D}(R,\alpha,\beta), \qquad |u_j| < \hat{r}_W \qquad (j = 0, 1, \cdots, n-1) \tag{A.2.3.15}$$

if W satisfies conditions (A.2.3.6) and (A.2.3.13) and $\hat{r}_W$ is a sufficiently small positive number. Furthermore

$$F(z,\vec{0}) = 0. \tag{A.2.3.16}$$

Now we can complete the proof of Theorem A.2.3.2 by means of the lemma below.

LEMMA A.2.3.3 (J.-P. Ramis–Y. Sibuya [129]). *There exist a sectorial domain* $W = \mathscr{D}(R,\alpha,\beta)$ *and a function* $\phi_1(z) \in \mathscr{A}_{s,0}(R,\alpha,\beta)$ *such that*

(i) *W satisfies conditions* (A.2.3.6) *and* (A.2.3.13),
(ii) $\alpha < \theta < \beta$,
(iii) *the function ϕ_1 satisfies differential equation* (A.2.3.12) *on W.*

In fact, if we set

$$\phi(z) = \phi_0(z) + \phi_1(z),$$

on W of Lemma A.2.3.3, the sectorial domain W and the function ϕ satisfy all of the requirements given in Theorem A.2.3.2. □

REMARK A.2.3.4.

(1) The hardest part of the proof of Lemma A.2.3.3 is the choice of the sectorial domain W. The reader will find a profound analysis concerning such a sectorial domain W in M. Hukuhara [67] (cf. also M. Iwano [70] and Ramis–Sibuya [129]). In Ramis–Sibuya [129], Lemma A.2.3.3 was reduced to the problem of finding a bounded solution by means of the Phragmén-Lindelöf theorem.

(2) If $G(z,\vec{y})$ of differential equation (A.2.3.4) is holomorphic and bounded in $(z,\vec{y})$ on a domain

$$|z| > R_0, \qquad |y_j| < r_0 \qquad (j = 0, 1, \cdots, n-1),$$

then G satisfies condition (iii) of Assumption A.2.3.1 for any non-negative number s.

(3) If $H(z, y_0, \cdots, y_n)$ is a polynomial in $(y_0, \cdots, y_n)$ with coefficients in $\mathscr{A}_s(R_0, a, b)$, the differential equation

$$P[y] = z^{-\hat{M}}\ H(z, y, \delta y, \cdots, \delta^{n-1} y, \delta^n y) \qquad \text{(A.2.3.17)}$$

can be changed to differential equation (A.2.3.4) by means of the implicit function Theorem A.2.2.9 if $A_{sp}(a_n) \neq 0$ and the integer $\hat{M}$ is sufficiently large. Furthermore G satisfies condition (iii) of Assumption A.2.3.4. Hence Theorem A.2.3.2 applies to (A.2.3.17) if $A_{sp}(a_n) \neq 0$ and $\hat{M}$ is sufficiently large.

(4) Let

$$F(z, y_0, y_1, \ldots, y_n) = \sum_{0 \leq |\vec{p}| \leq N} F_{\vec{p}}(z)\ \vec{y}^{\vec{p}}$$

be a polynomial in $y_0, \cdots, y_n$ with coefficients $F_{\vec{p}}(z)$ in $\mathscr{A}_s(R_0, a, b)$. Set

$$\hat{F}(z, \vec{y}) = \sum_{0 \leq |\vec{p}| \leq N} A_{sp}(F_{\vec{p}})\ \vec{y}^{\vec{p}}.$$

Suppose that the differential equation

$$\hat{F}(z, v, \delta v, \cdots, \delta^n v) = 0$$

has a solution $f = \sum_{m \geq 0} c_m z^{-m} \in \mathbf{C}[[z^{-1}]]_s$ such that

$$\frac{\partial \hat{F}}{\partial y_n}(z, f, \delta f, \cdots, \delta^n f) \neq 0.$$

Then by means of a transformation $v = \sum_{0 \leq m \leq q} c_m\ z^{-m} + z^{-r}\ y$, we can change the differential equation

$$F(z, v, \delta v, \cdots, \delta^n v) = 0 \qquad \text{(A.2.3.18)}$$

to differential equation (A.2.3.17). Furthermore we can make $\hat{M}$ of (A.2.3.17) as large as we want by choosing q and r in a suitable way. The operator P of (A.2.3.17) depends on the choice of q and r. However the shape of the Newton polygon $\mathscr{N}(P)$ becomes independent of q and r if q and r are sufficiently large (cf. Y. Sibuya–S. Sperber [141, 142] and K. Mahler [92, 93]).

(5) In condition (v) of Assumption A.2.3.1, we required that f should be in $z^{-1}\mathbf{C}[[z^{-1}]]_s$. In the Poincaré asymptotics, f is supposed to be in $z^{-1}\mathbf{C}[[z^{-1}]]$. To understand the relation between asymptotic solutions in the Poincaré sense and Gevrey asymptotic solutions, we must study the Maillet Theorem A.2.0.1 more carefully (cf. Theorem A.2.4.2).

Let $\phi_1(z)$ and $\phi_2(z)$ be two solutions of differential equation (A.2.3.4) satisfying all of the requirements of Theorem A.2.3.2. Set

$$\psi(z) = \phi_1(z) - \phi_2(z)$$

on $W = \mathscr{D}(R, \alpha, \beta)$. Then

$$\begin{aligned}
&P[\psi] \\
&= z^{-M}\{G(z, \phi_2 + \psi, \delta[\phi_2 + \psi], \cdots, \delta^{n-1}[\phi_2 + \psi]) \\
&\quad - G(z, \phi_2, \delta[\phi_2], \cdots, \delta^{n-1}[\phi_2])\} \\
&= z^{-M} \sum_{0 \le h \le n-1} \left[\int_0^1 \frac{\partial G}{\partial y_h}(z, \cdots, \delta_h[\phi_2 + t\psi], \cdots) dt \right] \delta^h \psi \\
&= z^{-M} \sum_{0 \le h \le n-1} \left[\int_0^1 \frac{\partial G}{\partial y_h}(z, \cdots, \delta^h[t\phi_1 + (1-t)\phi_2], \cdots) dt \right] \delta^h \psi.
\end{aligned}$$

If we put

$$R = \sum_{0 \le h \le n-1} \left[\int_0^1 \frac{\partial G}{\partial y_h}(z, \cdots, \delta^h[t\phi_1 + (1-t)\phi_2], \cdots) dt \right] \delta^h,$$

then

$$\begin{cases} (P - z^{-M}R)[\psi] = 0, \\ \mathscr{N}(P - z^{-M}R) = \mathscr{N}(P), \\ \psi \in \mathscr{A}_{s,0}(R, \alpha, \beta). \end{cases} \tag{A.2.3.19}$$

Let

$$0 \le k_1 < k_2 < \cdots < k_p < +\infty \tag{A.2.3.20}$$

be all of the distinct slopes of the Newton polygon $\mathscr{N}(P)$ and let

$$\{ (X, Y_j + k_j(X - X_j)) \; ; \; X_j \le X \le X_{j+1} \}$$

be the side of $\mathscr{N}(P)$ whose slope is k_j, where $Y_{j+1} = Y_j + k_j(X_{j+1} - X_j)$. Note that $X_1 = 0$, $X_{p+1} = N$, and $X_{j+1} - X_j \in \mathbf{Z}^+$. Set $n_j = X_{j+1} - X_j$ $(j = 1, 2, \cdots, p)$. Then, as it was mentioned in Remark A.1.4.2, the differential equation $A_{sp}(P - z^{-M}R)[y] = 0$ admits n_j linearly independent solutions of the form

$$y_{j,h}(z) = z^{\mu_{j,h}} \exp[\lambda_{j,h} z^{k_j}\{1 + o(1)\}] f_{j,h}(z) \qquad (h = 1, \cdots, n_j), \tag{A.2.3.21}$$

where the $\mu_{j,h}$ are constants, the $\lambda_{j,h}$ are nonzero constants and $f_{j,h} \in \mathbf{C}[[z^{-\frac{1}{\sigma}}]][z^{\frac{1}{\sigma}}, \log z]$ for some positive integer σ. On the other hand, there exist a nonnegative number $\mu(R, \alpha, \beta)$ and a positive number $\lambda(R, \alpha, \beta)$ such that

$$|\psi(z)| \le \mu(R, \alpha, \beta) \exp\{-\lambda(R, \alpha, \beta) \, |z|^{\frac{1}{s}}\}$$

for $z \in W = \mathscr{D}(R, \alpha, \beta)$ (cf. Theorem A.2.0.4).

Let ℓ be the total number of pairs (j, h) such that

$$k_j \geq \frac{1}{s}, \qquad \Re(\lambda_{j,h}\, z^{k_j}) < 0 \quad \text{on} \quad W. \tag{A.2.3.22}$$

Now concerning the uniqueness of Gevrey asymptotic solutions, we can state the theorem below.

THEOREM A.2.3.5 (Ramis–Sibuya [129]). *The general solution ϕ of differential equation* (A.2.3.4) *that satisfies all of the requirements of Theorem* A.2.3.2 *on a sectorial domain $W = \mathscr{D}(R, \alpha, \beta)$ contains ℓ arbitrary constants.*

§A.2.4. Applications of Lemma A.2.0.5

In this section, we shall explain a Gevrey-asymptotics version of Theorem 6.4.1 (or Theorem 6.9.7) and a result concerning the Gevrey properties of formal solutions which is more precise than the Maillet Theorem A.2.0.1. In both cases, we shall utilize Lemma A.2.0.5.

THEOREM A.2.4.1. *Let s be a positive number. Assume that N sectorial domains*

$$\mathscr{S}_\ell = \mathscr{D}(R, a_\ell, b_\ell) \qquad (\ell = 1, \cdots, N) \tag{A.2.4.1}$$

form a covering at $z = \infty$ and that $n \times n$ matrices $F_{k\ell}(z) \in gl(n, \mathscr{A}_s(\mathscr{S}_k \cap \mathscr{S}_\ell))$ are defined for all (k, ℓ) satisfying the condition

$$\mathscr{S}_k \cap \mathscr{S}_\ell \neq \varnothing. \tag{A.2.4.2}$$

Also assume that, if

$$\mathscr{S}_h \cap \mathscr{S}_k \cap \mathscr{S}_\ell \neq \varnothing, \tag{A.2.4.3}$$

the relation

$$F_{hk}(z) F_{k\ell}(z) = F_{h\ell}(z) \qquad (z \in \mathscr{S}_h \cap \mathscr{S}_k \cap \mathscr{S}_\ell) \tag{A.2.4.4}$$

holds and that

$$A_{sp}(F_{k\ell}) = I, \tag{A.2.4.5}$$

where I is the $n \times n$ identity matrix. Then there exist a positive number R' ($\geq R$) and $n \times n$ matrices $Q_\ell(z)$ ($\ell = 1, \cdots, N$) such that

(i) *$Q_\ell(z)$ and $Q_\ell(z)^{-1}$ belong to $gl(n, \mathscr{A}_s(R', a_\ell, b_\ell))$,*

(ii) *$\lim_{z \to \infty} Q_\ell(z) = I$,*

(iii) *the relation*

$$F_{k\ell}(z) = Q_k(z)^{-1} Q_\ell(z) \qquad (z \in \mathscr{S}'_k \cap \mathscr{S}'_\ell) \tag{A.2.4.6}$$

holds whenever condition (A.2.4.2) *is satisfied, where $\mathscr{S}'_\ell = \mathscr{D}(R', a_\ell, b_\ell)$.*

SKETCH OF PROOF. We may assume that $\{\mathscr{S}_1, \mathscr{S}_2, \cdots, \mathscr{S}_N\}$ is a good covering at $z = \infty$. Condition (A.2.4.5) implies that there exist a nonnegative number γ and a positive number λ such that the inequalities

$$\begin{cases} |F_{\ell\ \ell+1}(z) - I| \leq \gamma \exp\left\{-\lambda\, |z|^{\frac{1}{s}}\right\}, \\ |F_{\ell+1\ \ell}(z) - I| \leq \gamma \exp\left\{-\lambda\, |z|^{\frac{1}{s}}\right\} \end{cases} \tag{A.2.4.7}$$

hold on $\mathscr{S}_\ell \cap \mathscr{S}_{\ell+1}$ respectively.

By virtue of Theorem 6.4.1, there exist a positive number R' ($\geq R$) and $n \times n$ matrices $Q_\ell(z) \in gl(n, \mathscr{A}(R', a_\ell, b_\ell))$ ($\ell = 1, \cdots, N$) such that

(1) $Q_\ell(z)^{-1} \in gl(n, \mathscr{A}(R', a_\ell, b_\ell))$,
(2) $\lim_{z\to\infty} Q_\ell(z) = I$,
(3) the relations

$$\begin{cases} F_{\ell+1\ \ell}(z) = Q_{\ell+1}(z)^{-1} Q_\ell(z), \\ F_{\ell\ \ell+1}(z) = Q_\ell(z)^{-1} Q_{\ell+1}(z) \end{cases}$$

hold on $\mathscr{S}'_\ell \cap \mathscr{S}'_{\ell+1}$, where $\mathscr{S}'_\ell = \mathscr{D}(R', a_\ell, b_\ell)$.

Observe that

$$\begin{cases} Q_\ell(z) - Q_{\ell+1}(z) = Q_\ell(z)\ \{\ I - F_{\ell\ \ell+1}(z)\ \}, \\ Q_\ell(z)^{-1} - Q_{\ell+1}(z)^{-1} = \{\ I - F_{\ell+1\ \ell}(z)\ \}\ Q_\ell(z)^{-1}. \end{cases} \tag{A.2.4.8}$$

Hence

$$\begin{cases} |Q_\ell(z) - Q_{\ell+1}(z)| \leq |Q_\ell(z)|\ \gamma \exp\left\{-\lambda\, |z|^{\frac{1}{s}}\right\}, \\ |Q_\ell(z)^{-1} - Q_{\ell+1}(z)^{-1}| \leq |Q_\ell(z)^{-1}|\ \gamma \exp\left\{-\lambda\, |z|^{\frac{1}{s}}\right\} \end{cases} \tag{A.2.4.9}$$

on $\mathscr{S}'_\ell \cap \mathscr{S}'_{\ell+1}$. Therefore by utilizing Lemma A.2.0.5, we conclude that $Q_\ell(z)$ and $Q_\ell(z)^{-1}$ belong to $gl(n, \mathscr{A}_s(R', a_\ell, b_\ell))$. □

[This proof is due to B. Malgrange.]

Consider a differential equation of form (A.2.3.17)

$$P[y] = z^{-M}\ H(z, y, \delta y, \cdots, \delta^{n-1} y, \delta^n y), \tag{A.2.4.10}$$

where

$$\begin{cases} P = \sum_{h=0}^{n} a_h(z)\, \delta^h \qquad (a_h \in \mathbf{C}\{z^{-1}\}), \\ a_h \neq 0, \\ H(z, y_0, y_1, \cdots, y_{n-1}, y_n) \in \mathbf{C}\{z^{-1}\}[y_0, y_1, \cdots, y_{n-1}, y_n], \\ M \in \mathbf{N}. \end{cases} \tag{A.2.4.11}$$

Let

$$0 < k_1 < k_2 < \cdots < k_p < +\infty \tag{A.2.4.12}$$

be all of the positive slopes of the Newton polygon $\mathcal{N}(P)$. As in §A.2.3, we denote by $M(P)$ the nonnegative integer such that $\mathcal{N}(P+z^{-M}R) = \mathcal{N}(P)$ for any integer M not less than $M(P)$ and any linear differential operator $R = \sum_{0\leq h\leq n} b_h(z)\,\delta^h \quad (b_h \in \mathbf{C}[[z^{-1}]])$.

THEOREM A.2.4.2. *If* $f \in \mathbf{C}[[z^{-1}]]$ *satisfies differential equation* (A.2.4.10) *and if* $M \geq M(P)$, *then*

$$f \in \mathbf{C}[[z^{-1}]]_s, \tag{A.2.4.13}$$

where

$$s \in \left\{0, \frac{1}{k_1}, \frac{1}{k_2}, \cdots, \frac{1}{k_p}\right\}. \tag{A.2.3.14}$$

REMARK A.2.4.3.

(1) Suppose that a formal power series $f = \sum_{m\geq 0} c_m\, z^{-m} \in \mathbf{C}[[z^{-1}]]$ satisfies the differential equation

$$F(z, v, \delta v, \cdots, \delta^n v) = 0, \tag{A.2.4.15}$$

where

$$F(z, y_0, y_1, \ldots, y_n) \in \mathbf{C}\{z^{-1}\}[y_0, y_1, \ldots, y_n].$$

In such a case, if

$$\frac{\partial F}{\partial y_n}(z, f, \delta f, \cdots, \delta^n f) \neq 0, \tag{A.2.4.16}$$

we can change differential equation (A.2.4.15) to differential equation (A.2.4.10) by means of a transformation $v = \sum_{0\leq m\leq q} c_m\, z^{-m} + z^{-r}\, y$. Furthermore we can make M of (A.2.4.10) as large as we want by choosing q and r in a suitable way. Also if we set

$$P_0 = \sum_{h=0}^{n} \left(\frac{\partial F}{\partial y_h}(z, f, \delta f, \cdots, \delta^n f)\right)\, \delta^h, \tag{A.2.4.17}$$

then the Newton polygon $\mathcal{N}(P)$ of the operator P of differential equation (A.2.4.10) and $\mathcal{N}(P_0)$ have the same shape if q and r are sufficiently large (cf. (4) of Remark A.2.3.4). If condition (A.2.4.16) is not satisfied, we can replace F by another F satisfying (A.2.4.16) (cf. Y. Sibuya–S. Sperber [141, 142], K. Mahler [92, 93], and B. Malgrange [103]). Therefore, Theorem A.2.4.2 is a refinement of the Maillet Theorem A.2.0.1.

(2) J.-P. Ramis [123] proved Theorem A.2.4.2 in the case when differential equation (A.2.4.10) is linear. See, also, H. Gingold [60].
(3) B. Malgrange [103] showed that $s \leq \frac{1}{k_1}$. [For the general case, see Y. Sibuya [140].]
(4) Conclusion (A.2.4.14) of Theorem A.2.4.2 means that if $s < \frac{1}{k_j}$, then $s \leq \frac{1}{k_{j+1}}$.
(5) In the case when $a_h \in \mathbf{C}[[z^{-1}]]_s$ and $H \in \mathbf{C}[[z^{-1}]]_s[y_0, \cdots, y_n]$, set

$$\sigma_j = \max\left\{s, \frac{1}{k_j}\right\} \qquad (j = 1, \cdots, p)$$

then we can conclude that

$$f \in \mathbf{C}[[z^{-1}]]_\sigma,$$

where

$$0 \leq \sigma \leq s \quad \text{or} \quad \sigma \in \{0, \sigma_1, \cdots, \sigma_p\}$$

(cf. Y. Sibuya [140]; see also Theorem A.2.1.9).

