

Translations of
**MATHEMATICAL
MONOGRAPHS**

Volume 88

Fewnomials

A. G. Khovanskii



American Mathematical Society



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Providence, Rhode Island

А. Г. ХОВАНСКИЙ
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Dedicated to the memory of
ALLEN LOWELL SHIELDS
by the author and the translator

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Contents

Introduction	1
CHAPTER I. An Analogue of the Bezout Theorem for a System of Real Elementary Equations	9
§1.1. General estimate of the number of roots of a system of equations	10
§1.2. Estimate of the number of solutions of a system of quasipolynomials	12
§1.3. A version of the general estimate of the number of roots of a system of equations	13
§1.4. Estimate of the number of solutions of a system of trigonometric quasipolynomials	14
§1.5. Elementary functions of many real variables	16
§1.6. Estimate of the number of solutions of a system of elementary equations	17
§1.7. Remarks	18
CHAPTER II. Two Simple Versions of the Theory of Fewnomials	21
§2.1. Rolle's theorem for dynamical systems	21
§2.2. Algebraic properties of P -curves	23
§2.3. One more version of the theory of fewnomials	27
CHAPTER III. Analogues of the Theorems of Rolle and Bezout for Separating Solutions of Pfaff Equations	31
§3.1. Coorientation and linking index	33
§3.2. Separating submanifolds	36
§3.3. Separating solutions of Pfaff equations	39
§3.4. Separating solutions on 1-dimensional manifolds; an analogue of Rolle's theorem	42
§3.5. Higher-dimensional analogues of Rolle's estimate	45
§3.6. Ordered systems of Pfaff equations, their separating solutions and characteristic sequences	51

§3.7. Estimate of the number of points in a zero-dimensional separating solution of an ordered system of Pfaff equations via the generalised number of zeroes of the characteristic sequence of the system	56
§3.8. The virtual number of zeroes	62
§3.9. Representative families of divisorial sequences	65
§3.10. The virtual number of zeroes on a manifold equipped with a volume form	68
§3.11. Estimate of the virtual number of zeroes of a sequence with isolated singular points via their order and index	73
§3.12. A series of analogues of Rolle's estimate and Bezout's theorem	76
§3.13. Fewnomials in a complex region and Newton polyhedra	81
§3.14. Estimate of the number of connected components and the sum of the Betti numbers of higher-dimensional separating solutions	89
CHAPTER IV. Pfaff Manifolds	95
§4.1. Simple affine Pfaff manifolds	96
§4.2. Affine Pfaff manifolds	98
§4.3. Pfaff A -manifolds	103
§4.4. Pfaff manifolds	107
§4.5. Pfaff functions in Pfaff domains in \mathbf{R}^n	110
§4.6. Results	112
CHAPTER V. Real-Analytic Varieties with Finiteness Properties and Complex Abelian Integrals	115
§5.1. Basic analytic varieties	116
§5.2. Analytic Pfaff manifolds	117
§5.3. Finiteness theorems	119
§5.4. Abelian integrals	120
Conclusion	123
Appendix. Pfaff equations and limit cycles, by Yu. S. Il'yashenko	129
Bibliography	133
Index	137

APPENDIX

Pfaffian Equations and Limit Cycles¹

The above estimates of the number of solutions of Pfaffian functional systems may be applied to investigations of bifurcations of limit cycles of planar vector fields. This observation is due to S. Yu. Yakovenko; together with Yu. S. Il'yashenko he obtained the theorem, stated below. We begin with some definitions.

A singular point of a planar vector field is called *elementary* if it has at least one nonzero eigenvalue. A *polycycle* of the planar vector field is a separatrix polygon: a connected finite union of singular points and phase curves, coming from and tending to some of these points. A polycycle is called *elementary* if all its singular points are elementary.

THEOREM. *In a typical finite-parameter family of smooth vector fields in the plane the only elementary polycycles which can occur are those which generate a finite number of limit cycles under the bifurcation in this family. Moreover, for any natural number n a number $E(n)$ exists such that any polycycle in the typical n -parameter family generates no more than $E(n)$ limit cycles under bifurcation in this family.*

The first step of the proof is to give a list of finitely smooth normal forms of local families, obtained from the perturbations of the elementary singular points in typical finiteparameter families (Yu. S. Il'yashenko and S. Yu. Yakovenko, Russian Math. Surveys **46**(1991), no. 1). These normal forms have two crucial properties: they are polynomial and integrable. The corresponding change of coordinates becomes smoother as the neighbourhood of the critical value of the parameter in the base of the family becomes smaller. As the neighbourhood collapses to a point, the rate of smoothness tends to infinity.