Sketch of Proof of Theorem A.2.4.2. Utilizing the Poincaré asymptotics, we can find

(a) a good covering $\{\mathcal{S}_1, \cdots, \mathcal{S}_p\}$ at $z = \infty$,
(b) p functions $\phi_1(z), \cdots, \phi_p(z)$ such that

$$\begin{cases} \phi_j \in \mathscr{A}(\mathcal{S}_j), \\ A_{sp}(\phi_j) = f, \\ P[\phi_j] = z^{-M}\, H(z, \phi_j, \delta\phi_j, \cdots, \delta^n\phi_j) \qquad \text{on } \mathcal{S}_j, \end{cases} \qquad (j = 1, \cdots, p).$$

Set

$$u_j = \phi_j - \phi_{j+1} \qquad \text{on} \qquad \mathcal{S}_j \cap \mathcal{S}_{j+1}.$$

Then

$$\begin{cases} u_j \in \mathscr{A}_{+\infty,0}(\mathcal{S}_j \cap \mathcal{S}_{j+1}) \quad (\text{i.e., } A_{sp}(u_j) = 0), \\ (P - z^{-M}R)[u_j] = 0 \qquad \text{on} \qquad \mathcal{S}_j \cap \mathcal{S}_{j+1}, \end{cases} \qquad (j = 1, \cdots, p),$$

where

$$R = \sum_{0 \leq h \leq n} \left[\int_0^1 \frac{\partial H}{\partial y_h}(z, \cdots, \delta^h[t\phi_j + (1-t)\phi_{j+1}], \cdots)dt\right] \delta^h.$$

(cf. §A.2.3). Since $\mathcal{N}(P - z^{-M}R) = \mathcal{N}(P)$, we must have

$$|u_j(z)| \leq \gamma \exp\left\{-\lambda\, |z|^k\right\} \qquad \text{on} \qquad \mathcal{S}_j \cap \mathcal{S}_{j+1}$$

for a nonnegative number γ and a positive number λ, where

$$k \in \{k_1, \cdots, k_p\}.$$

Now, by utilizing Lemma A.2.0.5 we conclude that

$$\phi_j \in \mathscr{A}_s(\mathscr{S}_j) \qquad (j = 1, \cdots, p),$$

where

$$s \in \left\{0, \frac{1}{k_1}, \cdots, \frac{1}{k_p}\right\}.$$

Hence

$$f = A_{sp}(\phi) \in \mathbf{C}[[z^{-1}]]_s. \quad \square$$

REMARK A.2.4.4. The proof of Theorem A.2.4.2 also showed that every asymptotic solution of differential equation (A.2.4.10) in the Poincaré sense is a Gevrey asymptotic solution.
[The sketch of the proof given above involves some ambiguities. Such ambiguities will be removed by choosing the covering $\{\mathscr{S}_1, \cdots, \mathscr{S}_N\}$ prudently.]

§A.2.5. Phragmén–Lindelöf theorem in a cohomological form

Lemma A.2.0.5 is a corollary of the theorem below.

THEOREM A.2.5.1. *Let $\{\mathscr{S}_1, \mathscr{S}_2, \cdots, \mathscr{S}_p\}$ be a good covering at $z = \infty$. Assume that p functions $\delta_1(z), \delta_2(z), \cdots, \delta_p(z)$ satisfy the following conditions*:

(i) $\delta_j(z) \in \mathscr{O}(\mathscr{S}_j \cap \mathscr{S}_{j+1})$,
(ii) $|\delta_j(z)| \leq \gamma \exp\{-\lambda\, |z|^k\}$ *on* $\mathscr{S}_j \cap \mathscr{S}_{j+1}$, *where* $\gamma \geq 0$, $\lambda > 0$ *and* $k > 0$ *are suitable numbers.*

Then, there exist p functions $\psi_1(z), \psi_2(z), \cdots, \psi_p(z)$ such that

(a) $\psi_j(z) \in \mathscr{A}_{\frac{1}{k}}(\mathscr{S}_j)$.
(b) $\delta_j(z) = \psi_j(z) - \psi_{j+1}(z)$ *on* $\mathscr{S}_j \cap \mathscr{S}_{j+1}$.

The following is a sketch of the proof of Lemma A.2.0.5 by means of Theorem A.2.5.1.
Applying Theorem A.2.5.1 to N functions $\delta_\ell(z) = \phi_\ell(z) - \phi_{\ell+1}(z)$, we derive

$$\phi_\ell(z) - \phi_{\ell+1}(z) = \psi_\ell(z) - \psi_{\ell+1}(z) \qquad \text{on} \quad \mathscr{S}_\ell \cap \mathscr{S}_{\ell+1}.$$

Hence

$$\phi_\ell(z) - \psi_\ell(z) = \phi_{\ell+1}(z) - \psi_{\ell+1}(z) \qquad \text{on} \quad \mathscr{S}_\ell \cap \mathscr{S}_{\ell+1}.$$

Set

$$\phi(z) = \phi_\ell(z) - \psi_\ell(z) \qquad \text{on} \quad \mathscr{S}_\ell \qquad (\ell = 1, \cdots, N).$$

Then $\phi(z)$ is holomorphic and bounded on $\cup_{1\leq\ell\leq N}\mathscr{S}_\ell$. Therefore

$$\phi_\ell = \psi_\ell + \phi \in \mathscr{A}_{\frac{1}{k}}(\mathscr{S}_\ell) \qquad (\ell = 1, \cdots, N). \quad \square$$

In Appendix 3, we shall restate Theorem A.2.5.1 in terms of cohomologies (cf. Theorem A.3.0.4).

If we replace conditions (1) and (2) of Lemma A.2.0.5 by the condition

$$\phi_\ell \in \mathscr{A}_{+\infty,0}(\mathscr{S}_\ell) \qquad (\ell = 1,\ 2,\ \cdots,\ N),$$

then we conclude that

$$\phi_\ell \in \mathscr{A}_{\frac{1}{k},0}(\mathscr{S}_\ell) \qquad (\ell = 1,\ 2,\ \cdots,\ N).$$

Actually we can derive a more precise conclusion.

THEOREM A.2.5.2 (Y. Sibuya [136]). *Assume that a covering* $\{\ \mathscr{S}_\ell\ ;\ \ell = 1,\ \cdots,\ N\ \}$ *at* $z = \infty$ *is good and that N functions* $\phi_1(z),\ \cdots,\ \phi_N(z)$ *satisfy the conditions*

(1) $\phi_\ell(z) \in \mathscr{A}_{+\infty,0}(\mathscr{S}_\ell)$ *and bounded on* $\mathscr{S}_\ell$,

(2)

$$|\phi_\ell(z)\ -\ \phi_{\ell+1}(z)|\ \leq\ \gamma\ \exp\{\ -\ \lambda\ |z|^k\} \qquad on \quad \mathscr{S}_\ell \cap \mathscr{S}_{\ell+1},$$

where $\gamma \geq 0$, $\lambda > 0$ *and* $k > 0$ *are suitable numbers.*

Then there exists a nonnegative number μ *such that*

$$|\phi_\ell(z)| \leq\ \mu\ \exp\{-\lambda\ |z|^k\} \qquad on \quad \mathscr{S}_\ell \qquad (\ell = 1, \cdots, N). \qquad (A.2.5.1)$$

REMARK A.2.5.3.

(1) To understand the Matkowsky condition for the Ackerberg–O'Malley resonance (cf. Ackerberg-O'Malley [35], Y. Sibuya [136], B. J. Matkowsky [107], N. Kopell [84, 85], and C.-H. Lin [87]), we must estimate flat functions precisely. In particular, we must show an exponential decay of such functions. In such a study, it is important that λ of (A.2.5.1) and λ of condition (2) of Theorem A.2.5.2 are the same.

(2) The constant μ of (A.2.5.1) depends on λ. However for a sufficiently small λ we can choose μ independent of λ so that μ remains constant as $\lambda \to 0$ (cf. C.-H. Lin [86]).

So far $\{\mathscr{S}_1,\ \cdots,\ \mathscr{S}_N\}$ has been a covering at $z = \infty$. This means that $\cup_{1\leq\ell\leq N}\mathscr{S}_\ell = \{\ z;\ |z| > R_0\ \}$. We can prove a theorem similar to Theorem A.2.5.2 also in the case when $\cup_{1\leq\ell\leq N}\mathscr{S}_\ell$ is a sectorial domain.

ASSUMPTION A.2.5.4. *A set of sectorial domains* $\{\mathcal{S}_\ell = \mathscr{D}(R_0, a_\ell, b_\ell)\ ;\ \ell = 1, 2, \cdots, N\}$ *satisfies the following conditions*:

(i) $a_\ell < a_{\ell+1} \qquad (\ell = 1, \cdots, N-1)$,

(ii) $b_\ell - a_\ell < \pi \qquad (\ell = 1, \cdots, N)$,

(iii) $\mathcal{S}_\ell \cap \mathcal{S}_{\ell+1} \neq \emptyset \quad (\ell = 1, \cdots, N-1)$ *and* $\mathcal{S}_\ell \cap \mathcal{S}_k = \emptyset$ *otherwise if* $\ell \neq k$,

(iv) $-\pi < a_1 < b_N < \pi$.

Note that

$$\bigcup_{1 \le \ell \le N} \mathcal{S}_\ell = \mathscr{D}(R_0, a_1, b_N) \tag{A.2.5.2}$$

THEOREM A.2.5.5 (C.-H. Lin [86]). *Let* $\{\mathcal{S}_1, \cdots, \mathcal{S}_N\}$ *be a set of sectorial domains satisfying assumption* A.2.5.4. *Assume that N functions* $\phi_1(z), \cdots, \phi_N(z)$ *satisfy the following conditions*:

(i) $\phi_\ell(z) \in \mathscr{A}_{+\infty,0}(\mathcal{S}_\ell)$ *and bounded on* $\mathcal{S}_\ell$,

(ii) $\phi_1(z)$ *is continuous and*

$$|\phi_1(z)| \le K_m\, |z|^{-m} \qquad (m = 0, 1, 2, \cdots)$$

on a sectorial domain

$$|z| > R_0, \quad a_1 \le \arg z \le a_1 + \varepsilon, \tag{A.2.5.3}$$

where the K_m *are suitable nonnegative numbers and* ε *is a small positive number*,

(iii) $\phi_N(z)$ *is continuous and*

$$|\phi_N(z)| \le K_m\, |z|^{-m} \qquad (m = 0, 1, 2, \cdots)$$

on a sectorial domain

$$|z| > R_0, \quad b_N - \varepsilon \le \arg z \le b_N, \tag{A.2.5.4}$$

(iv)

$$|\phi_\ell(z) - \phi_{\ell+1}(z)| \le \gamma \exp\{-\lambda\, |z|^k\} \qquad on \quad \mathcal{S}_\ell \cap \mathcal{S}_{\ell+1},$$

where $\mu \ge 0$, $\lambda > 0$, *and* $k > 0$ *are suitable numbers*,

(v)

$$\begin{cases} |\phi_1(t\, e^{i\, a_1})| \le \gamma \exp\{-\lambda\, t^k\}, \\ |\phi_N(t\, e^{i\, b_N})| \le \gamma \exp\{-\lambda\, t^k\}, \end{cases} \qquad for \quad t \ge R_0. \tag{A.2.5.5}$$

Then there exists a nonnegative number μ *such that*

$$|\phi_\ell(z)| \le \mu \exp\{-\lambda\, |z|^k\} \qquad on \quad \mathcal{S}_\ell \qquad (\ell = 1, \cdots, N). \tag{A.2.5.6}$$

Furthermore, for a sufficiently small λ *we can choose* μ *independent of* λ *so that* μ *remains constant as* $\lambda \to 0$.

C.-H. Lin [86] proved the following theorem by using Theorem A.2.5.5.

Phragmén–Lindelöf Theorem in a cohomological form (C.-H. Lin). *Let $\{\mathcal{S}_1, \cdots, \mathcal{S}_N\}$ be a set of sectorial domains satisfying Assumption A.2.5.4. Assume that*

$$-\frac{\pi}{2} < a_1 < b_N < \frac{\pi}{2}.$$

Assume also that N functions $\phi_1(z), \cdots, \phi_N(z)$ satisfy the following conditions:

(i) $\phi_\ell(z) \in \mathcal{O}(\mathcal{S}_\ell)$ *and continuous on the closure of* $\mathcal{S}_\ell$,

(ii)

$$|\phi_\ell(z)| \leq K \exp\{c|z|\} \qquad on \quad \mathcal{S}_\ell$$

for positive numbers K and c,

(iii)

$$|\phi_\ell(z) - \phi_{\ell+1}(z)| \leq L_0 \qquad on \quad \mathcal{S}_\ell \cap \mathcal{S}_{\ell+1},$$

where L_0 is a positive number,

(iv)

$$\begin{cases} |\phi_1(t\, e^{i\, a_1})| \leq L_0, \\ |\phi_N(t\, e^{i\, b_N})| \leq L_0, \end{cases} \qquad for \quad t \geq R_0.$$

Then there exists a positive number L such that

$$|\phi_\ell(z)| \leq L \qquad on \quad \mathcal{S}_\ell \qquad (\ell = 1, \cdots, N).$$

APPENDIX 3

Cohomological Methods

§A.3.0. Preliminaries

In §6.11, we gave our main result concerning the holomorphic classification of differential equations within an equivalence class with respect to the formal equivalence (cf. Theorem 6.11.1). B. Malgrange [99, 101] explained such a classification in terms of sheaf-cohomologies, utilizing various sheaves over the unit-circle S^1 in the complex plane.

OBSERVATION A.3.0.1. For an open connected arc V on S^1, let

$$\mathcal{S}(\rho(\theta), V) = \{z\ ; \arg z \in V,\ |z| > \rho(\arg z)\},$$

where $\rho(\theta)$ is a positive-valued continuous function on the arc V. We set

$$\mathcal{A}_s(V) = \{f \in \mathcal{A}_s(\mathcal{S}(\rho(\theta), V)) \quad \text{for some } \rho(\theta) > 0\},$$

where $0 \leq s \leq +\infty$ and $\mathcal{A}_{+\infty}(\mathcal{S}) = \mathcal{A}(\mathcal{S})$. Similarly we set

$$\mathcal{A}_{s,0}(V) = \{f \in \mathcal{A}_{s,0}(\mathcal{S}(\rho(\theta), V)) \quad \text{for some } \rho(\theta) > 0\}.$$

Thus we define two kinds of sheaves $\mathcal{A}_s$ and $\mathcal{A}_{s,0}$ over S^1. We can regard these sheaves either as sheaves of commutative differential algebras over $\mathbf{C}$ or as sheaves of $\mathbf{C}$-linear spaces. For a point θ of S^1, we denote by $\mathcal{A}_s(\theta)$ and $\mathcal{A}_{s,0}(\theta)$ the stalks of the sheaves $\mathcal{A}_s$ and $\mathcal{A}_{s,0}$ at θ respectively.

OBSERVATION A.3.0.2. Utilizing the following homomorphisms of commutative differential algebras over $\mathbf{C}$:

$$\begin{cases} A_{sp} : \mathcal{A}_s(V) \to \mathbf{C}[[z^{-1}]]_s, \\ \mathrm{id}_{\mathcal{A}_{s,0}(V)} : \mathcal{A}_{s,0}(V) \to \mathcal{A}_s(V), \end{cases}$$

we can define exact sequences

$$0 \to \mathcal{A}_{s,0}(V) \to \mathcal{A}_s(V) \to \mathbf{C}[[z^{-1}]]_s \to 0 \tag{A.3.0.1}$$

if the size of the arc V is properly restricted, where $0 \leq s \leq +\infty$ and $\mathbf{C}[[z^{-1}]]_{+\infty} = \mathbf{C}[[z^{-1}]]$. In particular, we have exact sequences

$$0 \to \mathcal{A}_{s,0}(\theta) \to \mathcal{A}_s(\theta) \to \mathbf{C}[[z^{-1}]]_s \to 0 \tag{A.3.0.2}$$

for every point θ of S^1. Hence we have exact sequences of homomorphisms of sheaves

$$0 \to \mathscr{A}_{s,0} \to \mathscr{A}_s \to \mathbf{C}[[z^{-1}]]_s \to 0, \qquad \text{(A.3.0.3)}$$

where $\mathbf{C}[[z^{-1}]]_s$ are constant sheaves over S^1 with $\mathbf{C}[[z^{-1}]]_s(V) = \mathbf{C}[[z^{-1}]]_s$ for every open connected arc V of S^1 respectively.

Observation A.3.0.3. Since S^1 is a real one-dimensional manifold,

$$H^k(S^1, \mathscr{A}_{s,0}) = 0, \quad H^k(S^1, \mathscr{A}_s) = 0, \quad H^k(S^1, \mathbf{C}[[z^{-1}]]_s) = 0 \quad \text{for } k \geq 2.$$

Also we have

$$H^0(S^1, \mathscr{A}_{s,0}) = 0, \quad H^0(S^1, \mathscr{A}_s) = \mathbf{C}\{z^{-1}\}, \quad H^0(S^1, \mathbf{C}[[z^{-1}]]_s) = \mathbf{C}[[z^{-1}]]_s.$$

Thus we derive exact sequences

$$0 \to \mathbf{C}\{z^{-1}\} \to \mathbf{C}[[z^{-1}]]_s \to H^1(S^1, \mathscr{A}_{s,0}) \to H^1(S^1, \mathscr{A}_s) \to H^1(S^1, \mathbf{C}[[z^{-1}]]_s) \to 0 \qquad \text{(A.3.0.4)}$$

of $\mathbf{C}$-linear maps.

Now let us rephrase theorem A.2.5.1 as follows.

Theorem A.3.0.4. *For $0 \leq s \leq +\infty$, we have*

$$\mathrm{Image}(H^1(S^1, \mathscr{A}_{s,0}) \to H^1(S^1, \mathscr{A}_s)) = 0. \qquad \text{(A.3.0.5)}$$

[The case when $s = +\infty$ was explained in §6.5.] By virtue of Theorem A.3.0.4, we can derive from (A.3.0.4) exact sequences

$$0 \to \mathbf{C}\{z^{-1}\} \to \mathbf{C}[[z^{-1}]]_s \to H^1(S^1, \mathscr{A}_{s,0}) \to 0. \qquad \text{(A.3.0.6)}$$

Thus we arrive at the result below.

Theorem A.3.0.5 (B. Malgrange [99] and J.-P. Ramis [123]). *We have*

$$H^1(S^1, \mathscr{A}_{s,0}) \simeq \mathbf{C}[[z^{-1}]]_s / \mathbf{C}\{z^{-1}\} \qquad \text{(A.3.0.7)}$$

for $0 \leq s \leq +\infty$, where $\simeq$ means that the both sides of (A.3.0.7) *are isomorphic to each other as* $\mathbf{C}$*-linear spaces.*

§A.3.1. Irregularity

Let us consider a differential operator

$$P = \sum_{h=0}^{n} a_h(z)\, \delta^h \qquad (a_h \in \mathbf{C}\{z^{-1}\};\ a_n \neq 0).$$

We can regard P as a $\mathbf{C}$-linear map in various ways:

$$P\begin{cases} : \mathbf{C}[[z^{-1}]]_s \to \mathbf{C}[[z^{-1}]]_s, \\ : \mathscr{A}_s(V) \to \mathscr{A}_s(V), \\ : \mathscr{A}_{s,0}(V) \to \mathscr{A}_{s,0}(V). \end{cases} \qquad (0 \le s \le +\infty).$$

Setting

$$\mathscr{K}_s(V) = \mathrm{Kernel}(P : \mathscr{A}_{s,0}(V) \to \mathscr{A}_{s,0}(V)),$$

we define sheaves $\mathscr{K}_s$ over S^1 for $0 \le s \le +\infty$. Then, if the size of V is properly restricted, we have exact sequences

$$0 \to \mathscr{K}_s(V) \to \mathscr{A}_{s,0}(V) \to \mathscr{A}_{s,0}(V) \to 0 \qquad (0 \le s \le +\infty).$$

In particular we have exact sequences

$$0 \to \mathscr{K}_s(\theta) \to \mathscr{A}_{s,0}(\theta) \to \mathscr{A}_{s,0}(\theta) \to 0 \qquad (0 \le s \le +\infty)$$

at every point θ of S^1. Thus we arrive at exact sequences of homomorphisms of sheaves

$$0 \to \mathscr{K}_s \to \mathscr{A}_{s,0} \to \mathscr{A}_{s,0} \to 0 \qquad (0 \le s \le +\infty). \tag{A.3.1.1}$$

Now (A.3.1.1) implies that

$$0 \to H^1(S^1, \mathscr{K}_s) \to H^1(S^1, \mathscr{A}_{s,0}) \to H^1(S^1, \mathscr{A}_{s,0}) \to 0 \tag{A.3.1.2}$$

are exact. Utilizing Theorem A.3.0.5 we rewrite exact sequences (A.3.1.2) as

$$0 \to H^1(S^1, \mathscr{K}_s) \to \mathbf{C}[[z^{-1}]]_s/\mathbf{C}\{z^{-1}\} \to \mathbf{C}[[z^{-1}]]_s/\mathbf{C}\{z^{-1}\} \to 0. \tag{A.3.1.3}$$

We can summarize what we have explained above in the theorem below.

THEOREM A.3.1.1. *For* $0 \le s \le +\infty$, *we have*

$$H^1(S^1, \mathscr{K}_s) \simeq \mathrm{Kernel}\left(P : \mathbf{C}[[z^{-1}]]_s/\mathbf{C}\{z^{-1}\} \to \mathbf{C}[[z^{-1}]]_s/\mathbf{C}\{z^{-1}\}\right) \tag{A.3.1.4}$$

as $\mathbf{C}$*-linear spaces.*

Set

$$\ell_s = \dim_{\mathbf{C}} H^1(S^1, \mathscr{K}_s) \qquad (0 \le s \le +\infty). \tag{A.3.1.5}$$

THEOREM A.3.1.2 (J.-P. Ramis [123]). *Let*

$$0 < k_1 < k_2 < \cdots < k_p \tag{A.3.1.6}$$

be all of the positive slopes of the Newton polygon $\mathscr{N}(P)$. *Then* ℓ_s *satisfies the conditions below:*

(i) $0 \le \ell_s \le \ell_{+\infty} < +\infty \quad (0 \le s \le +\infty)$,

(ii) $\ell_{s_1} \leq \ell_{s_2}$ *if* $s_1 < s_2$,
(iii)

$$\ell_s = \begin{cases} 0 & if \ s < \dfrac{1}{k_p}, \\ \ell_{\frac{1}{k_{j+1}}} & if \ \dfrac{1}{k_{j+1}} \leq s < \dfrac{1}{k_j}, \\ \ell_{\frac{1}{k_1}} & if \ \dfrac{1}{k_1} \leq s \leq +\infty. \end{cases} \tag{A.3.1.7}$$

OBSERVATION A.3.1.3. Let $J : \mathbf{C}[[z^{-1}]]_s \to \mathbf{C}[[z^{-1}]]_s/\mathbf{C}\{z^{-1}\}$ be the natural $\mathbf{C}$-linear map such that $f \in J(f)$ for every $f \in \mathbf{C}[[Z^{-1}]]_s$. Then $f \in \mathbf{C}[[z^{-1}]]_s$ satisfies the condition:

$$J(f) \in \text{Kernel}(P : \mathbf{C}[[z^{-1}]]_s/\mathbf{C}\{z^{-1}\} \to \mathbf{C}[[z^{-1}]]_s/\mathbf{C}\{z^{-1}\}),$$

if and only if

$$P[f] \in \mathbf{C}\{z^{-1}\}. \tag{A.3.1.8}$$

Condition (A.3.1.8) means that f satisfies a linear differential equation $P[y] = \phi$ for some ϕ in $\mathbf{C}\{z^{-1}\}$. Note that the dimension ℓ_s of

$$\text{Kernel}(P : \mathbf{C}[[z^{-1}]]_s/\mathbf{C}\{z^{-1}\} \to \mathbf{C}[[z^{-1}]]_s/\mathbf{C}\{z^{-1}\})$$

is continuous in s except possibly for $s \in \{\frac{1}{k_1}, \cdots, \frac{1}{k_p}\}$. By utilizing this fact, J.-P. Ramis proved Theorem A.2.4.2 in the case when differential equation (A.2.4.10) is linear (cf. (2) of remark A.2.4.3).

OBSERVATION A.3.1.4. There are two ways to show that $\ell_{+\infty}$ is finite.

METHOD 1. Let us look at

$$\ell_{+\infty} = \dim_{\mathbf{C}} \text{Kernel}\left(P : \mathbf{C}[[z^{-1}]]/\mathbf{C}\{z^{-1}\} \to \mathbf{C}[[z^{-1}]]/\mathbf{C}\{z^{-1}\}\right).$$

To evaluate $\ell_{+\infty}$, we start with the commutative diagram below:

$$\begin{array}{ccccccccc} 0 \to & \mathbf{C}\{z^{-1}\} \to & \mathbf{C}[[z^{-1}]] \to & \mathbf{C}[[z^{-1}]]/\mathbf{C}\{z^{-1}\} \to & 0 \\ & \downarrow P & \downarrow P & \downarrow P & \\ 0 \to & \mathbf{C}\{z^{-1}\} \to & \mathbf{C}[[z^{-1}]] \to & \mathbf{C}[[z^{-1}]]/\mathbf{C}\{z^{-1}\} \to & 0 \end{array}.$$

Set

$$\begin{cases} K_1 = \text{Kernel}(P : \mathbf{C}\{z^{-1}\} \to \mathbf{C}\{z^{-1}\}), \\ K_2 = \text{Kernel}(P : \mathbf{C}[[z^{-1}]] \to \mathbf{C}[[z^{-1}]]), \\ K_3 = \text{Kernel}(P : \mathbf{C}[[z^{-1}]]/\mathbf{C}\{z^{-1}\} \to \mathbf{C}[[z^{-1}]]/\mathbf{C}\{z^{-1}\}), \\ CK_1 = \text{Cokernel}(P : \mathbf{C}\{z^{-1}\} \to \mathbf{C}\{z^{-1}\}), \\ CK_2 = \text{Cokernel}(P : \mathbf{C}[[z^{-1}]] \to \mathbf{C}[[z^{-1}]]), \\ CK_3 = \text{Cokernel}(P : \mathbf{C}[[z^{-1}]]/\mathbf{C}\{z^{-1}\} \to \mathbf{C}[[z^{-1}]]/\mathbf{C}\{z^{-1}\}), \end{cases}$$

Then

$$0 \to K_1 \to K_2 \to K_3 \to CK_1 \to CK_2 \to CK_3 \to 0$$

is exact. Note that K_1 and K_2 are finite-dimensional over $\mathbf{C}$ and that $CK_3 = 0$ (cf. (A.3.1.3)). Also it is not difficult to show that CK_2 is finite-dimensional over $\mathbf{C}$. Hence if CK_1 is finite-dimensional over $\mathbf{C}$, we can conclude that $\ell_{+\infty}$ is finite. To show that CK_1 is finite-dimensional over $\mathbf{C}$, we look at the diagram below:

$$\begin{array}{ccccccccc} 0 \to & z^{-N}\mathbf{C}\{z^{-1}\} \to & \mathbf{C}\{z^{-1}\} \to & \mathbf{C}\{z^{-1}\}/z^{-N}\mathbf{C}\{z^{-1}\} \to & 0 \\ & \downarrow P & \downarrow P & \downarrow P & \\ 0 \to & z^{-M}\mathbf{C}\{z^{-1}\} \to & \mathbf{C}\{z^{-1}\} \to & \mathbf{C}\{z^{-1}\}/z^{-M}\mathbf{C}\{z^{-1}\} \to & 0 \end{array},$$

where M and N are suitable positive integers such that $M \geq N$. It can be shown that there exists a nonnegative integer N_0 such that

$$\text{Kernel}(P : z^{-N}\mathbf{C}\{z^{-1}\} \to z^{-N-N_0}\mathbf{C}\{z^{-1}\}) = 0$$

for all sufficiently large integers N. By utilizing a method due to W. A. Harris–Y. Sibuya–L. Weinberg [63] we can show that, if a positive integer N is sufficiently large,

$$\dim_{\mathbf{C}} \text{Cokernel}(P : z^{-N}\mathbf{C}\{z^{-1}\} \to z^{-N-N_0}\mathbf{C}\{z^{-1}\})$$

is finite and can be evaluated explicitly. Hence CK_1 is finite-dimensional over $\mathbf{C}$.

METHOD 2. We look at

$$\ell_{+\infty} = \dim_{\mathbf{C}} H^1(S^1, \mathscr{K}_{+\infty}).$$

Set

$$\ell_{+\infty}(\theta) = \dim_{\mathbf{C}} \mathscr{A}_{+\infty,0}(\theta) \tag{A.3.1.9}$$

Then the function $\ell_{+\infty}(\theta)$ is piecewise continuous on S^1 with a finite number of jumps. The discontinuity of $\ell_{+\infty}(\theta)$ takes place only at those values of θ which define the so-called *Stokes lines*. By virtue of an idea due to P. Deligne [55] we can show that

$$\ell_{+\infty} = \frac{1}{2} \text{Var}(\theta; \ell_{+\infty}(\theta)), \tag{A.3.1.10}$$

where $\text{Var}(\theta; \ell_{+\infty}(\theta))$ denotes the total sum of the absolute values of jumps of $\ell_{+\infty}(\theta)$ over S^1 (cf. J.-P. Ramis [123], D. Bertrand [51], and D. G. Babbitt–V. S. Varadarajan [40]).

J.-P. Ramis [123] also gave the formulas

$$\ell_s = \frac{1}{2} \text{Var}(\theta; \ell_s(\theta)) \tag{A.3.1.11}$$

for $0 \leq s < +\infty$, where $\ell_s(\theta) = \dim_{\mathbf{C}} \mathscr{A}_{s,0}(\theta)$.

REMARK A.3.1.5.

(1) Method 2 utilizes the theory of asymptotic solutions, but Method 1 does not. Note that our proof of the refined theorem of Maillet (cf. Theorem A.2.4.2) of §A.2.4 utilizes the theory of asymptotic solutions. However the proofs given by E. Maillet, K. Mahler, H.Gingold and B. Malgrange (cf. Remark A.2.4.3) do not use such asymptotic analysis.
(2) The evaluation of ℓ_s amounts to the evaluation of indices of the operator P in various situations such as shown at the beginning of this section (cf. B. Malgrange [98]). Generally speaking such a study can be done by two methods, one of which is based on asymptotic analysis but the other is not. J.-P. Ramis treated the indices of P by means of asymptotic analysis in his paper [123] and without utilizing aysmptotic analysis in his paper [125]. Those indices are explicitly given by means of the Newton polygon $\mathcal{N}(P)$ of P.
(3) The main purpose of the work of W. A. Harris–Y. Sibuya–L. Weinberg [63] was to estimate the number of linearly independent convergent power-series solutions of a system of linear differential equations at $z = 0$ (cf. O. Perron [121], F. Lettenmeyer [90]; see also P. Hartman [66]). Their method does not involve asymptotic analysis. M. Hukuhara–M. Iwano [68] and M. Iwano [69] studied similar problems by utilizing asymptotic analysis. The idea of P. Deligne is quite similar to the idea of Hukuhara–Iwano.

DEFINITION A.3.1.6 (B. Malgrange [98]). The integer $\ell_{+\infty}$ is called the **irregularity** of the operator P at $z = \infty$.

REMARK A.3.1.7. The point at $z = \infty$ is a regular singular point of P if and only if $\ell_{+\infty} = 0$.

§A.3.2. Classification by means of cohomologies

To explain Malgrange's idea of classifying analytic linear differential equations by means of cohomologies, we shall make some observations.

OBSERVATION A.3.2.1. For a given open connected arc V of S^1, let us denote by $\mathcal{G}_s(V)$ the set of all $n \times n$ matrices F such that $F \in gl(n, \mathcal{A}_s(V))$ and $F^{-1} \in gl(n, \mathcal{A}_s(V))$. Also set $\mathcal{G}_{s,0}(V) = \{F \in \mathcal{G}_s(V);\ A_{sp}(F) = I_n\}$, where I_n is the $n \times n$ identity matrix. Let $\mathcal{G}_s$ and $\mathcal{G}_{s,0}$ be the sheaves over S^1 defined by $\mathcal{G}_s(V)$ and $\mathcal{G}_{s,0}(V)$, respectively. We denote by $\mathcal{G}_s(\theta)$ and $\mathcal{G}_{s,0}(\theta)$ the stalks of $\mathcal{G}_s$ and $\mathcal{G}_{s,0}$ at θ, respectively. If we use the following homomorphisms of noncommutative groups:

$$\begin{cases} A_{sp} : \mathcal{G}_s(V) & \to & GL(n, \mathbf{C}[[z^{-1}]]_s), \\ \mathrm{id}_{\mathcal{G}_{s,0}(V)} : \mathcal{G}_{s,0}(V) & \to & \mathcal{G}_s(V), \end{cases}$$

we can derive exact sequences

$$0 \to \mathcal{G}_{s,0}(V) \to \mathcal{G}_s(V) \to GL(n, \mathbf{C}[[z^{-1}]]_s) \to 0$$

for a sufficiently small arc V and

$$0 \to \mathcal{G}_{s,0}(\theta) \to \mathcal{G}_s(\theta) \to GL(n, \mathbf{C}[[z^{-1}]]_s) \to 0$$

for every $\theta \in S^1$. Hence we have exact sequences of homomorphisms of sheaves

$$0 \to \mathcal{G}_{s,0} \to \mathcal{G}_s \to GL(n, \mathbf{C}[[z^{-1}]]_s) \to 0$$

for $0 \leq s \leq +\infty$, where $GL(n, \mathbf{C}[[z^{-1}]]_s)$ are constant sheaves.

OBSERVATION A.3.2.2. Since S^1 is a real one-dimensional manifold,

$$\begin{cases} H^k(S^1, \mathcal{G}_{s,0}) = 0, \quad H^k(S^1, \mathcal{G}_s) = 0, \\ H^k(S^1, GL(n, \mathbf{C}[[z^{-1}]]_s)) = 0, \end{cases} \qquad (k \geq 2),$$

and also

$$\begin{cases} H^0(S^1, \mathcal{G}_{s,0}) = 0, \quad H^0(S^1, \mathcal{G}_s) = GL(n, \mathbf{C}\{z^{-1}\}), \\ H^0(S^1, GL(n, \mathbf{C}[[z^{-1}]]_s)) = GL(n, \mathbf{C}[[z^{-1}]]_s). \end{cases}$$

For each s ($0 \leq s \leq +\infty$), utilizing the results above, we can define a map

$$J_s : GL(n, \mathbf{C}[[z^{-1}]]_s)/GL(n, \mathbf{C}\{z^{-1}\}) \;\to\; H^1(S^1, \mathcal{G}_{s,0}) \qquad \text{(A.3.2.1)}$$

as follows: Let $\{V_1, \cdots, V_N\}$ be a covering of S^1 by open connected arcs of reasonable sizes. Given $F \in GL(n, \mathbf{C}[[z^{-1}]]_s)$, there exist $F_\ell \in \mathcal{G}_s(V_\ell)$ ($\ell = 1, \cdots, N$) such that $A_{sp}(F_\ell) = F$. Set $F_{\ell k} = F_\ell F_k^{-1}$ whenever $V_\ell \cap V_k \neq \emptyset$. Then $F_{\ell k} \in \mathcal{G}_{s,0}(V_\ell \cap V_k)$ and the assignment

$$F \;\longmapsto\; \{F_{\ell k};\; V_\ell \cap V_k \neq \emptyset\} \qquad \text{(A.3.2.2)}$$

induces map (A.3.2.1).

THEOREM A.3.2.3 (B. Malgrange [99] and J.-P. Ramis [123]). *The maps J_s are bijective.*

In particular, the surjectivity of the J_s follows from Theorem 6.4.1 (or Theorem 6.9.7) and Theorem A.2.4.1.

OBSERVATION A.3.2.4. Let us consider two differential equations

$$\frac{dW}{dz} = A_0(z)W \qquad \text{(A.3.2.3)}$$

and

$$\frac{dV}{dz} = A(z)V, \tag{A.3.2.4}$$

where W and V are $n \times n$ unknown matrices and $A_0(z)$ and $A(z) \in gl(n, \mathscr{L})$. Here $\mathscr{L}$ denotes $\mathbf{C}\{z^{-1}\}[z]$.

DEFINITION A.3.2.5. Two differential equations (A.3.2.3) and (A.3.2.4) are said to be **formally equivalent of Gevrey order** s if there exists an $n \times n$ matrix $\hat{P}(z) \in GL(n, \mathbf{C}[[z^{-1}]]_s)$ such that the transformation

$$W = \hat{P}(z)V \tag{A.3.2.5}$$

changes (A.3.2.3) to (A.3.2.4).

Since $A = \hat{P}^{-1}A_0\hat{P} - \hat{P}^{-1}\frac{d\hat{P}}{dz}$, let us look at

$$G_s[A_0] = \{\hat{P} \in GL(n, \mathbf{C}[[z^{-1}]]_s);\ \hat{P}[A_0] \in gl(n, \mathscr{L})\}, \tag{A.3.2.6}$$

where

$$\hat{P}[A_0] = \hat{P}^{-1}A_0\hat{P} - \hat{P}^{-1}\frac{d\hat{P}}{dz}. \tag{A.3.2.7}$$

Note that $G_s[A_0]$ is a subset of $GL(n, \mathbf{C}[[z^{-1}]]_s)$ such that

$$G_s[A_0]GL(n, \mathbf{C}\{z^{-1}\}) \subset G_s[A_0].$$

Hence the map J_s (cf. (A.3.2.1)) can be restricted to $G_s[A_0]/GL(n, \mathbf{C}\{z^{-1}\})$. We must investigate

$$\text{Image}(J_s : G_s[A_0]/GL(n, \mathbf{C}\{z^{-1}\}) \rightarrow H^1(S^1, \mathscr{G}_{s,0})).$$

Let $\{V_1, \cdots, V_N\}$ be a covering of S^1 by open connected arcs of reasonable sizes. Given $F \in G_s[A_0]$, we can choose $F_\ell \in \mathscr{G}_s(V_\ell)$ $(\ell = 1, \cdots, N)$ such that

$$\begin{cases} A_{sp}(F_\ell) = F, \\ F[A_0] = F_\ell^{-1}A_0F_\ell - F_\ell^{-1}\dfrac{dF_\ell}{dz}. \end{cases} \tag{A.3.2.8}$$

Set $F_{\ell k} = F_\ell F_k^{-1}$ whenever $V_\ell \cap V_k \neq \varnothing$. Then $F_{\ell k}$ belongs to $\mathscr{G}_{s,0}(V_\ell \cap V_k)$ and $X = F_{\ell k}$ satisfy the condition

$$A_0 = X^{-1}A_0X - X^{-1}\frac{dX}{dz}. \tag{A.3.2.9}$$

Note that condition (A.3.2.9) and (6.11.8) are the same. Note also that if X satisfies condition (A.3.2.9) and if $n \times n$ matrices P and Q satisfy the condition $XP = Q$, then

$$P^{-1}A_0P - P^{-1}\frac{dP}{dz} = Q^{-1}A_0Q - Q^{-1}\frac{dQ}{dz}.$$

For an open connected arc V of S^1 we set

$$\mathcal{G}_{s,0}[A_0](V) = \left\{F \in \mathcal{G}_{s,0}(V)\ ;\ A_0 = F^{-1}A_0F - F^{-1}\frac{dF}{dz}\right\} \quad \text{(A.3.2.10)}$$

and we define sheaves $\mathcal{G}_{s,0}[A_0]$ over S^1. Then B. Malgrange [99] and J.-P. Ramis [124] gave the theorem below.

THEOREM A.3.2.6. *For $0 \le s \le +\infty$, there are bijective maps*

$$J_s[A_0] : G_s[A_0]/GL(n, \mathbf{C}\{z^{-1}\}) \ \to\ H^1(S^1, \mathcal{G}_{s,0}[A_0]) . \qquad \text{(A3.2.11)}$$

This is the classification theorem due to the idea of Malgrange.