The correspondence map of the hyperbolic sector of the singular point is the map from the entrance semi-interval to the exit one along the phase curves (Figure 1). The integrability of the normal form of the family allows one to calculate the related correspondence maps. They are elementary

¹ This Appendix, written by Yu. S. Il'yashenko, is similar to part of a forthcoming paper of Yu. S. Il'yashenko and S. Yu. Yakovenko, "Bifurcations of elementary polycycles in typical families".

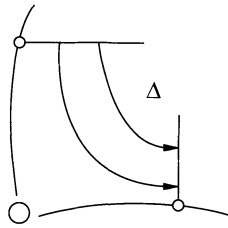


FIGURE 1

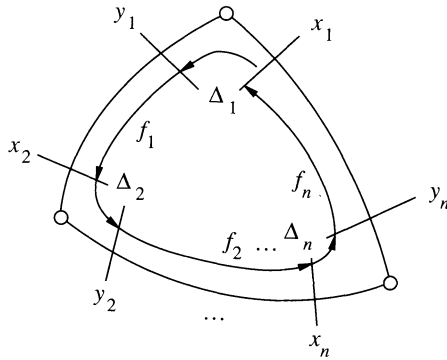


FIGURE 2

transcendental functions, satisfying some polynomial Pfaffian equations. For instance, the perturbation of a nonresonance saddle with ratio of eigenvalues equal to $-\lambda(\varepsilon)$ (ε being the parameter of the family) has the orbital normal form $\dot{x} = x, \dot{y} = -y\lambda(\varepsilon)$. Its correspondence map is equal to $y = x^{\lambda(\varepsilon)}$ and satisfies the Pfaffian equation $x dy - \lambda(\varepsilon)y dx = 0$.

The investigation of the number of limit cycles, generated by the perturbation of the polycycle, is reduced to the study of the Pfaffian functional system in the following way. Take the elementary polycycle (Figure 2) and separate each of its vertices by two intervals transversal to the polycycle with coordinates x_i (at the entrance) and y_i (the exit). Denote the correspondence map from the subset of the former to the latter by $y_i = \Delta_i(x_i, \varepsilon)$, and denote the map of the exit interval to the next entrance interval along the phase curves by $x_{i+1} = f_i(y_i, \varepsilon)$. We can consider our family to be normalized near each singular point, x_i and y_i being the restrictions onto the transversal intervals of the normalizing coordinates near the i th singular point. So the first series of equations is given by some standard transcendental functions, and may be replaced by polynomial Pfaffian equations that are consequences of previous ones. For instance, the equation $y_i = x_i^{\lambda(\varepsilon)}$ is replaced by $x_i dy_i - \lambda(\varepsilon)y_i dx_i = 0$.

The x_i, y_i coordinates of the intersections of the limit cycle with the transversals satisfy the system of equations

$$y_i = \Delta_i(x_i, \varepsilon) \quad (1)$$

$$x_{i+1} = f_i(y_i, \varepsilon), \quad i = 1, \dots, n, \quad (2)$$

where n is the number of vertexes of the polycycle, and the numeration of variables is cyclic modulo n . This system is replaced with the system (3), (2), with (3) equal to

$$\omega_i(x_i, y_i, \varepsilon) = 0, \quad i = 1, \dots, n. \quad (3)$$

Modulo some technical details, the system (1) gives the manifold which is the dividing solution Γ of the system (3).

It is not difficult to prove that the number of limit cycles (e.g., the number of solutions of the system (1, 2)) is no larger than the upper number of preimages for the map $\Gamma \rightarrow \mathbf{R}^n$ given by the functions $x_{i+1} - f_i(y_i, \varepsilon)$ from (2).