OBSERVATION A.3.2.7. Let $\{V_1, \cdots, V_N\}$ be a covering of S^1 by open connected arcs of reasonable sizes and $\{F_{\ell k} \in \mathcal{G}_{s,0}[A_0](V_\ell \cap V_k); V_\ell \cap V_k \neq \emptyset\}$ be a one-cocycle defined by the sheaf $\mathcal{G}_{s,0}[A_0]$. Also let $\{\mathcal{S}_\ell, \Phi_\ell(z)\}$ be a system of fundamental solutions of (A.3.2.3) in the neighborhood of $z = \infty$, where

$$\mathcal{S}_\ell = \mathcal{S}(\rho(\theta), V_\ell) = \{z\ ; \arg z \in V_\ell,\ |z| > \rho(\arg z)\}.$$

Here $\rho(\theta)$ is a positive-valued continuous function on the arc V. Condition (A.3.2.9) implies that there exist $n \times n$ matrices $C_{\ell k} \in GL(n, \mathbf{C})$ such that

$$F_{\ell k}(z) = \Phi_\ell(z) C_{\ell k} \Phi_k(z)^{-1} \qquad \text{on} \quad \mathcal{S}_\ell \cap \mathcal{S}_k.$$

Since $\{F_{\ell k}\}$ is a one-cocycle, the matrices $C_{\ell k}$ satisfy the condition below:

$$C_{hk} C_{k\ell} = C_{h\ell} \qquad \text{whenever} \quad \mathcal{S}_h \cap \mathcal{S}_k \cap \mathcal{S}_\ell \neq \emptyset.$$

Furthermore, $F_{\ell k} \in \mathcal{G}_{s,0}[A_0](V_\ell \cap V_k)$ means that

$$A_{sp}(F_{\ell k}) \ = \ I_n, \qquad \text{(A.3.2.12)}$$

where I_n is the $n \times n$ identity matrix. This implies that $\{F_{\ell k}\}$ defines a trivial asymptotic system at $z = \infty$ (cf. Definition 6.9.4, Definition 6.9.5 and Theorem 6.9.7). Let us compare Definition 6.10.3 of Stokes phenomena and what we have explained above.

DEFINITION 6.10.3. We call $\{A_0(z), \{\mathcal{S}_\ell, \Phi_\ell(z)\}, C_{k\ell}\}$ a **Stokes phenomenon** at $z = \infty$ if the following conditions are satisfied:

(1) the differential equation (6.10.1) (i.e., $\frac{dW_0}{dz} = A_0(z)W_0$) belongs to the equivalence class $\mathcal{E}$,

(2) $\{\mathcal{S}_1, \cdots, \mathcal{S}_N\}$ is a covering at $z = \infty$ satisfying the conditions of Theorem 6.10.1,

(3) $\{\mathcal{S}_\ell, \Phi_\ell(z)\}$ is a system of fundamental solutions of (6.10.1) in the neighborhood of $z = \infty$,

(4) the $C_{k\ell}$ are matrices in $GL(n, \mathbf{C})$ which are defined whenever $\mathcal{S}_k \cap \mathcal{S}_\ell \neq \emptyset$,

(5) $C_{hk}C_{k\ell} = C_{h\ell}$ whenever $\mathcal{S}_h \cap \mathcal{S}_k \cap \mathcal{S}_\ell \neq \emptyset$,
(6) $\Phi_k(z)\ C_{k\ell}\ \Phi_\ell(z)^{-1} \in GL(n, \mathcal{A}(\mathcal{S}_k \cap \mathcal{S}_\ell))$ whenever $\mathcal{S}_k \cap \mathcal{S}_\ell \neq \emptyset$,
(7) the asymptotic system $\{\Phi_k(z)\ C_{k\ell}\ \Phi_\ell(z)^{-1}\}$ is trivial at $z = \infty$.

Right after this definition, we made the following remark.

REMARK 6.10.4. Condition (7) can be replaced by

(7′) the asymptotic system $\{\Phi_k(z)\ C_{k\ell}\ \Phi_\ell(z)^{-1}\}$ is formally trivial.

In particular, if $A_{sp}(\Phi_k(z)\ C_{k\ell}\ \Phi_\ell(z)^{-1}) = I$ whenever $\mathcal{S}_k \cap \mathcal{S}_\ell \neq \emptyset$, condition (7) is satisfied.

Note that we did not assume condition (A.3.2.12). Instead, we assumed condition (7). To go from $G_s[A_0]/GL(n, \mathbf{C}\{z^{-1}\})$ to the moduli space of analytic linear differential equations with respect to the holomorphic classification, we must classify $G_s[A_0]/GL(n, \mathbf{C}\{z^{-1}\})$ further. For details, see D. G. Babbitt–V. S. Varadarajan [39, 40].

REMARK A.3.2.8. The idea of B. Malgrange applies also to nonlinear differential equations (cf. J. Martinet–J.-P. Ramis [105, 106]).

§A.3.3. Analytization

The main problem of Chapter 3 was to explain the following theorem of G. D. Birkhoff.

THEOREM 3.3.1 (G. D. Birkhoff). *Given an integer k and a matrix* $A(\zeta) \in gl(n, \mathcal{O}(\Delta(0, \frac{1}{R_0})))$, *we consider the differential equation*

$$\frac{dW}{dz} = z^k A\left(\frac{1}{z}\right) W. \tag{3.3.3}$$

Then, there exists a matrix $P(\zeta) \in GL(n, \mathcal{O}(\Delta(0, \frac{1}{R_0})))$ *such that the transformation*

$$W = P\left(\frac{1}{z}\right) V \tag{3.3.4}$$

changes (3.3.3) *to a differential equation*

$$\frac{dV}{dz} = z^k B\left(\frac{1}{z}\right) V \tag{3.3.5}$$

with a matrix $B(\frac{1}{z})$ *whose entries are polynomials in* $\frac{1}{z}$ *with complex constant coefficients. Moreover we can choose such a* $P(\zeta)$ *so that* $z = 0$ *is at worst a regular singular point of* (3.3.5).

In this section we shall explain how to relax the requirement that A should belong to $\in gl(n, \mathcal{O}(\Delta(0, \frac{1}{R_0})))$. The main result is the following theorem.

THEOREM A.3.3.1 (J.-P. Ramis [124]). *Given an s* ($0 \leq s \leq +\infty$) *and an operator* $L = \frac{d}{dz} - z^k \hat{A}(z)$, *where k is an integer and* $\hat{A}$ *is an* $n \times n$ *matrix* $\in gl(n, \mathbf{C}[[z^{-1}]]_s)$, *there exists an* $n \times n$ *matrix* $\hat{P} \in GL(n, \mathbf{C}[[z^{-1}]]_s)$ *such that*

$$\hat{P}^{-1} \circ L \circ \hat{P} = \frac{d}{dz} - z^k A(z) \qquad \textit{with} \quad A \in gl(n, \mathbf{C}\{z^{-1}\}). \tag{A.3.3.1}$$

REMARK A.3.3.2. $\hat{P}^{-1} \circ L \circ \hat{P}$ denotes the composition of three operators $\hat{P}^{-1}$, L and $\hat{P}$.

SKETCH OF PROOF OF THEOREM A.3.3.1. Let $\{V_1, \cdots, V_N\}$ be a *good* covering of S^1 by open connected arcs of reasonable sizes. Then there exist $A_\ell \in gl(n, \mathscr{A}_s(V_\ell))$ and $P_{\ell h} \in \mathscr{G}_{s,0}(V_\ell \cap V_h)$, ($V_\ell \cap V_h \neq \emptyset$) such that

$$\begin{cases} A_{sp}(A_\ell) = \hat{A} & (\ell = 1, \cdots, N), \\ \dfrac{dP_{\ell h}(z)}{dz} = z^k\{A_\ell(z)P_{\ell h}(z) - P_{\ell h}(z)A_h(z)\} & (z \in V_\ell \cap V_h \neq \emptyset), \\ P_{\ell j}P_{jh} = P_{\ell h} \quad \text{if} \quad V_\ell \cap V_j \cap V_h \neq \emptyset. \end{cases}$$

Set $\mathscr{L}_\ell = \frac{d}{dz} - z^k A_\ell(z)$. Then

$$\mathscr{L}_h = P_{\ell h}^{-1} \circ \mathscr{L}_\ell \circ P_{\ell h} \qquad \text{if} \quad V_\ell \cap V_h \neq \emptyset.$$

By virtue of Theorem 6.9.7 and Theorem A.2.4.1, there exist $P_\ell \in \mathscr{G}_s(V_\ell)$, ($\ell = 1, \cdots, N$) such that

$$\begin{cases} A_{sp}(P_\ell - I_n) \in z^{-1}gl(n, \mathbf{C}[[z^{-1}]]_s), \\ P_{\ell h}(z) = P_\ell(z)P_h(z)^{-1} \qquad \text{if} \quad V_\ell \cap V_h \neq \emptyset. \end{cases}$$

Hence

$$P_\ell^{-1} \circ \mathscr{L}_\ell \circ P_\ell = P_h^{-1} \circ \mathscr{L}_h \circ P_k \qquad \text{on} \quad V_\ell \cap V_h \neq \emptyset.$$

Now if we set $A_{sp}(P_\ell) = \hat{P}$ and $\hat{P}^{-1} \circ \mathscr{L} \circ \hat{P} = \frac{d}{dz} - z^k A(z)$, we have

$$\begin{cases} \hat{P} \in GL(n, \mathbf{C}[[z^{-1}]]_s), \\ A \in gl(n, \mathbf{C}\{z^{-1}\}). \quad \square \end{cases}$$

[The idea of this proof is due to B. Malgrange (cf. J.-P. Ramis [124]). The author of this book also wrote a paper [138] on analytization which is not quite complete but not totally useless.]

§A.3.4. Injectivity of D-modules $\mathscr{A}_{s,0}(\theta)$

Set

$$D = \left\{ \sum_{h=0}^{m} a_h(z)\, \delta^h \; ; \; a_h \in \mathbf{C}\{z^{-1}\}, \; m \in \mathbf{N} \right\}. \tag{A.3.4.1}$$

We can regard D as a noncommutative ring. Then, for every $\theta \in S^1$, the stalks $\mathcal{A}_{s,0}(\theta)$, $\mathcal{A}_s(\theta)$, and $\mathbf{C}[[z^{-1}]]_s$ are D-modules and we have exact sequences of homomorphisms of D-modules

$$0 \to \mathcal{A}_{s,0}(\theta) \to \mathcal{A}_s(\theta) \to \mathbf{C}[[z^{-1}]]_s \to 0, \tag{A.3.4.2}$$

where

$$\begin{cases} A_{sp} : \mathcal{A}_s(\theta) & \to \quad \mathbf{C}[[z^{-1}]]_s, \\ \mathrm{id}_{\mathcal{A}_{s,0}(\theta)} : \mathcal{A}_{s,0}(\theta) & \to \quad \mathcal{A}_s(\theta). \end{cases}$$

The main result of this section is the theorem below.

THEOREM A.3.4.1 (Y. Sibuya [137]). *Exact sequences* (A.3.4.2) *of homomorphisms of D-modules split.*

The proof of this theorem is based on the observations below:

OBSERVATION A.3.4.2.

I: A D-module M is said to be *divisible* if the following conditions are satisfied:
 (i) if $z^{-1}a = 0$ for an $a \in M$, then $a = 0$,
 (ii) for every $a \in M$ and every $P \in D$, there exists $b \in M$ such that $Pb = a$.

II: A D-module M is said to be *injective* if for every diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M_1 & \xrightarrow{\ i\ } & M_2 \\ & & \big\downarrow \lambda & & \\ & & M & & \end{array}$$

of homomorphisms of D-modules, where the horizontal sequence is exact, there exists $\mu \in \mathrm{Hom}_D(M_2, M)$ such that $\lambda = \mu \circ i$.

III: If a D-module M is divisible , then M is injective (cf. R. Godement [61] and Y. Sibuya [137]).

OBSERVATION A.3.4.3. Lemma A.2.3.3 implies that the D-modules $\mathcal{A}_{s,0}(\theta)$ are divisible and hence injective. In particular, if we consider the diagram below

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{A}_{s,0}(\theta) & \xrightarrow{\ \mathrm{id}\ } & \mathcal{A}_s(\theta) \\ & & \big\downarrow \mathrm{id} & & \\ & & \mathcal{A}_{s,o}(\theta) & & \end{array}$$

there exist homomorphisms $\mu_{s,\theta} : \mathcal{A}_s(\theta) \to \mathcal{A}_{s,0}(\theta)$ satisfying the requirement

$$\mu_{s,\theta}\Big|_{\mathcal{A}_{s,0}(\theta)} = \mathrm{id}_{\mathcal{A}_{s,0}(\theta)}.$$

Note that $\mu_{s,\theta}$ are not unique (cf. Y. Sibuya [137]).

OBSERVATION A.3.4.4. For a given $\hat{f} \in \mathbf{C}[[z^{-1}]]_s$, choose $f \in \mathscr{A}_s(\theta)$ such that $A_{sp}(f) = \hat{f}$ and set

$$\phi_{s,\theta}(\hat{f}) = f - \mu_{s,\theta}(f). \tag{A.3.4.3}$$

Then $\phi_{s,\theta}(\hat{f})$ are independent of the choice of f. Furthermore the maps

$$\phi_{s,\theta} : \mathbf{C}[[z^{-1}]]_s \to \mathscr{A}_s(\theta)$$

are homomorphisms of D-modules such that

$$A_{sp} \circ \phi_{s,\theta} = \mathrm{id}_{\mathbf{C}[[z^{-1}]]_s}. \tag{A.3.4.4}$$

This proves Theorem A.3.4.1. Note that

$$\begin{cases} \mu_{s,\theta}\Big|_{\mathbf{C}\{z^{-1}\}} = 0, & \phi_{s,\theta}\Big|_{\mathbf{C}\{z^{-1}\}} = \mathrm{id}_{\mathbf{C}\{z^{-1}\}}, \\ \mu_{s,\theta} \circ \phi_{s,\theta} = 0, & \phi_{s,\theta} \circ A_{sp} = \mathrm{id}_{\mathscr{A}_s(\theta)} - \mu_{s,\theta}. \end{cases} \tag{A.3.4.5}$$

APPLICATION A.3.4.5:. Let us consider the following system of linear differential equations of higher order (cf. Y. Sibuya [137]):

$$\sum_{h \in N(j)} p_{jh}\, u_h = f_j \qquad (j \in \mathscr{J}), \tag{A.3.4.6}$$

where $\mathscr{H}$ and $\mathscr{J}$ are nonempty sets, $N(j)$ are nonempty subsets of $\mathscr{H}$, $p_{jh} \in D$, $u_h \in \mathscr{A}_s(\theta)$ are unknowns, and $f_j \in \mathscr{A}_s(\theta)$. We shall look at system (A.3.4.6) under the assumption below.

ASSUMPTION A.3.4.6. *There exist $\hat{g}_h \in \mathbf{C}[[z^{-1}]]_s$ $(h \in \mathscr{H})$ such that*

$$\sum_{h \in N(j)} p_{jh}\, \hat{g}_h = A_{sp}(f_j) \qquad (j \in \mathscr{J}), \tag{A.3.4.7}$$

i.e., $u_h = \hat{g}_h$ $(h \in \mathscr{H})$ is a formal solution of system (A.3.4.6).

First of all, the following theorem is a basic result.

THEOREM A.3.4.7. *System* (A.3.4.6) *admits a solution*

$$u_h = \eta_h \in \mathscr{A}_s(\theta) \qquad (h \in \mathscr{H})$$

satisfying the condition

$$A_{sp}(\eta_h) = \hat{g}_h \qquad (h \in \mathscr{H}) \tag{A.3.4.8}$$

if and only if the system

$$\sum_{h \in N(j)} p_{jh}\, v_h = \mu_{s,\theta}(f_j) \qquad (j \in \mathscr{J}) \tag{A.3.4.9}$$

admits a solution

$$v_h = \sigma_h \in \mathcal{A}_{s,0}(\theta) \qquad (h \in \mathcal{H}).$$

SKETCH OF PROOF. Assumption (A.3.4.7) implies that

$$\sum_{h \in N(j)} p_{jh}\, \phi_{s,\theta}(\hat{g}_h) = f_j - \mu_{s,\theta}(f_j) \qquad (j \in \mathcal{J}).$$

Hence

$$\eta_h = \phi_{s,\theta}(\hat{g}_h) + \sigma_h \qquad (h \in \mathcal{H}). \tag{A.3.4.10}$$

□

Since $\mu_{s,\theta}(f) = 0$ if $f \in \mathbf{C}\{z^{-1}\}$, we derive the following corollary of Theorem A.3.4.7.

COROLLARY A.3.4.8. *If* $f_j \in \mathbf{C}\{z^{-1}\}$ $(j \in \mathcal{J})$, *then*

$$\eta_h = \phi_{s,\theta}(\hat{g}_h) \in \mathcal{A}_s(\theta) \qquad (h \in \mathcal{H})$$

satisfy system (A.3.4.6) *and condition* (A.3.4.8).

To investigate system (A.3.4.9), the following lemma is helpful.

LEMMA A.3.4.9. *If* $f_j \in \mathcal{A}_{s,0}(\theta)$ $(j \in \mathcal{J})$, *then system* (A.3.4.6) *admits a solution* $u_h \in \mathcal{A}_{s,0}(\theta)$ $(h \in \mathcal{H})$ *if and only if every finite subsystem of* (A.3.4.6) *admits a solution in* $\mathcal{A}_{s,0}(\theta)$.

This lemma is a particular case of the theorem below.

THEOREM A.3.4.10 (S. Gacsalyi [58]). *Let R be an associative ring* (*with unit*) *and M be an injective R-module. A system of equations*

$$\sum r_{jk}\, X_k = m_j, \qquad (\, r_{jk} \in R,\ m_j \in M\,),$$

where, for each j, $r_{jk} = 0$ for almost all k, has a solution in M if and only if any finite subsystem has a solution in M.

A proof of this theorem is found in L. Fuchs [57] (cf. also Y. Sibuya [137]).

APPENDIX 4

k-Summability

§A.4.0. Preliminaries

In Appendix 2 we studied injectivity and surjectivity of the maps $A_{sp} : \mathscr{A}_s(V) \to \mathbf{C}[[z^{-1}]]_s$ where V is an open connected arc of the unit circle S^1 in the complex plane (cf. Theorems A.2.2.11 and A.2.2.14). In this appendix we shall study the case when V is large so that A_{sp} is injective but not surjective. One of the basic results is a theorem (cf. Theorem A.4.0.3) due to F. Nevanlinna [111] (see also G. N. Watson [148]).

OBSERVATION A.4.0.1. For a given $f \in \mathscr{A}_s\left(R_0, -\frac{\beta_0}{2}, \frac{\beta_0}{2}\right)$, set $A_{sp}(f) = \sum_{m\geq 0} a_m\ z^{-m} \in \mathbf{C}[[z^{-1}]]_s$. This means that, for every positive number R greater than R_0 and every positive number β less than β_0, there exist two nonnegative numbers $K_{R,\beta}$ and $B_{R,\beta}$ such that

$$\left| f(z) - \sum_{m=0}^{N} a_m\ z^{-m} \right| \leq K_{R,\beta}\ ((N+1)!)^s\ \left(B_{R,\beta}\right)^{N+1}\ |z|^{-(N+1)}$$

on the sectorial domain $\mathscr{D}(R, -\frac{\beta}{2}, \frac{\beta}{2})$ for all nonnegative integers N. Throughout this appendix, we assume that

$$+\infty\ >\ \beta_0\ >\ s\pi\ >\ 0. \tag{A.4.0.1}$$

Also we denote $\frac{1}{s}$ by k, i.e.

$$k\ =\ \frac{1}{s}. \tag{A.4.0.2}$$

OBSERVATION A.4.0.2. For a given $g = \sum_{m\geq 0} a_m\ z^{-m} \in \mathbf{C}[[z^{-1}]]$ let us write

$$\phi_g(z)\ =\ \sum_{m\geq 0} \frac{a_m}{\Gamma\left(1+\frac{m}{k}\right)}\ z^{-m}. \tag{A.4.0.3}$$

Then $g \in \mathbf{C}[[z^{-1}]]_s$ if and only if $\phi_g \in \mathbf{C}\{z^{-1}\}$. F. Nevanlinna [111] characterized

$$\text{Image}\left(\ A_{sp} : \mathscr{A}_s\left(R_0, -\frac{\beta_0}{2}, \frac{\beta_0}{2}\right)\ \to\ \mathbf{C}[[z^{-1}]]_s\right)$$

by means of the following result.

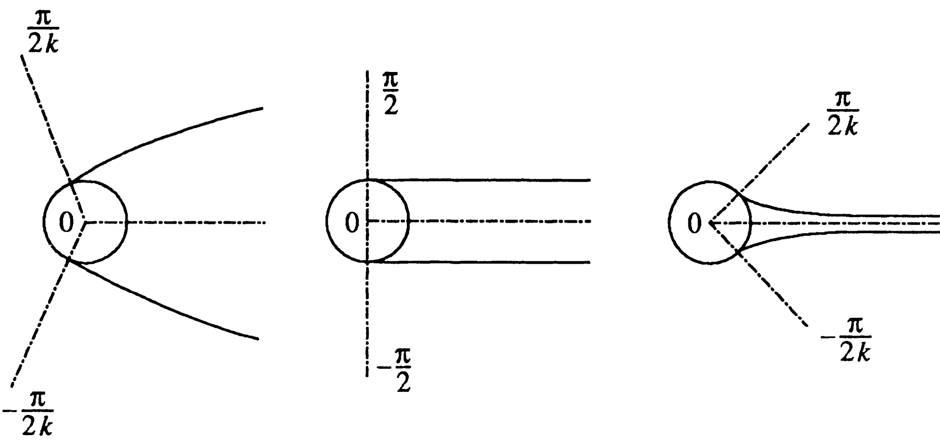

FIGURE 10

THEOREM A.4.0.3 (F. Nevanlinna). *Assume that β_0 satisfies condition* (A.4.0.1). *Suppose also that $f \in \mathscr{A}_s\left(R_0, -\frac{\beta_0}{2}, \frac{\beta_0}{2}\right)$ and $g \in \mathbf{C}[[z^{-1}]]_s$ satisfy the condition*

$$A_{sp}(f) = g.$$

Then $\phi_g(\frac{1}{t})$ is holomorphic and satisfies the estimate

$$\left|\phi_g\left(\frac{1}{t}\right)\right| \leq C \exp\left\{\lambda^k \Re(t^k)\right\} \tag{A.4.0.4}$$

on a domain

$$\{t\ ;\ |t| < r_0\} \cup \{t\ ;\ |\Im(t^k)| < r_0^k,\ \Re(t^k) > 0\}, \tag{A.4.0.5}$$

where $C \geq 0$, $\lambda > 0$ and $r_0 > 0$ are suitable numbers (*cf.* Figure 10). *Furthermore*

$$f(z) = k\ z^k \int_0^{+\infty} \phi_g\left(\frac{1}{t}\right) \exp\left\{-(tz)^k\right\}\ t^{k-1}\ dt \tag{A.4.0.6}$$

for $\Re(z^k) > \rho$ for a sufficiently large positive number ρ.

REMARK A.4.0.4.

(1) Theorem A.4.0.3 also implies that the maps A_{sp} are injective but not surjective.
(2) Under assumption (A.4.0.1) the domain $\{z\ ;\ \Re(z^k) > \rho\}$ is contained in the sectorial domain $\mathscr{D}(R_0, -\frac{\beta_0}{2}, \frac{\beta_0}{2})$ if ρ is sufficiently large.

(3) The right-hand side of (A.4.0.6) can be writen as $\ell_{+\infty,k}(\phi_g)$ if we use Definition (A.2.2.15) of the incomplete Leroy transforms of order k.

In the following definition, we assume that β_0 satisfies condition (A.4.0.1).

DEFINITION A.4.0.5 (J.-P. Ramis [124]).

I: If $f \in \mathscr{A}_s\left(R_0, \alpha - \frac{\beta_0}{2}, \alpha + \frac{\beta_0}{2}\right)$ and $g \in \mathbf{C}[[z^{-1}]]_s$ satisfy the condition

$$A_{sp}(f) = g,$$

the power series g is said to be **k-summable in the direction** α and the function f is called the sum of the power series g in the direction α, where $k = \frac{1}{s}$ (cf. (A.4.0.2)).

II: $g \in \mathbf{C}[[z^{-1}]]_s$ is said to be **k-summable** if g is k-summable in every direction α except for a finite number of directions $\alpha_1, \cdots, \alpha_p$. The finite set

$$\Sigma(g) = \{\alpha_1, \cdots, \alpha_p\} \qquad \text{(A.4.0.7)}$$

is called the singular support of g.

REMARK A.4.0.6.

(a) If k is a positive integer and if a power series $g \in \mathbf{C}[[z^{-1}]]$ is k-summable in the directions $\alpha + \left(\frac{2\pi}{k}\right) h \quad (h = 0, 1, \cdots, k-1)$, then the sum of g in the direction α has an inverse factorial series expansion

$$\sum_{p=0}^{k-1} \frac{1}{z^p} \sum_{n=0}^{\infty} \frac{b_{p,n}}{(ze^{-i\alpha})^k[(ze^{-i\alpha})^k + \omega][(ze^{-i\alpha})^k + 2\omega] \cdots [(ze^{-i\alpha})^k + n\omega]},$$

where the $b_{p,n}$ are suitable numbers and ω is a sufficiently large positive number. This series expansion is uniformly convergent if $\Re((ze^{-i\alpha})^k)$ is sufficiently large (cf. F. Nevanlinna [111] and G. N. Watson [148]).

(b) If g is k-summable in every direction α belonging to an open interval $\alpha_1 < \alpha < \alpha_2$, the sum f_α of g in the directions α all together define a function $f \in \mathscr{A}_s\left(\rho(\theta), \alpha_1 - \frac{\varepsilon_1+\pi}{2k}, \alpha_2 + \frac{\varepsilon_2+\pi}{2k}\right)$, where $\rho(\theta)$ is a positive-valued continuous function on the open interval $\alpha_1 - \frac{\varepsilon_1+\pi}{2k} < \theta < \alpha_2 + \frac{\varepsilon_2+\pi}{2k}$ and ε_1 and ε_2 are nonnegative numbers.

(c) If g is k-summable and the singular support $\Sigma(g)$ is empty, then $g \in \mathbf{C}\{z^{-1}\}$.

(d) If we denote by $\mathbf{C}\{z^{-1}\}_s$ the set of all k-summable power series in $\mathbf{C}[[z^{-1}]]$, where k and s are related by (A.4.0.2), then $\mathbf{C}\{z^{-1}\}_s$ is a commutative differential algebra over $\mathbf{C}$. Furthermore we have

$$\mathbf{C}\{z^{-1}\} \subset \mathbf{C}\{z^{-1}\}_s \subset \mathbf{C}[[z^{-1}]]_s. \tag{A.4.0.8}$$

(e) A formal power series $g \in \mathbf{C}[[z^{-1}]]$ is k-summable with the singular support $\Sigma(g) = \{\alpha_1, \cdots, \alpha_p\}$ if and only if

$$g = g_0 + \sum_{j=1}^{p} g_j \tag{A.4.0.9}$$

for some $g_0 \in \mathbf{C}\{z^{-1}\}$ and $g_j \in \mathbf{C}\{z^{-1}\}_s$ with $\Sigma(g_j) = \{\alpha_j\}$ $(j = 1, \cdots, p)$. The g_j $(j = 1, \cdots, p)$ are uniquely determined modulo $\mathbf{C}\{z^{-1}\}$. In fact suppose that $g_1 \in \mathbf{C}\{z^{-1}\}_s$ and $\tilde{g}_1 \in \mathbf{C}\{z^{-1}\}_s$ satisfy the condition

$$\begin{cases} \Sigma(g_1) = \Sigma(\tilde{g}_1) = \{\alpha_1\}, \\ \Sigma(g - g_1) = \Sigma(g - \tilde{g}_1) = \{\alpha_2, \cdots, \alpha_p\}. \end{cases}$$

Then $g_1 - \tilde{g}_1 \in \mathbf{C}\{z^{-1}\}_s$ and

$$\Sigma(g_1 - \tilde{g}_1) \subset \{\alpha_1\} \cap \{\alpha_2, \cdots, \alpha_p\} = \varnothing.$$

Hence $g_1 - \tilde{g}_1 \in \mathbf{C}\{z^{-1}\}$. Therefore we can use induction on p if we can find a $g_1 \in \mathbf{C}\{z^{-1}\}_s$ satisfying the condition

$$\Sigma(g_1) = \{\alpha_1\}, \qquad \Sigma(g - g_1) = \{\alpha_2, \cdots, \alpha_p\}. \tag{A.4.0.10}$$

We can construct such a g_1 by means of Theorem A.2.5.1 (or Theorem A.3.0.4).

(f) If a formal power series g is k_1-summable and k_2-summable for two distinct numbers k_1 and k_2, then g is convergent, i.e.,

$$\mathbf{C}\{z^{-1}\}_{s_1} \cap \mathbf{C}\{z^{-1}\}_{s_2} = \mathbf{C}\{z^{-1}\} \quad \text{if} \quad s_1 \neq s_2. \tag{A.4.0.11}$$

As a matter of fact

$$\mathbf{C}\{z^{-1}\}_s \cap \mathbf{C}[[z^{-1}]]_\sigma = \mathbf{C}\{z^{-1}\} \quad \text{if} \quad \sigma < s \tag{A.4.0.12}$$

(cf. J.-P. Ramis [126] and J.-P. Ramis–Y. Sibuya [129]).

§A.4.1. *k*-summability of formal solutions

Let us look at a differential equation:

$$\delta \vec{y} = z^q \left\{ \vec{g}_0(z) + \Phi(z)\,\vec{y} + \sum_{|\vec{p}|\geq 2} \vec{g}_{\vec{p}}(z)\,\vec{y}^{\vec{p}} \right\}, \tag{A.4.1.1}$$

where q is a positive integer and

$$\vec{g}_0 \in z^{-1}\left(\mathbf{C}\{z^{-1}\}_s\right)^n, \qquad \Phi \in gl(n, \mathbf{C}\{z^{-1}\}), \qquad \vec{g}_{\vec{p}} \in \mathbf{C}\{z^{-1}\}^n.$$

The series $\sum_{|\vec{p}|\geq 2} \vec{g}_{\vec{p}}(z)\, \vec{y}^{\vec{p}}$ is uniformly and absolutely convergent on a domain $|z| > R_0$, $|\vec{y}| < r_0$, where $|\vec{y}|$ is the maximum of absolute values of entries of the vector $\vec{y}$, and R_0 and r_0 are positive numbers.

OBSERVATION A.4.1.1. Assume that there exists a matrix $A_0 \in GL(n, \mathbf{C})$ such that $\Phi(z) - A_0 \in z^{-1} gl(n, \mathbf{C}\{z^{-1}\})$. By virtue of Theorem A.2.1.9 differential equation (A.4.1.1) admits a unique solution $\vec{g} \in z^{-1}\left(\mathbf{C}[[z^{-1}]]_\sigma\right)^n$, where

$$\sigma = \max\left\{s, \frac{1}{q}\right\}. \tag{A.4.1.2}$$

The matrix A_0 has nonzero eigenvalues $\lambda_1, \cdots, \lambda_n$. For each j let Σ_j be the set of all directions α such that

$$\Im(\lambda_j\, e^{iq\alpha}) = 0, \qquad \Re(\lambda_j\, e^{iq\alpha}) < 0. \tag{A.4.1.3}$$

THEOREM A.4.1.2 (J.-P. Ramis–Y. Sibuya [129]). *The unique solution* $\vec{g} \in z^{-1}\left(\mathbf{C}[[z^{-1}]]_\sigma\right)^n$ *is q-summable if*

$$s = \frac{1}{q}. \tag{A.4.1.4}$$

Furthermore

$$\Sigma(\vec{g}) \subset \Sigma(\vec{g}_0) \cup \left(\bigcup_{j=1}^{n} \Sigma_j\right). \tag{A.4.1.5}$$

EXPLANATION. This result is based on the following facts:

(i) If $\alpha \notin \Sigma(\vec{g}_0)$, there exist two positive numbers R and ε such that the sum $\vec{f}_0$ of $\vec{g}_0$ in the direction α is in
$$z^{-1}\left(\mathscr{A}_s\left(R, \alpha - \varepsilon - \frac{s\pi}{2}, \alpha + \varepsilon + \frac{s\pi}{2}\right)\right)^n.$$

(ii) If $s \geq \frac{1}{q}$ and $\alpha \notin \Sigma(\vec{g}_0) \cup \left(\bigcup_{j=1}^{n} \Sigma_j\right)$, there exist two positive numbers $\tilde{R}$, $\tilde{\varepsilon}$ and
$$\vec{f} \in z^{-1}\left(\mathscr{A}_s\left(\tilde{R}, \alpha - \tilde{\varepsilon} - \frac{\pi}{2q}, \alpha + \tilde{\varepsilon} + \frac{\pi}{2q}\right)\right)^n$$
such that
$$\begin{cases} \delta\vec{f}(z) = z^q\left\{\vec{f}_0(z) + \Phi(z)\vec{f}(z) + \displaystyle\sum_{|\vec{p}|\geq 2} \vec{g}_{\vec{p}}(z)\, \vec{f}(z)^{\vec{p}}\right\}, \\ A_{sp}(\vec{f}) = \vec{g}. \end{cases}$$

(iii) If $0 < s < \frac{1}{q}$, then $\vec{g} \in z^{-1}\left(\mathbf{C}[[z^{-1}]]_{\frac{1}{q}}\right)^n$. Since $\mathscr{D}(R, \alpha - \varepsilon - \frac{s\pi}{2}, \alpha + \varepsilon + \frac{s\pi}{2})$ is too small, $\vec{g}$ is in general not q-summable in the direction α.

(iv) If $s > \frac{1}{q}$, then $\vec{g} \in z^{-1}\left(\mathbf{C}[[z^{-1}]]_s\right)^n$. Since $\mathscr{D}(\tilde{R}, \alpha-\tilde{\varepsilon}-\frac{\pi}{2q}, \alpha+\tilde{\varepsilon}+\frac{\pi}{2q})$ is too small, $\vec{g}$ is in general not $\frac{1}{s}$-summable.

REMARK A.4.1.3. To prove the existence of $\vec{f}$ of (ii), we must study the existence and uniqueness of Gevrey asymptotic solutions more carefully (cf. (1) of Remark A.2.3.4).

REMARK A.4.1.4. Since $\mathbf{C}\{z^{-1}\} \subset \mathbf{C}\{z^{-1}\}_s$ for every positive s, theorem A.4.1.2 applies to the case when $\vec{g}_0 \in \left(\mathbf{C}\{z^{-1}\}\right)^n$.

Next let us look at the differential equation

$$P[y] = z^{-M}\, G(z, y, \delta y, \cdots, \delta^{n-1} y), \tag{A.4.1.6}$$

where

$$\begin{cases} P = \sum_{h=0}^{n} a_h(z)\, \delta^h, \\ G(z, y_0, y_1, \cdots, y_{n-1}) = G_0(z) + \sum_{|\vec{p}|\geq 1} G_{\vec{p}}(z)\, \vec{y}^{\vec{p}}. \end{cases} \tag{A.4.1.7}$$

We assume that

(1) $a_h \in \mathbf{C}\{z^{-1}\}$, $G_0 \in \mathbf{C}\{z^{-1}\}_s$ and $G_{\vec{p}} \in \mathbf{C}\{z^{-1}\}$,

(2) $a_n \neq 0$,

(3) the series G is uniformly and absolutely convergent on a domain $|z| > R_0$, $|y_h| < r_0$ $(h = 0, 1, \cdots, n-1)$, where R_0 and r_0 are two positive numbers.

OBSERVATION A.4.1.5. Let

$$0 < k_1 < k_2 < \cdots < k_p < +\infty \tag{A.4.1.8}$$

be all of the distinct positive slopes of sides of the Newton polygon $\mathscr{N}(P)$ and let

$$\{(X, Y_j + k_j(X - X_j))\ ;\ X_j \leq X \leq X_{j+1}\}$$

be the side of $\mathscr{N}(P)$ whose slope is k_j, where $Y_{j+1} = Y_j + k_j(X_{j+1} - X_j)$. Then the $X_{j+1} - X_j$ are positive integers. Set

$$n_j = X_{j+1} - X_j \qquad (j = 1, 2, \cdots, p).$$

The linear homogeneous differential equation $P[y] = 0$ admits n_j linearly independent formal solutions of the form

$$y_{j,h}(z) = z^{\mu_{j,h}} \exp[\lambda_{j,h} z^{k_j}\{1 + o(1)\}]\, f_{j,h}(z) \qquad (h = 1, \cdots, n_j),$$

where the $\mu_{j,h}$ are constants, the $\lambda_{j,h}$ are nonzero constants and $f_{j,h} \in \mathbf{C}[[z^{-\frac{1}{\sigma}}]][z^{\frac{1}{\sigma}}, \log z]$ for some positive integer σ (cf. Remark A.1.4.2.).

For every (p, h) $(h = 1, \cdots, n_p)$ we denote by Σ_h the set of all directions α such that

$$\Im(\lambda_{p,h} z^{k_p}) = 0, \qquad \Re(\lambda_{p,h} z^{k_p}) < 0.$$

Assuming that the positive integer M on the right-hand side of differential equation (A.4.1.6) is sufficiently large, we can prove the theorem below.

THEOREM A.4.1.6 (J.-P. Ramis–Y. Sibuya [129]). *If* $g \in z^{-1}\mathbf{C}[[z^{-1}]]_s$ *satisfies differential equation* (A.4.1.6) *and if* $0 < s \leq \frac{1}{k_p}$, *then* g *is* $\frac{1}{s}$-*summable and*

$$\Sigma(g) \subset \begin{cases} \Sigma(G_0) \cup \left(\bigcup_{h=1}^{n_p} \Sigma_h \right) & \text{if} \quad s = \dfrac{1}{k_p}, \\ \Sigma(G_0) & \text{if} \quad 0 < s < \dfrac{1}{k_p}. \end{cases}$$

The main ideas of the proof are similar to those of the proof of Theorem A.4.1.2.

EXAMPLE A.4.1.7. J.-P. Ramis showed that

$$g = \sum_{m=0}^{\infty} (-1)^m \, m! \, x^{m+1} + \sum_{m=0}^{\infty} (-1)^m \, m! \, x^{2m+2}$$

satisfies the differential equation

$$\begin{cases} P[y] = 0, \\ P = D^5[x^5(2-x)D^2 - x^2(2x^3 - 5x^2 + 4)D + 2(x^2 - x + 2)], \end{cases}$$

where $D = \frac{d}{dx}$. He also showed that $g\left(\frac{1}{z}\right)$ is not k-summable for any $k > 0$ (cf. J.-P. Ramis–Y. Sibuya [129]).

§A.4.2. The Turrittin problem

In this section, we shall consider a differential equation

$$\delta W + A(z)W = 0 \qquad \left(\delta = z\frac{d}{dz}\right), \tag{A.4.2.1}$$

where W is an $n \times n$ unknown matrix and $A(z) \in gl(n, \mathbf{C}\{z^{-1}\}[z])$. Let

$$\tilde{\delta} U + B(\zeta)U = 0 \qquad \left(\tilde{\delta} = \zeta\frac{d}{d\zeta}\right) \tag{A.4.2.2}$$

be a differential equation in the standard form given in the Hukuhara–Turrittin Theorem 6.8.1. By virtue of Theorem 6.8.1 differential equation (A.4.2.1) can be changed to the differential equation (A.4.2.2) by a transformation

$$z = \zeta^\sigma, \qquad W = P(\zeta)U, \tag{A.4.2.3}$$

where σ is a positive integer, $P \in gl(n, \mathbf{C}[[\zeta^{-1}]])$, and $\det(P) \neq 0$ as an element of $\mathbf{C}[[z^{-1}]]$. In §6.8 we gave a sketch of the proof of Theorem 6.8.1 in terms of three cases. The matrix P of transformation (A.4.2.3) can be written as a product

$$P(\zeta) = \hat{P}_1(\zeta)\hat{P}_2(\zeta)\cdots\hat{P}_N(\zeta)$$

of a finite number of matrices of transformations $W_j = \hat{P}_j(\zeta)U_j$ $(j = 1, \cdots, N)$ each of which is of one of those three cases (cf. Remark 6.8.2). The main concern of this section is to show that each $\hat{P}_j$ is k_j-summable as a formal power series in ζ^{-1} for some positive integer k_j.