One can reduce system (3) to the following:

$$F(x, y, f', f'', \dots, f^{(n)}, \varepsilon) = 0. \quad (4)$$

Here $x = x_1, \dots, x_n$; $y = y_1, \dots, y_n$; $f = f_1, \dots, f_n$; $\varepsilon \in \mathbf{R}^k$; $F : \mathbf{R}^N \rightarrow \mathbf{R}^n$, $N = n^2 + 2n + k$, is a polynomial map in all its variables.

The following fact is useful in the end of the proof.

THEOREM. *For any polynomial map $\mathcal{F} : \mathbf{R}^N \rightarrow \mathbf{R}^m$ of rank m at a generic point there is a number ε such that the upper number of preimages for the composition $\mathcal{F} \circ g$ of \mathcal{F} with a typical smooth map $g : (B^m, 0) \rightarrow (\mathbf{R}^N, 0)$ is no larger than ε . Here B^m is a small ball in \mathbf{R}^m , centered at 0.*

The proof of this theorem uses the ideas of A. M. Gabrielov (*The formal relations between analytic functions*, Functional Anal. Appl. 5(1971), no. 4, 64–65).

In fact, we need a similar fact for some special maps of the kind $\mathcal{F} = (F, \text{id}) : \mathbf{R}^N \times \mathbf{R}^k \rightarrow \mathbf{R}^n \times \mathbf{R}^k = \mathbf{R}^m$, $g(x, \varepsilon) = (x, f(x, \varepsilon), \dots, f^{(n)}(x, \varepsilon), \varepsilon) \in \mathbf{R}^N$, $(x, \varepsilon) \in B^m$, with only f being generic. This g is not generic in the space of all maps $B^m \rightarrow \mathbf{R}^N$. Yet the transversality arguments in the proof of the previous theorem may be modified so that the theorem would be applicable to the class of maps g described above. This observation is due to O. V. Shelkovnikov and completes the proof.

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Bibliography

1. V. I. Arnol'd, *Ordinary differential equations*, "Nauka", Moscow, 1971; English transl., The M.I.T. Press, Cambridge, MA, 1973, 1978.
2. V. I. Arnol'd, A. N. Varchenko, and S. M. Gusein-Zade, *Singularities of differentiable mappings*, Vol. I, "Nauka", Moscow, 1982; English transl., *Singularities of differentiable maps*. Vol. I. *The classification of critical points, caustics and wave fronts*. Monographs in Mathematics, Vol. 82, Birkhäuser Boston, Inc., Boston, MA, 1985.
3. —, *Singularities of differentiable mappings*, Vol. II, "Nauka", Moscow, 1982; English transl., *Singularities of differentiable maps*. Vol. II. *Monodromy and asymptotics of integrals*. Monographs in Mathematics, Vol. 83, Birkhäuser Boston, Inc., Boston, MA, 1988.
4. V. I. Arnol'd and O. A. Oleinik, *The topology of real algebraic varieties*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1979**, no. 6, 7–17; English transl. in Moscow Univ. Math. Bull. **34** (1979).
5. F. S. Berezovskaya, *The index of a stationary point of a vector field in the plane*, Funktsional. Anal. i Prilozhen. **13** (1979), no. 2, 77; English transl. in Functional Anal. Appl. **13** (1979), no. 2.
6. D. N. Bernshtein, *The number of roots of a system of equations*, Funktsional. Anal. i Prilozhen. **9** (1975), no. 3, 1–4; English transl. in Functional Anal. Appl. **9** (1975), no. 3.
7. D. N. Bernshtein, A. G. Kushnirenko, and A. G. Khovanskii, *Newton polyhedra*, Uspekhi Mat. Nauk **31** (1976), no. 3(189), 201–202. (Russian)
8. —, *Mixed Minkowski volume of Newton polyhedra, Milnor numbers, and the number of solutions of a system of algebraic equations*, Uspekhi Mat. Nauk **31** (1976), no. 3, 196. (Russian)
9. E. Brieskorn, *Die Monodromie der isolierten Singularitäten von Hyperflächen*, Manuscripta Math. **2** (1970), 103–161.
10. A. D. Brjuno [Bryuno], *Local methods in nonlinear differential equations*, "Nauka", Moscow, 1979; English transl., Springer-Verlag, Berlin and New York, 1989.
11. H. Busemann, *Convex surfaces*, Interscience, New York, 1958.
12. A. N. Varchenko, *Newton polyhedra and estimates of oscillatory integrals*, Funktsional. Anal. i Prilozhen. **10** (1976), no. 3, 13–138; English transl. in Functional Anal. Appl. **10** (1976), no. 3.
13. —, *The characteristic polynomial of the monodromy operator, and the Newton diagram of a singularity*, Uspekhi Mat. Nauk **32** (1977), no. 2, 63–64. (Russian)
14. —, *Estimation of the number of zeros of an abelian integral depending on a parameter, and limit cycles*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 2, 14–25; English transl. in Functional Anal. Appl. **18** (1984), no. 2.
15. A. N. Varchenko and A. G. Khovanskii, *Asymptotic behavior of integrals over vanishing cycles and the Newton polyhedron*, Dokl. Akad. Nauk SSSR **283** (1985), no. 3, 521–525; English transl. in Soviet Math. Dokl. **32** (1985), no. 1.
16. V. A. Vasil'ev, *Asymptotics of exponential integrals, the Newton diagram and classification of minimum points*, Funktsional. Anal. i Prilozhen. **11** (1977), no. 3, 1–11; English transl. in Functional Anal. Appl. **11** (1977), no. 3.
17. —, *Asymptotic behavior of exponential integrals in the complex domain*, Funktsional. Anal. i Prilozhen. **13** (1979), no. 4, 1–12, 96; English transl. in Functional Anal. Appl. **13** (1979), no. 4.

18. È. B. Vinberg, *The nonexistence of crystallographic reflection groups in Lobachevsky spaces of large dimension*, Funktsional. Anal. i Prilozhen. **15** (1981), no. 2, 67–68; English transl. in Functional Anal. Appl. **15** (1981), no. 2.
19. —, *The absence of crystallographic groups of reflections in Lobachevsky spaces of large dimension*, Trudy Moskov. Mat. Obshch. **47** (1984), 68–102, 246; English transl. in Trans. Moscow Math. Soc. **1985**, 75–112.
20. A. M. Gabrièlov, *Projections of semianalytic sets*, Funktsional. Anal. i Prilozhen. **2** (1968), no. 4, 18–30; English transl. in Functional Anal. Appl. **2** (1968), no. 4.
21. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, NJ, 1965.
22. O. A. Gel'fond, *Zeros of systems of almost periodic polynomials*, Preprint, Fiz. Inst. Akad. Nauk SSSR, Moscow, 1978.
23. —, *The mean index of an almost periodic vector field*, Preprint, Fiz. Inst. Akad. Nauk SSSR, Moscow, 1981.
24. —, *On the average number of roots of systems of holomorphic almost periodic equations*, Uspekhi Mat. Nauk **39** (1984), no. 1, 123–124. (Russian)
25. O. A. Gel'fond and A. G. Khovanskii, *Real Liouville functions*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 2, 52–53; English transl. in Functional Anal. Appl. **14** (1980), no. 2.
26. R. G. Gurevich, *Decidability of the equational theory of positive numbers with exponentiation*, Sibirsk. Mat. Zh. **25** (1984), no. 2, 216–219; English transl. in Siberian Math. J. **25** (1984), no. 2.
27. V. I. Danilov, *The geometry of toric varieties*, Uspekhi Mat. Nauk **33** (1978), no. 2, 85–134, 247; English transl. in Russian Math. Surveys **33** (1978), no. 2, 1–52.
28. —, *Newton polyhedra and vanishing cohomology*, Funktsional. Anal. i Prilozhen. **13** (1979), no. 2, 32–47, 96; English transl. in Functional Anal. Appl. **13** (1979), no. 2.
29. V. I. Danilov and A. G. Khovanskii, *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 5, 925–945; English transl. in Math. USSR Izv. **29** (1987), no. 2.
30. B. Ya. Kazarnovskii, *On the zeros of exponential sums*, Dokl. Akad. Nauk SSSR **257** (1981), no. 4, 804–808; English transl. in Soviet Math. Dokl. **23** (1981), no. 2.
31. —, *Newton polyhedra and roots of systems of exponential sums*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 4, 40–49, 96; English transl. in Functional Anal. Appl. **18** (1984), no. 4.
32. A. G. Kushnirenko, *The Newton polyhedron and the number of solutions of a system of k equations in k unknowns*, Uspekhi Mat. Nauk **30** (1975), no. 2(182), 266–267. (Russian)
33. —, *The Newton polyhedron and Milnor numbers*, Funktsional. Anal. i Prilozhen. **9** (1976), no. 1, 74–75; English transl. in Functional Anal. Appl. **9** (1976), no. 1.
34. —, *The Newton polyhedron and Bézout's theorem*, Funktsional. Anal. i Prilozhen. **10** (1976), no. 3, 82–83; English transl. in Functional Anal. Appl. **10** (1976), no. 3.
35. J. Milnor, *Morse theory*, Ann. of Math. Studies, No. 51, Princeton Univ. Press, Princeton, NJ, 1963.
36. —, *Singular points of complex hypersurfaces*, Ann. of Math. Studies, No. 61, Princeton Univ. Press, Princeton, NJ; Univ. of Tokyo Press, Tokyo, 1968.
37. V. V. Nikulin, *On arithmetic groups generated by reflections in Lobachevsky spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 3, 637–669, 719–720; English transl. in Math. USSR Izv. **16** (1981), no. 3.
38. —, *On the classification of arithmetic groups generated by reflections in Lobachevsky spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 1, 113–142, 240; English transl. in Math. USSR Izv. **18** (1982), no. 1.
39. G. S. Petrov, *The number of zeros of complete elliptic integrals*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 2, 73–74; English transl. in Functional Anal. Appl. **18** (1984), no. 2.
40. —, *Elliptic integrals and their nonoscillation*, Funktsional. Anal. i Prilozhen. **20** (1986), no. 1, 46–49, 96; English transl. in Functional Anal. Appl. **20** (1986), no. 1.
41. I. G. Petrovskii and O. A. Oleinik, *On the topology of real algebraic surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **18** (1949), 389–402; English transl., Amer. Math. Soc. Transl. (1) **7** (1962), 399–417.