OBSERVATION A.4.2.1. We are not going to prove Theorem 6.8.1 again by another method. We want to find out more information concerning transformation (A.4.2.3). The two matrices P and B are related through the equation

$$\tilde{\delta}P(\zeta) + \sigma\, A(\zeta^{\sigma})P(\zeta) - P(\zeta)B(\zeta) = 0.$$

Theorem 6.8.1 guarantees the existence of P and B. We can write P in the form

$$P(\zeta) = P_0(\zeta)\left[I + \zeta^{-M}\sum_{m\geq 0}\zeta^{-m}P_m\right],$$

where $P_0 \in GL(n, \mathbf{C}[\zeta, \zeta^{-1}])$, I is the $n \times n$ identity matrix, M is a positive integer, and $P_m \in gl(n, \mathbf{C})$. The transformation

$$z = \zeta^{\sigma}, \qquad W = P_0(\zeta)V, \tag{A.4.2.4}$$

changes differential equation (A.4.2.1) to

$$\tilde{\delta}V + [\, B(\zeta) + \zeta^{-L}E(\zeta)\,]\, V = 0, \tag{A.4.2.5}$$

where $E \in gl(n, \mathbf{C}\{\zeta^{-1}\})$ and L is a positive integer. The integer L tends to $+\infty$ as M tends to $+\infty$. The transformation

$$V = \left[I + \zeta^{-M}\sum_{m\geq 0}\zeta^{-m}P_m\right] U \tag{A.4.2.6}$$

changes equation (A.4.2.5) to equation (A.4.2.2). Hence the matrix

$$X = I + \zeta^{-M}\sum_{m\geq 0}\zeta^{-m}P_m \tag{A.4.2.7}$$

satisfies the differential equation

$$\tilde{\delta}X + [\, B(\zeta) + \zeta^{-L}E(\zeta)\,]X - X\,B(\zeta) = 0. \tag{A.4.2.8}$$

OBSERVATION A.4.2.2. Let

$$0 < k_1 < k_2 < \cdots < k_\ell < +\infty \tag{A.4.2.9}$$

be all of the distinct positive slopes of the Newton polygon of the differential equation

$$\tilde{\delta} Y + \sigma\{A(\zeta^\sigma)\, Y \;-\; Y\, A(\zeta^\sigma)\} \;=\; 0. \tag{A.4.2.10}$$

The Newton polygons of differential equations (A.4.2.8) and (A.4.2.10) have the same shape.

THEOREM A.4.2.3. *Suppose that L is sufficiently large. Then matrix* (A.4.2.7) *can be written as a product*

$$X \;=\; P_\ell(\zeta)P_{\ell-1}(\zeta)\cdots P_1(\zeta), \tag{A.4.2.11}$$

where

$$P_j \;\in\; gl(n,\mathbf{C}\{\zeta^{-1}\}_{\frac{1}{k_j}}) \qquad (j = 1,\ \cdots,\ \ell). \tag{A.4.2.12}$$

OBSERVATION A.4.2.4. To prove Theorem A.4.2.3, it suffices to prove the following refinement of Lemma 5.5.2. Let us assume that the matrix $A(z)$ of differential equation (A.4.2.1) has the form

$$A(z) \;=\; z^q\{A_0 \;+ z^{-1}\tilde{A}(z)\},$$

where q is a positive integer, $\tilde{A} \in gl(n,\mathbf{C}\{z^{-1}\})$, and $A_0 \in gl(n,\mathbf{C})$. We also assume that A_0 has the form

$$A_0 = \begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & E_p \end{bmatrix}, \tag{A.4.2.13}$$

where

(i) for each j, E_j is an $n_j \times n_j$ matrix and $\sum_{p\geq j\geq 1} n_j \;=\; n$,
(ii) for each j, there exists a complex number λ_j such that $E_j - \lambda_j I_j$ is nilpotent; here I_j is the $n_j \times n_j$ identity matrix,
(iii) $p \geq 2$,
(iv) $\lambda_j \neq \lambda_k$ if j and k are distinct.

LEMMA A.4.2.5. *There exists an $n \times n$ matrix $T(z)$ such that*

(1) $T \;-\; I \in z^{-1}\; gl(n,\mathbf{C}\{z^{-1}\}_{\frac{1}{q}})$, *where I is the $n \times n$ identity matrix,*
(2) *the transformation*

$$W = T(z)V \tag{A.4.2.14}$$

changes differential equation (A.4.2.1) *to a differential equation*

$$\delta V \;+\; z^q B(z)V \;=\; 0 \tag{A.4.2.15}$$

with a matrix $B(z)$ of the form

$$B(z) = \begin{bmatrix} F_1(z) & 0 & \cdots & 0 \\ 0 & F_2(z) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & F_p(z) \end{bmatrix}, \qquad \text{(A.4.2.16)}$$

where, for each j,

$$F_j(z) - E_j \in z^{-1} gl(n_j, \mathbf{C}\{z^{-1}\}). \qquad \text{(A.4.2.17)}$$

REMARK A.4.2.6.

(a) We can prove Lemma A.4.2.5 by means of the argument given in §5.5 together with Theorem A.2.1.9, the analytization Theorem A.3.3.1, and Theorem A.4.1.2.

(b) The uniqueness of factors $P_1, \cdots, P_\ell$ on the right-hand side of factorization (A.4.2.11) can be explained as follows. Suppose that another factorization of X is given as

$$X = Q_\ell(\zeta)Q_{\ell-1}(\zeta)\cdots Q_1(\zeta),$$

where

$$Q_j \in gl(n, \mathbf{C}\{\zeta^{-1}\}_{\frac{1}{k_j}}) \qquad (j = 1, \cdots, \ell).$$

Then

$$Q_1 P_1^{-1} = \{P_\ell \cdots P_2\}\{Q_\ell \cdots Q_2\}^{-1}. \qquad \text{(A.4.2.18)}$$

The left-hand side of (A.4.2.18) belongs to $GL(n, \mathbf{C}\{\zeta^{-1}\}_{\frac{1}{k_1}})$, whereas the right-hand side of (A.4.2.18) belongs to $GL(n, \mathbf{C}[[\zeta^{-1}]]_{\frac{1}{k_2}})$. Hence by virtue of (A.4.0.2) (cf. (f) of Remark A.4.0.6) we have $Q_1 P_1^{-1} \in GL(n, \mathbf{C}\{z^{-1}\})$. In this way we can prove that the factors P_j are unique modulo $GL(n, \mathbf{C}\{z^{-1}\})$, i.e.,

$$Q_j = P_j R_j, \qquad R_j \in GL(n, \mathbf{C}\{z^{-1}\}) \qquad (j = 1, \cdots, \ell).$$

Note that the R_j are not independent of each other.

(c) Each P_j has a convergent inverse factorial series expansion (cf. (a) of Remark A.4.0.6).

(d) Matrix (A.4.2.7) always has a formal inverse factorial series expansion. H. L. Turrittin [147] showed that such an expansion of the matrix X is convergent if $\ell = 1$. He left other cases open. B. Malgrange and J.-P. Ramis gave an answer to the Turrittin problem by proving Theorem A.4.2.3 (cf. J.-P. Ramis [124, 127]). J.-P. Ramis [126, 128] applied factorization (A.4.2.11) to the study of Galois groups of linear differential equations. Note that, if a direction α is not in any of the singular supports of P_j, the sums F_j of P_j define an analytic solution $F_\ell F_{\ell-1} \cdots F_1$ of differential equation (A.4.2.8). J.-P.Ramis used this fact to establish a canonical isomorphism between the formal Galois group and the analytic Galois group.

(e) For some other information concerning factorization (A.4.2.11) see also W. Jurkat [77], W. Balser [44, 45, 47] and Balser–Jurkat–Lutz [50].

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APPENDIX 5

Differential Equations Containing Parameters

§A.5.0. Preliminaries

In the case when a given linear differential equation contains auxiliary parameters $\vec{t} = (t_1, \cdots, t_p)$, the structure of analytic continuation (or of monodromy) and the Stokes structure defined by this equation also depend on $\vec{t}$. Generally speaking, it takes much harder work than what is usually required to solve *good exercises* if we try to find the dependence of those structures on the parameters $\vec{t}$ (cf. for example Y. Sibuya [9, 29, 30, 31], W. Balser [46], D. G. Babbitt–V. S. Varadarajan [37, 38, 39, 40, 41] and R. Schäfke [131]). In this appendix we shall explain some aspects of

(1) isomonodromic deformations (§A.5.2),
(2) isoformal deformations (§A.5.3), and
(3) singular perturbations (§A.5.4).

REMARK A.5.0.1. Given a particular situation in which we study a problem of asymptotic analysis containing some auxiliary parameters, we must use a definition of asymptotic expansions in many variables that is most suitable for the situation. H. Majima generalized Theorem 6.4.1 to the case of several complex variables by utilizing a concept of *strong asymptotic expansions* in many variables. We shall explain the Majima asymptotics in §A.5.1. For details of the Majima asymptotics, see H. Majima [95, 96, 97].

§A.5.1. The Majima asymptotics

In any asymptotic analysis, it is very important to define *flatness* in a suitable way. Let us consider a sectorial domain

$$\mathscr{D}(\vec{R}, \vec{a}, \vec{b}) = \left\{ \prod_{m=1}^{n_0} \mathscr{D}(R_m, a_m, b_m) \right\} \times \left\{ \prod_{m=n_0+1}^{n} \mathscr{D}(R_m) \right\} \quad \text{(A.5.1.1)}$$

in $\mathbf{C}^n$, where $\vec{R} = (R_1, \cdots, R_n)$, $\vec{a} = (a_1, \cdots, a_{n_0})$, $\vec{b} = (b_1, \cdots, b_{n_0})$, and

$$\begin{cases} \mathscr{D}(R_m, a_m, b_m) &= \{z_m \ ; \ a_m < \arg z_m < b_m, \ |z_m| > R_m\}, \\ \mathscr{D}(R_m) &= \{z_m \ ; \ |z_m| > R_m\}. \end{cases}$$

Here a point in $\mathbf{C}^n$ is denoted by $\vec{z} = (z_1, \cdots, z_n)$. H. Majima [95] defines a flat function $f(\vec{z})$ as follows. An $f \in \mathscr{O}(\mathscr{D}(\vec{R}, \vec{a}, \vec{b}))$ is said to be flat as $(z_1, \cdots, z_{n_0}) \to (\infty, \cdots, \infty)$ uniformly on $\prod_{m=n_0+1}^{n} \mathscr{D}(R_m)$ if, for every n_0-tuple $\vec{N} = (N_1, \cdots, N_{n_0})$ of nonnegative integers and every $(\vec{\rho}, \vec{\alpha}, \vec{\beta})$ satisfying the conditions

$$\begin{cases} \rho_m > R_m & (m = 1, \cdots, n), \\ a_m < \alpha_m < \beta_m < b_m & (m = 1, \cdots, n_0), \end{cases} \tag{A.5.1.2}$$

there exists a nonnegative constant $K[\vec{N}, \vec{\rho}, \vec{\alpha}, \vec{\beta}]$ such that

$$|f(\vec{z})| \leq K[\vec{N}, \vec{\rho}, \vec{\alpha}, \vec{\beta}] \prod_{m=1}^{n_0} |z_m|^{-N_m} \quad \text{for} \quad \vec{z} \in \mathscr{D}(\vec{\rho}, \vec{\alpha}, \vec{\beta}). \tag{A.5.1.3}$$

Note that, if f is flat in this sense, then its integral $\int_a^b f(\vec{z})\, dz_j$ with respect to an entry z_j of $\vec{z}$ is also flat as a function of other entries $(\cdots, z_{j-1}, z_{j+1}, \cdots)$.

Majima defines strongly asymptotically developable functions as follows: Denote by $\mathscr{G}$ the set of all nonempty subsets of $\{1, \cdots, n_0\}$ and by $\sigma(J)$ $(J \in \mathscr{G})$ the number of elements of J. Let us identify with $\mathbf{C}^{n-\sigma(J)}$ the subspace of $\mathbf{C}^n$ defined by $z_j = \infty$ $(j \in J)$, where $\mathbf{C}^0 = \{0\}$. Also we define maps

$$p_J : \mathbf{C}^n \to \mathbf{C}^{n-\sigma(J)}$$

by

$$p_J(\vec{z}) = \vec{z}\big|_{z_j = \infty \text{ for } j \in J} \tag{A5.1.4}$$

respectively. Set

$$p_J(\mathscr{D}(\vec{R}, \vec{a}, \vec{b})) = \left\{ \prod_{m \notin J} \mathscr{D}(R_m, a_m, b_m) \right\} \times \left\{ \prod_{m=n_0+1}^{n} \mathscr{D}(R_m) \right\}.$$

DEFINITION A.5.1.1. A function $f(\vec{z})$ is said to be **strongly asymptotically developable** on $\mathscr{D}(\vec{R}, \vec{a}, \vec{b})$ if

(i) $f \in \mathscr{O}(\mathscr{D}(\vec{R}, \vec{a}, \vec{b}))$ and bounded on $\mathscr{D}(\vec{R}, \vec{a}, \vec{b})$,

(ii) there exists a set of functions

$$\mathscr{F} = \{f_J(p_J(\vec{z}) \ ; \ \vec{\ell}) \ ; \ J \in \mathscr{G}, \ \vec{\ell} \in N^{\sigma(J)} \} \tag{A.5.1.5}$$

such that

(1) $f_J(p_J(\vec{z})\,;\,\vec{\ell})$ is holomorphic and bounded on $p_J(\mathscr{D}(\vec{R},\vec{a},\vec{b}))$,

(2)

$$\lim_{t_j\to 0\ (j\in J)} \left[\prod_{j\in J}\left(\frac{\partial}{\partial t_j}\right)^{\ell_j}\right] f\left(\frac{1}{\vec{t}}\right) = \left(\prod_{j\in J}(\ell_j!)\right) f_J\left(p_J\left(\frac{1}{\vec{t}}\right);\,\vec{\ell}\right)$$

uniformly on every sectorial domain $\mathscr{D}(\vec{\rho},\vec{\alpha},\vec{\beta})$ (of $\vec{z}=\frac{1}{\vec{t}}$) satisfying condition (A.5.1.2), where N is the set of all non-negative integers, the ℓ_j $(j\in J)$ are the entries of $\vec{\ell}$ and $\frac{1}{\vec{t}}=\left(\frac{1}{t_1},\cdots,\frac{1}{t_n}\right)$. The set $\mathscr{F}$ is called **the set of coefficients of asymptotic expansion** of f on $\mathscr{D}(\vec{R},\vec{a},\vec{b})$.

DEFINITION A.5.1.2. A set of functions

$$\mathscr{F} = \{f_J(p_J(\vec{z})\,;\,\vec{\ell})\,;\ J\in\mathscr{G},\ \vec{\ell}\in N^{\sigma(J)}\}$$

is said to be **consistent** on $\mathscr{D}(\vec{R},\vec{a},\vec{b})$ if

(i) $f_J(p_J(\vec{z})\,;\,\vec{\ell})\in\mathscr{O}(p_J(\mathscr{D}(\vec{R},\vec{a},\vec{b})))$ and bounded on $p_J(\mathscr{D}(\vec{R},\vec{a},\vec{b}))$,

(ii)

$$\lim_{t_j\to 0\ (j\in J-K)} \left[\prod_{j\in J-K}\left(\frac{\partial}{\partial t_j}\right)^{\ell_j}\right] f_K\left(p_K\left(\frac{1}{\vec{t}}\right);\,\vec{k}\right)$$

$$= \left(\prod_{j\in J-K}(\ell_j!)\right) f_J\left(p_J\left(\frac{1}{\vec{t}}\right);\,(\vec{\ell},\vec{k})\right),$$

whenever $K\subset J$, uniformly on every sectorial domain $\mathscr{D}(\vec{\rho},\vec{\alpha},\vec{\beta})$ (of $\vec{z}=\frac{1}{\vec{t}}$) satisfying condition (A.5.1.2), where $\vec{\ell}\in N^{\sigma(J-K)}$, $\vec{k}\in N^{\sigma(K)}$ and $(\vec{\ell},\vec{k})\in N^{\sigma(J)}$.

OBSERVATION A.5.1.3. Let us denote by $\mathscr{A}(\vec{R},\vec{a},\vec{b})$ the set of all strongly asymptotically developable functions on $\mathscr{D}(\vec{R},\vec{a},\vec{b})$. Then $\mathscr{A}(\vec{R},\vec{a},\vec{b})$ is a commutative differential algebra over $\mathbf{C}$. Furthermore, for every consistent set $\mathscr{F}$ on $\mathscr{D}(\vec{R},\vec{a},\vec{b})$, there exists an $f\in\mathscr{A}(\vec{R},\vec{a},\vec{b})$ such that $\mathscr{F}$ is the set of coefficients of asymptotic expansion of f on $\mathscr{D}(\vec{R},\vec{a},\vec{b})$. Note also that, if $f\in\mathscr{A}(\vec{R},\vec{a},\vec{b})$, then the set $\mathscr{F}$ of coefficients of asymptotic expansion of f is consistent on $\mathscr{D}(\vec{R},\vec{a},\vec{b})$.

REMARK A.5.1.4. If $\sigma(J) = n_0$ (*i.e.*, $J = \{1, \cdots, n_0\}$), then $f_J(p_J(\vec{z}) ; \vec{\ell}) \in \mathscr{O}(\prod_{m=n_0+1}^{n} \mathscr{D}(R_m))$. Furthermore , if $n_0 = n$, then f_J is a constant. Conversely, for any power series

$$\sum_{\vec{\ell} \in N^{n_0}} a_{\vec{\ell}}(z_{n_0+1}, \cdots, z_n)\ z_1^{-\ell_1} \cdots z_{n_0}^{-\ell_{n_0}} \in \mathbf{C}\{z_{n_0+1}^{-1}, \cdots, z_n^{-1}\}[[z_1^{-1}, \cdots, z_{n_0}^{-1}]]$$

and any sectorial domain $\mathscr{D}(\vec{R}, \vec{a}, \vec{b})$, there exists a set $\mathscr{F}$ of functions

$$\mathscr{F} = \{f_J(p_J(\vec{z}) ; \vec{\ell}) ; J \in \mathscr{G}, \vec{\ell} \in N^{\sigma(J)}\}$$

such that

(1) $\mathscr{F}$ is consistent on $\mathscr{D}(\vec{R}, \vec{a}, \vec{b})$,

(2) if $J = \{1, \cdots, n_0\}$, we have $f_J(p_J(\vec{z}) ; \vec{\ell}) = a_{\vec{\ell}}(z_{n_0+1}, \cdots, z_n)$ for all $\vec{\ell} \in N^{n_0}$.

OBSERVATION A.5.1.5. A function $f \in \mathscr{A}(\vec{R}, \vec{a}, \vec{b})$ is flat in the sense of Majima if all coefficients $f_J(p_J(\vec{z}) ; \vec{\ell})$ of asymptotic expansion of f are identically equal to zero on $\mathscr{D}(\vec{R}, \vec{a}, \vec{b})$.

OBSERVATION A.5.1.6. For $f \in \mathscr{A}(\vec{R}, \vec{a}, \vec{b})$ we can derive various expansions by computing the integrals of

$$\left[\prod_{j \in J} \left(\frac{\partial}{\partial t_j}\right)^{\ell_j}\right] f\left(\frac{1}{\vec{t}}\right)$$

with respect to t_j $(j \in J)$. For example, let us consider the case when $n = n_0 = 2$. In this case the set $\mathscr{F}$ of coefficients of asymptotic expansion of $f \in \mathscr{A}(\vec{R}, \vec{a}, \vec{b})$ is given by

$$\mathscr{F} = \{f_\ell(z_1), g_k(z_2), a_{k\ell} ; \ell, k \in N\},$$

where $f_\ell \in \mathscr{A}(R_1, a_1, b_1)$, $g_k \in \mathscr{A}(R_2, a_2, b_2)$, $a_{\ell k} \in \mathbf{C}$. We can show that

$$\begin{cases} \lim_{t_2 \to 0} \dfrac{\partial^\ell}{\partial t_2^\ell} f\left(\dfrac{1}{t_1}, \dfrac{1}{t_2}\right) = (\ell!)\ f_\ell\left(\dfrac{1}{t_1}\right), \\[2ex] \lim_{t_1 \to 0} \dfrac{\partial^k}{\partial t_1^k} f\left(\dfrac{1}{t_1}, \dfrac{1}{t_2}\right) = (k!)\ g_k\left(\dfrac{1}{t_2}\right), \\[2ex] \lim_{(t_1, t_2) \to (0,0)} \dfrac{\partial^{k+\ell}}{\partial t_1^k \partial t_2^\ell} f\left(\dfrac{1}{t_1}, \dfrac{1}{t_2}\right) = (k!)(\ell!)\ a_{k\ell}, \\[2ex] A_{sp}(f_\ell) = \displaystyle\sum_{k \geq 0} a_{k\ell}\ z_1^{-k}, \qquad A_{sp}(g_k) = \sum_{\ell \geq 0} a_{k\ell}\ z_2^{-\ell}. \end{cases}$$

Furthermore, for example, we can derive

$$\begin{cases} | f(z_1, z_2) - f_0(z_1) | = O(|z_2|^{-1}), \\ | f(z_1, z_2) - g_0(z_2) | = O(|z_1|^{-1}), \\ | f(z_1, z_2) - f_0(z_1) - g_0(z_2) + a_{00} | = O(|z_1|^{-1}|z_2|^{-1}), \\ | f(z_1, z_2) - f_0(z_1) - z_2^{-1} f_1(z_1) - g_0(z_2) - z_1^{-1} g_1(z_2) \\ + a_{00} + a_{10} z_1^{-1} + a_{01} z_2^{-1} + a_{11} z_1^{-1} z_2^{-1} | = O(|z_1|^{-2}|z_2|^{-2}), \text{ etc.} \end{cases}$$

Utilizing the concept of strong asymptotic developability, Majima generalized Theorem 6.4.1 to the case of several complex variables.

THEOREM A.5.1.7. *Assume that N sectorial domains*

$$\mathcal{S}_\ell = \mathcal{D}(\vec{R}, \vec{a}_\ell, \vec{b}_\ell) \qquad (\ell = 1, \cdots, N)$$

form a covering of the domain

$$\mathcal{D}(\vec{R}) = \prod_{m=1}^{n} \mathcal{D}(R_m).$$

and that $n \times n$ matrices $F_{k\ell}(\vec{z}) \in gl(n, \mathcal{A}(\mathcal{S}_k \cap \mathcal{S}_\ell))$ are defined for all (k, ℓ) satisfying the condition

$$\mathcal{S}_k \cap \mathcal{S}_\ell \neq \emptyset. \tag{A.5.1.6}$$

Also assume that, if

$$\mathcal{S}_h \cap \mathcal{S}_k \cap \mathcal{S}_\ell \neq \emptyset,$$

the relation

$$F_{hk}(\vec{z}) F_{k\ell}(\vec{z}) = F_{h\ell}(\vec{z}) \qquad (\vec{z} \in \mathcal{S}_h \cap \mathcal{S}_k \cap \mathcal{S}_\ell)$$

holds. Furthermore assume that the matrices

$$F_{k\ell} - I$$

are flat on $\mathcal{S}_k \cap \mathcal{S}_\ell$, where I is the $n \times n$ identity matrix. Then there exist positive numbers R'_m greater than R_m $(m = 1, \cdots, n)$ respectively and $n \times n$ matrices $Q_\ell(\vec{z})$ $(\ell = 1, \cdots, N)$ such that

(i) *$Q_\ell(\vec{z})$ and $Q_\ell(\vec{z})^{-1}$ belong to $gl(n, \mathcal{A}(\vec{R}', \vec{a}_\ell, \vec{b}_\ell))$,*
(ii) *$Q_\ell(\vec{z}) \to I$ as $(z_1, \cdots, z_{n_0}) \to (\infty, \cdots, \infty)$,*
(iii) *the relation*

$$F_{k\ell}(\vec{z}) = Q_k(\vec{z})^{-1} Q_\ell(\vec{z}) \qquad (\vec{z} \in \mathcal{S}'_k \cap \mathcal{S}'_\ell)$$

holds when condition (A.5.1.6) is satisfied, where $\mathcal{S}'_\ell = \mathcal{D}(\vec{R}', \vec{a}_\ell, \vec{b}_\ell)$.

REMARK A.5.1.8.

(i) For details of what we explained above, see H. Majima [95].

(ii) Majima derived existence and uniqueness of strongly asymptotically developable solutions of completely integrable Pfaffian systems in various cases. Also the Gevrey asymptotics were developed within the Majima asymptotics. For that information, see H. Majima [96, 97], Y. Haraoka [62] and T. Yagami [149].

§A.5.2. Isomonodromic deformations

Let $\mathscr{D}$ be a connected open set in the z-plane and let $A(z,\vec{t})$ be a matrix in $gl(n,\mathscr{O}(\mathscr{D}\times\mathscr{D}(\vec{r},\vec{\tau})))$, where $\vec{r}=(\rho_1,\cdots,\rho_p)$ is a p-tuple of positive numbers, $\vec{t}=(t_1,\cdots,t_p)\in\mathbf{C}^p$, and

$$\mathscr{D}(\vec{r},\vec{\tau}) = \prod_{h=1}^{p}\{t_h\ ;\ |t_h-\tau_h|<\rho_h\}.$$

Assume that $\{\Delta(z_j,r_j), W_j(z,\vec{t})\}$ is a system of fundamental solutions of the differential equation (cf. Definition 1.9.2)

$$\frac{dW}{dz}=A(z,\vec{t}) \tag{A.5.2.1}$$

for each fixed $\vec{t}\in\mathscr{D}(\vec{r},\vec{\tau})$, where $W_j(z,\vec{t})\in GL(n,\mathscr{O}(\Delta(z_j,r_j)\times\mathscr{D}(\vec{r},\vec{\tau})))$. Set

$$C_{jk}(\vec{t})=W_j(z,\vec{t})^{-1}W_k(z,\vec{t}) \tag{A.5.2.2}$$

whenever $\Delta(z_j,r_j)\cap\Delta(z_k,r_k)\neq\emptyset$. Then $C_{jk}(\vec{t})\in GL(n,\mathscr{O}(\mathscr{D}(\vec{r},\vec{\tau})))$. Moreover if $\Delta(z_j,r_j)\cap\Delta(z_k,r_k)\cap\Delta(z_h,r_h)\neq\emptyset$, we have

$$C_{jk}(\vec{t})C_{kh}(\vec{t})=C_{jh}(\vec{t}).$$

Hence $\{\Delta(z_j,r_j),C_{jk}(\vec{t})\}$ is a system of connection matrices on $\mathscr{D}$ for each fixed $\vec{t}\in\mathscr{D}(\vec{r},\vec{\tau})$ defined by the system $\{\Delta(z_j,r_j), W_j(z,\vec{t})\}$ of fundamental solutions of (A.5.2.1) (cf. Definition 1.10.1).

DEFINITION A.5.2.1. Differential equation (A.5.2.1) is said to be **isomonodromic** on $\mathscr{D}\times\mathscr{D}(\vec{r},\vec{\tau})$ if there exists a system $\{\Delta(z_j,r_j), W_j(z,\vec{t})\}$ of fundamental solutions of differential equation (A.5.2.1) such that

(i) $W_j(z,\vec{t})\in GL(n,\mathscr{O}(\Delta(z_j,r_j)\times\mathscr{D}(\vec{r},\vec{\tau})))$,

(ii) the connection matrices $C_{jk}=W_j(z,\vec{t})^{-1}W_k(z,\vec{t})$ defined on $\mathscr{D}$ by $\{\Delta(z_j,r_j), W_j(z,\vec{t})\}$ are independent of $\vec{t}\in\mathscr{D}(\vec{r},\vec{\tau})$.

REMARK A.5.2.2. Condition (ii) of Definition A.5.2.1 implies that the structure of analytic continuation (or of monodromy) defined on $\mathscr{D}$ by differential equation (A.5.2.1) is independent of $\vec{t}$. The author does not know whether the converse is also true. Note that the analyticity of $W_j(z,\vec{t})$ with respect to $\vec{t}$ is included in condition (i) of Definition A.5.2.1.

THEOREM A.5.2.3. *Differential equation* (A.5.2.1) *is isomonodromic on* $\mathscr{D} \times \mathscr{D}(\vec{r},\vec{\tau})$ *if and only if there exist p matrices* $B_h(z,\vec{t}) \in gl(n,\mathscr{O}(\mathscr{D} \times \mathscr{D}(\vec{r},\vec{\tau})))$ $(h = 1,\cdots,p)$ *such that the Pfaffian system*

$$\begin{cases} \dfrac{\partial W}{\partial z} = A(z,\vec{t})W, \\ \dfrac{\partial W}{\partial t_h} = B_h(z,\vec{t})W \qquad (h = 1, \cdots, p) \end{cases} \tag{A.5.2.3}$$

is completely integrable.

In fact, if $\{\Delta(z_j,r_j), W_j(z,\vec{t})\}$ satisfies all of the requirements of Definition A.5.2.1, we have

$$W_j(z,\vec{t})C_{jk} = W_k(z,\vec{t}), \qquad \frac{\partial W_j}{\partial t_h}(z,\vec{t})C_{jk} = \frac{\partial W_k}{\partial t_h}(z,\vec{t})$$

on $(\Delta(z_j,r_j) \cap \Delta(z_k,r_k)) \times \mathscr{D}(\vec{r},\vec{\tau})$. Hence

$$\frac{\partial W_j}{\partial t_h}(z,\vec{t})W_j(z,\vec{t})^{-1} = \frac{\partial W_k}{\partial t_h}(z,\vec{t})W_k(z,\vec{t})^{-1},$$

on $(\Delta(z_j,r_j) \cap \Delta(z_k,r_k)) \times \mathscr{D}(\vec{r},\vec{\tau})$. Therefore we can define $B_h(z,\vec{t})$ by

$$B_h(z,\vec{t}) = \frac{\partial W_j}{\partial t_h}(z,\vec{t})W_j(z,\vec{t})^{-1} \quad \text{on} \quad \Delta(z_j,r_j) \times \mathscr{D}(\vec{r},\vec{\tau}). \tag{A.5.2.4}$$

If Pfaffian system (A.5.2.3) is completely integrable, then we can define $\{\Delta(z_j,r_j), W_j(z,\vec{t})\}$ by solving system (A.5.2.3) on $\Delta(z_j,r_j) \times \mathscr{D}(\vec{r},\vec{\tau})$.

OBSERVATION A.5.2.4. Pfaffian system (A.5.2.3) is completely integrable if and only if

$$\begin{cases} \dfrac{\partial B_h}{\partial z} - \dfrac{\partial A}{\partial t_h} = [A, B_h] \qquad (h = 1, \cdots, p), \\[2ex] \dfrac{\partial B_h}{\partial t_k} - \dfrac{\partial B_k}{\partial t_h} = [B_k, B_h] \qquad (h,\ k = 1, \cdots, p), \end{cases} \tag{A.5.2.5}$$

where

$$[X,Y] = XY - YX.$$

REMARK A.5.2.5. To determine whether differential equation (A.5.2.1) is isomonodromic on $\mathscr{D} \times \mathscr{D}(\vec{r},\vec{\tau})$, we must try to find p matrices $B_h(z,\vec{t}) \in gl(n,\mathscr{O}(\mathscr{D} \times \mathscr{D}(\vec{r},\vec{\tau})))$ $(h = 1, \cdots, p)$ by solving equation (A.5.2.5). In some particular cases where more detailed information of the matrix $A(z,\vec{t})$ is available, we can find some necessary properties of $B_h(z,\vec{t})$ through (A.5.2.4). For example, if $A(z,\vec{t})$ is rational in z and if differential equation (A.5.2.1) is of Fuchsian type (cf. Definition 4.3.5), the matrices $B_h(z,\vec{t})$ are also rational in z. In such cases we might be able to derive necessary and sufficient conditions on the matrix $A(z,\vec{t})$ such that differential equation (A.5.2.1) is isomonodromic.

EXAMPLE A.5.2.6 (L. Schlesinger [132]). Let us consider differential equation (A.5.2.1) in the case when

$$A(z,\vec{t}) = \sum_{h=1}^{p} \frac{A_h(\vec{t})}{z - t_h}, \qquad \text{(A.5.2.6)}$$

where $A_h \in gl(n,\mathcal{O}(\mathcal{D}(\vec{r},\vec{\tau})))$. Set

$$\mathcal{D} = \{z;\ |z-\tau_h| > \rho_h \qquad (h = 1, \cdots, p)\}.$$

We assume that the positive numbers ρ_h are sufficiently small. Assume also that

(1) any two eigenvalues of $A_h(\vec{t})$ do not differ by integer,
(2) any two eigenvalues of $\sum_{1\leq h\leq p} A_h(\vec{t})$ do not differ by integer,
(3) the matrix $\sum_{1\leq h\leq p} A_h(\vec{t})$ is independent of $\vec{t}$.

Then, if differential equation (A.5.2.1) is isomonodromic, the matrices $B_h(z,\vec{t})$ defined by (A.5.2.4) must have the form

$$B_h(z,\vec{t}) = -\frac{A_h(\vec{t})}{z - t_h} + C_h(\vec{t}) \qquad (h = 1,\cdots,p), \qquad \text{(A.5.2.7)}$$

where $C_h(\vec{t}) \in gl(n,\mathcal{O}(\mathcal{D}(\vec{r},\vec{\tau})))$. Inserting (A.5.2.7) into equation (A.5.2.5), we derive

$$\frac{\partial C_h}{\partial t_k} - \frac{\partial C_k}{\partial t_h} = [C_k, C_h].$$

This means that the Pfaffian system

$$\frac{\partial R}{\partial t_h} = C_h(\vec{t})R \qquad (h = 1,\cdots,p)$$

is completely integrable on $\mathcal{D}(\vec{r},\vec{\tau})$. Therefore there exists a matrix $R \in GL(n,\mathcal{O}(\mathcal{D}(\vec{r},\vec{\tau})))$ such that

$$C_h(\vec{t}) = \frac{\partial R}{\partial t_h}R(\vec{t})^{-1} \qquad (h = 1,\cdots,p).$$

From this fact we can easily conclude that it will suffice to consider the case when all of the matrices $C_h(\vec{t})$ are identically equal to zero. Now we look at the Pfaffian system

$$\begin{cases} \dfrac{\partial W}{\partial z} = \displaystyle\sum_{h=1}^{p} \frac{A_h(\vec{t})}{z - t_h}\, W, \\[2ex] \dfrac{\partial W}{\partial t_h} = -\dfrac{A_h(\vec{t})}{z - t_h}\, W \qquad (h = 1,\cdots,p). \end{cases} \qquad \text{(A.5.2.8)}$$

THEOREM A.5.2.7 (L. Schlesinger [132]). *Pfaffian system* (A.5.2.8) *is completely integrable if and only if*

$$\begin{cases} \dfrac{\partial A_h}{\partial t_k} = \dfrac{[A_h, A_k]}{t_h - t_k} \quad (h \neq k), \\ \\ \dfrac{\partial A_h}{\partial t_h} = -\sum_{k \neq h} \dfrac{[A_h, A_k]}{t_h - t_k}. \end{cases} \qquad \text{(A.5.2.9)}$$

REMARK A.5.2.8.

(I) To prove Theorem A.5.2.7 it is not necessary to assume condition (3) given above. As a matter of fact, the Schlesinger equation (A.5.2.9) implies that $\sum_{h=1}^{p} A_h(\vec{t})$ is independent of $\vec{t}$.

(II) Pfaffian system (A.5.2.9) is completely integrable. Hence for any initial condition at an ordinary point, there exists a unique solution of (A.5.2.9) in a neighborhood of the initial point.

OBSERVATION A.5.2.9. In the case when differential equation (A.5.2.1) admits irregular singular points, the matrices $B_h(z, \vec{t})$ $(h = 1, \cdots, p)$ defined by (A.5.2.4) are not necessarily rational in z even if (A.5.2.1) is isomonodromic. For example if the entries of the matrix $A(z, \vec{t})$ are polynomials in z, fundamental solutions $W_j(z, \vec{t})$ are entire in z no matter what. Hence (A.5.2.1) is isomonodromic. However $B_h(z, \vec{t})$ $(h = 1, \cdots, p)$ defined by (A.5.2.4) are not necessarily polynomials in z. In order that the B_h have at worst poles at an irregular singular point ζ, it is necessary and sufficient that

$$\frac{\partial W_j}{\partial t_h} W_j^{-1} = O(|z - \zeta|^{-M})$$

for some positive number M in the neighborhood of ζ. To explain how to construct such a system $\{\Delta(z_j, r_j), W_j(z, \vec{t})\}$ of fundamental solutions of differential equation (A.5.2.1), let us make the following observation. Assuming that $z = \infty$ is an irregular singular point of (A.5.2.1), we shall try to define a Stokes phenomenon according to Definitions 6.10.2, 6.10.3 and 6.10.5.

Step I: We fix another differential equation

$$\frac{dU}{dz} = A_0(z, \vec{t})U \qquad \text{(A.5.2.10)}$$

so that $A_0 \in gl(n, \mathscr{O}(\mathscr{D}(R) \times \mathscr{D}(\vec{r}, \vec{t})))$, where $\mathscr{D}(R) = \{z;\ |z| > R\}$ and that there exists $\hat{P}(z, \vec{t}) \in GL(n, \mathscr{O}(\mathscr{D}(\vec{r}, \vec{t}))[[z^{-1}]][z])$ such that the transfomation

$$W = \hat{P}(z, \vec{t})U \qquad \text{(A.5.2.11)}$$

changes (A.5.2.1) to (A.5.2.10).

Step II: We fix a covering $\{\mathcal{S}_1, \cdots, \mathcal{S}_N\}$ at $z = \infty$ and fundamental solutions $\Phi_j(z, \vec{t})$ of (A.5.2.10) on $\mathcal{S}_j$ $(j = 1, \cdots, N)$ in such a way that

(a) $\Phi_j \in GL(n, \mathcal{O}(\mathcal{S}_j \times \mathcal{D}(\vec{r}, \vec{t})))$,

(b)

$$\frac{\partial \Phi_j}{\partial t_h}(z, \vec{t})\Phi_j(z, \vec{t})^{-1} = O(|z|^M) \qquad as \quad z \to \infty \text{ in } \mathcal{S}_j \quad (j = 1, \cdots, N), \tag{A.5.2.12}$$

where M is a positive number.

Step III: We fix a system $\{\mathcal{S}_j, P_j(z, \vec{t})\Phi_j(z, \vec{t}); j = 1, \cdots, N\}$ of fundamental solutions of differential equation (A.5.2.1) in the neighborhood of $z = \infty$ (cf. Definition 6.10.2), where $P_j(z, \vec{t})$ and $P_j(z, \vec{t})^{-1}$ are in $gl(n, \mathcal{A}(\mathcal{S}_j \times \mathcal{D}(\vec{r}, \vec{\tau})))$. [The definition of $\mathcal{A}(\mathcal{S}_j \times \mathcal{D}(\vec{r}, \vec{\tau}))$ is analogous to the definition of $\mathcal{A}(\vec{R}, \vec{a}, \vec{b})$ in §A.5.1 (cf. Observation A.5.1.3).]

Now let us set

$$W_j(z, \vec{t}) = P_j(z, \vec{t})\Phi_j(z, \vec{t}) \qquad (j = 1, \cdots, N).$$

Then we have

$$\frac{\partial W_j}{\partial t_h}(z, \vec{t})W_j(z, \vec{t})^{-1} = O(|z|^{M'})$$

for some positive number M' as $z \to \infty$ in $\mathcal{S}_j$. Note that if we set

$$W_j(z, \vec{t})^{-1}W_k(z, \vec{t}) = C_{jk}(\vec{t}) \tag{A.5.2.13}$$

whenever $\mathcal{S}_j \cap \mathcal{S}_k \neq \varnothing$, then $\{A_0(z, \vec{t}), \{\mathcal{S}_j, \Phi_j\}, C_{jk}(\vec{t})\}$ is the Stokes phenomenon defined at $z = \infty$ by the system $\{\mathcal{S}_j, W_j(z, \vec{t});\ j = 1, \cdots, N\}$ of fundamental solutions of differential equation (A.5.2.1). If we choose the system $\{\Delta(z_j, r_j), W_j(z, \vec{t})\}$ of fundamental solutions of Definition A.5.2.1 in this way and if the connection matrices C_{jk} are independent of $\vec{t}$, then the matrices B_h will be rational in z even if (A.5.2.1) admits irregular singular points.

REMARK A.5.2.10.

(α) Isomonodromic differential equations of form (A.5.2.1) are extensively studied by T. Miwa [110], Jimbo–Miwa [73, 74], Jimbo–Miwa–Ueno [76], K. Ueno [46] and B. Malgrange [102]. [See also Lin–Sibuya [88, 89]].

(β) Steps I and II of Observation A.5.2.9 can be worked out under certain assumptions by utilizing the Hukuhara–Turrittin Theorem 6.8.1 together with suitable asymptotic analysis (cf. K. Ueno [146] and Y. Sibuya [29]).

OBSERVATION A.5.2.11. Let us look at the second-order linear differential equation

$$\frac{d^2y}{dz^2} + p_1(z,t)\frac{dy}{dz} + p_2(z,t)y = 0 \tag{A.5.2.14}$$

where $(z,t) \in \mathbf{C}^2$ and $p_j \in \mathscr{O}(\mathscr{D}\times\mathscr{D}(\rho,\tau))$ $(j = 1,2)$. Differential equation (A.5.2.14) is equivalent to differential equation (A.5.2.1) if we set

$$A(z,t) = \begin{bmatrix} 0 & 1 \\ -p_2(z,t) & -p_1(z,t) \end{bmatrix}. \tag{A.5.2.15}$$

Pfaffian system (A.5.2.3) and integrability condition (A.5.2.5) become

$$\begin{cases} \dfrac{\partial W}{\partial z} = A(z,t)W, \\ \dfrac{\partial W}{\partial t} = B(z,t)W \end{cases} \tag{A.5.2.16}$$

and

$$\frac{\partial B}{\partial z} - \frac{\partial A}{\partial t} = [A,B], \tag{A.5.2.17}$$

respectively. If we set

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

condition (A.5.2.17) is equivalent to

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial^3 B_{12}}{\partial z^3} - 2p\dfrac{\partial B_{12}}{\partial z} - \dfrac{\partial p}{\partial z}B_{12} + \dfrac{\partial p}{\partial t} = 0, \\ \dfrac{\partial B_{11}}{\partial z} = \dfrac{1}{2}\left\{\dfrac{\partial(p_1B_{12})}{\partial z} - \dfrac{\partial^2 B_{12}}{\partial z^2} - \dfrac{\partial p_1}{\partial t}\right\}, \\ B_{21} = \dfrac{\partial B_{11}}{\partial z} - p_2B_{12}, \\ B_{22} = \dfrac{\partial B_{12}}{\partial z} + B_{11} - p_1B_{12}, \end{cases} \tag{A.5.2.18}$$

where $p = \frac{1}{2}\frac{\partial p_1}{\partial z} + \frac{1}{4}p_1^2 - p_2$.

OBSERVATION A.5.2.12. Assume that

$$\begin{cases} p_1(z,t) = \dfrac{1-\theta_0}{z} + \dfrac{1-\theta_1}{z-1} + \dfrac{1-\theta}{z-t} - \dfrac{1}{z-\lambda}, \\ p_2(z,t) = \dfrac{\kappa}{z(z-1)} + \dfrac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)} - \dfrac{t(t-1)H}{z(z-1)(z-t)}, \end{cases} \tag{A.5.2.19}$$

where θ_0, θ_1, θ, and κ are independent of t, while λ, μ, and H are functions of t. The points at $z = 0, 1, t$, and ∞ are regular singular points. We also assume that the point at $z = \lambda$ is an apparent singular point. This requirement fixes the function H as follows:

$$\begin{aligned} H = \frac{1}{t(t-1)}[\lambda(\lambda-1)(\lambda-t)\mu^2 - \{\theta_0(\lambda-1)(\lambda-t) + \theta_1\lambda(\lambda-t) \\ + (\theta-1)\lambda(\lambda-1)\}\mu + \kappa(\lambda-t)]. \end{aligned} \tag{A.5.2.20}$$

THEOREM A.5.2.13 (K. Okamoto [117]). *Under the assumption given above, differential equation* (A.5.2.14) *is isomonodromic if and only if* (λ, μ) *satisfies the Hamiltonian system*

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \qquad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda}. \tag{A.5.2.21}$$

Furthermore system (A.5.2.21) *is equivalent to the Painlevé equation* VI:

$$\frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}$$

$$+ \frac{\lambda(\lambda-1)(\lambda-t)}{2t^2(t-1)^2}\left[\theta_\infty^2 - \theta_0^2\frac{t}{\lambda^2} + \theta_1^2\frac{t-1}{(\lambda-1)^2} + (1-\theta^2)\frac{t(t-1)}{(\lambda-t)^2}\right],$$

where $\theta_\infty^2 = (\theta_0 + \theta_1 + \theta - 1)^2 - 4\kappa$.

REMARK A.5.2.14.

(1) By coalescing singular points in suitable ways, we can derive similar results for other Painlevé equations (cf. K. Okamoto [115, 117]).
(2) General solutions of Painlevé equations do not have *movable* branch points. Solutions to Pfaffian system (A.5.2.9) also do not have movable branch points. In more general situations, the Painlevé properties of isomonodromic differential equations were extensively investigated by T. Miwa [109] and B. Malgrange [102].
(3) In order to derive the Painlevé properties of isomonodromic differential equations, we must choose parameters in suitable ways. Generally speaking, as we mentioned at the end of Chapter 4 (§4.5), it becomes necessary to introduce some apparent singular points if we consider a Riemann problem over the Riemann sphere. In the case of a single nth-order linear differential equation

$$\sum_{0 \le h \le n} a_h(z)y^{(h)} = 0$$

of Fuchsian type, the number of apparent singular points that should be introduced to solve a given Riemann problem is known in terms of the number of prescribed singular points and the order n of the differential equation provided that the prescribed monodromy data be irreducible (cf. M. Ohtsuki [119]). Let $\vec{t} = (t_1, \cdots, t_p)$ be the prescribed singular points and let $\vec{\lambda} = (\lambda_1, \cdots, \lambda_k)$ be the introduced apparent singular points. Then, in certain cases of second-order linear differential equations, conditions that the given differential equation be isomonodromic are given by means of a system of differential equations on $\vec{\lambda}$ as a function of $\vec{t}$. Furthermore, such a system on $\vec{\lambda}$ are equivalent to a Hamiltonian system which is in turn equivalent to a 2×2 Schlesinger system (A.5.2.9) (cf. K. Okamoto [116, 117] and H. Kimura–K. Okamoto [79, 80]). There are many interesting open questions in this territory (cf. T. Kimura [78]).

(4) W. Dekkers [54] showed that, in the case of 2×2 matrices, it is not necessary to introduce any apparent singular points to solve the Riemann problem on the Riemann sphere.

(5) K. Okamoto [112, 113, 118] studied isomonodromic differential equations also on a torus. Recently, an intensive study on isomonodromic differential equations on a closed Riemann surface has been done by K. Iwasaki [71]. [See also D. A. Hejhal [19]].

(6) Although the Painlevé equations I $\sim$ VI and their relations with isomonodromic differential equations had been known for a long time, B. M. McCoy–C. A. Tracy–T. T. Wu [108] and Jimbo–Miwa–Sato–Môri [75] revived strong interest in that territory.

§A.5.3. Isoformal deformations

In §A.5.2 we explained isomonodromic differential equations with irregular singular points (cf. Observation A.5.2.9). There we suggested that, in Steps I and II, we may use the Hukuhara–Turrittin transformation (6.8.2) (cf. (β) of remark A.5.2.10). However, to do this, we must show that such a transformation is analytic with respect to the auxiliary variable $\vec{t}$. Generally speaking this is not at all a trivial enterprise. As a matter of fact, we can not expect such a property of the Hukuhara–Turrittin transformation except for very restricted cases. The main concern is to find a decent sufficient condition such that the Hukuhara–Turrittin transformation is analytic in $\vec{t}$. So far the sufficient condition due to D. G. Babbitt and V. S. Varadarajan [37, 38, 41] has been the best. Their condition applies to the case when the standard form (6.8.3) of **Hukuhara–Turrittin** is independent of $\vec{t}$, even though the matrix A of the given differential equation depends on $\vec{t}$ (i.e., the case when the given differential equation is *isoformal*). By utilizing such a result, Babbitt–Varadarajan [39, 40] investigated the analytic structure of the moduli space of analytic linear differential equations with respect to the meromorphic classification (cf. the end of §A.3.2). [Their method is based on the theory of Lie algebra. Recently, R. Schäfke [131] indicated that the Babbitt–Varadarajan transformation could be constructed by means of a more traditional method.]

§A.5.4. Singular perturbations and the Gevrey asymptotics

As we indicated in §A.2.5 some results concerning the Gevrey asymptotics were obtained through studies of singular perturbations. However, so far there have been only a limited number of significant applications of the Gevrey asymptotics to singular perturbations. We must realize that many works of singular perturbations do not utilize complex analysis other than results concerning turning points. Nevertheless, we believe that the Gevrey asymptotics are useful whenever we need sharp results.

In this section, we shall explain how to derive the Gevrey property of formal solutions of a singularly perturbed differential equation:

$$\frac{d\vec{y}}{dx} = \lambda^{\sigma}\left\{\vec{f}(x,\lambda) + A(x,\lambda)\vec{y} + \sum_{|\vec{p}|\geq 2} \vec{f}_{\vec{p}}(x,\lambda)\,\vec{y}^{\vec{p}}\right\}, \qquad \text{(A.5.4.1)}$$

where

(i) σ is a positive integer; $\vec{y}$, $\vec{f}$, and $\vec{f}_{\vec{p}}$ are n-dimensional column vectors; A is an $n \times n$ matrix; $\vec{p} = (p_1, \cdots, p_n)$ is an n-tuple of nonnegative integers; $|\vec{p}| = \sum_{1\leq h\leq n} p_h$; $\vec{y}^{\vec{p}} = y_1^{p_1}\cdots y_n^{p_n}$, and y_j is the jth entry of the column vector $\vec{y}$,

(ii) the entries of $\vec{f}$, $\vec{f}_{\vec{p}}$ and A belong to $\mathscr{O}(\mathscr{D}(r,R))$, where r and R are positive numbers and

$$\mathscr{D}(r,R) = \{(x,\lambda)\ ;\ |x| < r,\ \ |\lambda| > R\};$$

these functions are also assumed to be bounded on $\mathscr{D}(r,R)$,

(iii) the series $\sum_{|\vec{p}|\geq 2} \vec{f}_{\vec{p}}(x,\lambda)\,\vec{y}^{\vec{p}}$ is convergent uniformly on the domain

$$\mathscr{D}(r,R) \times \{\vec{y}\ ;\ |y_j| < \rho \qquad j = 1, \cdots, n\},$$

where ρ is a positive number,

(iv) $\vec{f}(0,\infty) = 0$,

(v) $A(,0,\infty)$ is invertible.

Under the assumptions given above, we shall explain how to prove the following theorem.

THEOREM A.5.4.1 (Y. Sibuya [139]). *Differential equation* (A.5.4.1) *admits a unique formal solution*

$$\vec{y} = \sum_{m\geq 0} \vec{a}_m(x)\lambda^{-m}, \tag{A.5.4.2}$$

such that $\vec{a}_0(0) = 0$ *and the entries of* $\vec{a}_m$ *belong to* $\mathscr{O}(\mathscr{D}(r_1))$, *where* r_1 *is a suitable positive number and* $\mathscr{D}(r_1) = \{x;\ |x| < r_1\}$. *Furthermore, the* $\vec{a}_m$ *satisfy the inequalities*:

$$|\vec{a}_m(x)| \leq C_0(m!)^{\frac{1}{\sigma}}\, C_1^m \qquad (m = 0, 1, \cdots) \tag{A.5.4.3}$$

on $\mathscr{D}(r_1)$, *where* C_0 *and* C_1 *are suitable positive numbers.*

OBSERVATION A.5.4.2. To illustrate the situation, let us look at the scalar equation

$$\frac{dy}{dx} - \lambda\{y - f_0(x)\} = 0.$$

The unique formal solution of this equation is given by

$$y = \sum_{m\geq 0} \frac{d^m f_0}{dx^m}(x)\, \lambda^{-m}.$$

Suppose that $f_0 \in \mathscr{O}(\mathscr{D}(r_0))$, $|f_0(x)| \leq M$ on $\mathscr{D}(r_0)$ and $0 < r_1 < r_0$. Then

$$\left|\frac{d^m f_0}{dx^m}(x)\right| \leq \left(\frac{r_0 M}{(r_0 - r_1)^{m+1}}\right)\, m!$$

on $\mathscr{D}(r_1)$. This example also shows that the assertion of Theorem A.5.4.1 is generally false if we do not assume analyticity of the entries of $\vec{f}_0$, $\vec{f}_{\vec{p}}$, and A with respect to x.

OBSERVATION A.5.4.3. The existence and uniqueness of formal solution (A.5.4.2) can be verified easily. The main problem is to verify inequalities (A.5.4.3). We can derive *recurrence formulas* to compute the $\vec{a}_m$ successively. However, since such formulas involve not only the $\vec{a}_m$ but also their derivatives with respect to x, it is extremely difficult to verify (A.5.4.3) by means of *recurrence formulas*.

OBSERVATION A.5.4.4. For every direction θ in the λ-plane, there exist a sectorial domain $\mathscr{S}(\theta) = \{\lambda\ ;\ |\arg\lambda - \theta| < \alpha(\theta),\ |\lambda| > R(\theta)\}$ and a disc $\Delta(\theta) = \{x\ ;\ |x| < r(\theta)\}$ such that differential equation (A.5.4.1) has a solution $\vec{y} = \vec{\phi}(x, \lambda; \theta)$ satisfying the inequalities:

$$\left|\vec{\phi}(x,\lambda;\theta) - \sum_{m=0}^{N} \vec{a}_m(x)\lambda^{-m}\right| \leq K_N\, |\lambda|^{-N-1} \tag{A.5.4.4}$$

on $\triangle(\theta) \times \mathscr{S}(\theta)$ for every nonnegative integer N, where $\alpha(\theta)$, $R(\theta)$, $r(\theta)$ and the K_N are suitable positive numbers (cf. Y. Sibuya [133]).

OBSERVATION A.5.4.5. We can choose a finite number of $\theta_1, \cdots, \theta_p$ in such a way that p sectorial domains $\mathscr{S}_j = \{\lambda ; |\arg\lambda - \theta_j| < \alpha(\theta_j), |\lambda| > R_0\}$ $(j = 1, \cdots, p)$ form a good covering at $\lambda = \infty$ (cf §A.2.0), where R_0 is a positive number greater than $R(\theta_1), \cdots, R(\theta_p)$. Set $\mathscr{D} = \cap_{1\leq j\leq p}\triangle(\theta_j)$. Now by utilizing differential equation (A.5.4.1), we can derive the following estimates:

$$|\vec{\phi}(x,\lambda;\theta_j) - \vec{\phi}(x,\lambda;\theta_{j+1})| < \gamma \exp(-\mu|\lambda|^\sigma) \quad \text{on} \quad \mathscr{D} \times (\mathscr{S}_j \cap \mathscr{S}_{j+1}), \tag{A.5.4.5}$$

where γ and λ are suitable positive numbers. In fact by setting

$$\vec{\delta}_j(x,\lambda) = \vec{\phi}(x,\lambda;\theta_j) - \vec{\phi}(x,\lambda;\theta_{j+1}),$$

we derive

$$|\delta_j(x,\lambda)| \leq 2K_N |\lambda|^{-N-1} \quad \text{on} \quad \mathscr{D} \times (\mathscr{S}_j \cap \mathscr{S}_{j+1}) \tag{A.5.4.6}$$

from (A.5.4.4) for every nonnegative integer N, and

$$\begin{aligned} &\frac{d\vec{\delta}_j(x,\lambda)}{dx} \\ &= \lambda^\sigma \Big\{A(x,\lambda)\vec{\delta}_j(x,\lambda) + \vec{F}(x,\lambda,\vec{\phi}(x,\lambda;\theta_{j+1}) + \vec{\delta}_j(x,\lambda)) \\ &\qquad -\vec{F}(x,\lambda,\vec{\phi}(x,\lambda;\theta_{j+1}))\Big\} \quad \text{on} \quad \mathscr{D} \times (\mathscr{S}_j \cap \mathscr{S}_{j+1}) \end{aligned} \tag{A.5.4.7}$$

from (A.5.4.1), where $\vec{F} = \sum_{|\vec{p}|\geq 2} \vec{f}_{\vec{p}}(x,\lambda)\, \vec{y}^{\vec{p}}$. As we did in §A.2.4, we can write (A.5.4.7) in the form

$$\frac{d\vec{\delta}_j(x,\lambda)}{dx} = \lambda^\sigma \{A(x,\lambda) + B(x,\lambda)\}\vec{\delta}_j(x,\lambda), \tag{A.5.4.8}$$

where

$$B(x,\lambda) = \int_0^1 \vec{F}_{\vec{y}}(x,\lambda,\vec{\phi}(x,\lambda;\theta_{j+1}) + t\vec{\delta}_j(x,\lambda))dt.$$

Here $\vec{F}_{\vec{y}}$ denotes the Jacobian matrix of $\vec{F}$ with respect to $\vec{y}$. Note that $B(0,\infty) = 0$. We can derive (A.5.4.5) from (A.5.4.6) and (A.5.4.8).

OBSERVATION A.5.4.6. To complete the proof of Theorem A.5.4.1, it suffices to generalize Lemma A.2.0.5 to the case when $\phi_1, \cdots, \phi_N$ depend on auxiliary parameters. In doing this, we can also improve estimates (A.5.4.4) as

$$|\vec{\phi}(x,\lambda;\theta) - \sum_{m=0}^{N} \vec{a}_m(x)\lambda^{-m}| \leq K(\theta)\, ((N+1)!)^{\frac{1}{\sigma}} L(\theta)^{N+1}|\lambda|^{-N-1} \tag{A.5.4.9}$$

for some positive numbers $K(\theta)$ and $L(\theta)$.

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This book is a translation of a 1976 book originally written in Japanese. The main attention is paid to intrinsic aspects of problems related to linear ordinary differential equations in complex domains. Examples of the problems discussed in the book include the Riemann problem on the Riemann sphere, a characterization of regular singularities, and a classification of meromorphic differential equations. Since the original book was published, many new ideas have developed, such as applications of D-modules, Gevrey asymptotics, cohomological methods, k-summability, and studies of differential equations containing parameters. Five appendices, added in the present edition, briefly cover these new ideas. In addition, more than 100 references have been added.

This book introduces the reader to the essential facts concerning the structure of solutions of linear differential equations in the complex domain and illuminates the intrinsic meaning of older results by means of more modern ideas. A useful reference for research mathematicians, this book would also be suitable as a textbook in a graduate course or seminar.

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