42. M. N. Prokhorov, *The absence of discrete reflection groups with noncompact fundamental polyhedron of finite volume in Lobachevsky space of large dimension*, *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), no. 2, 413–424; English transl. in *Math. USSR Izv.* **28** (1987), no. 2.
43. Henri Poincaré, *Mémoire sur les courbes définies par une équation différentielle*. I, II, J. *Math. Pures Appl.* (3) **7** (1881), 375–422; **8** (1882), 251–296; *Sur les courbes définies par les équations différentielles*. III, IV, J. *Math. Pures Appl.* (4) **1** (1885), 167–244; **2** (1886), 151–217; reprinted in *Oeuvres de Henri Poincaré*. Vol. I, Gauthier-Villars, Paris, 1928, pp. 3–84, 90–161, 167–221; Russian transl., GITTL, Moscow, 1947.
44. F. Pham, *Introduction à l'étude topologique des singularités de Landau*, Gauthier-Villars, Paris, 1967.
45. F. Hirzebruch, *Topological methods in algebraic geometry*, 3rd ed., Springer-Verlag, Berlin and New York, 1966.
46. A. G. Khovanskii, *Newton polyhedra and toral varieties*, *Funktsional. Anal. i Prilozhen.* **11** (1977), no. 4, 56–64; English transl. in *Functional Anal. Appl.* **11** (1977), no. 4.
47. —, *Newton polyhedra and the genus of complete intersections*, *Funktsional. Anal. i Prilozhen.* **12** (1978), no. 1, 51–61; English transl. in *Functional Anal. Appl.* **12** (1978), no. 1.
48. —, *Newton polyhedra and the Euler-Jacobi formula*, *Uspekhi Mat. Nauk* **33** (1978), no. 6, 237–238. (Russian)
49. —, *The geometry of convex polyhedra and algebraic geometry*, *Uspekhi Mat. Nauk* **34** (1979), no. 4, 160–161. (Russian)
50. —, *On a class of systems of transcendental equations*, *Dokl. Akad. Nauk SSSR* **255** (1980), no. 4, 804–807; English transl. in *Soviet Math. Dokl.* **22** (1980), no. 3.
51. —, *On the number of zeros of real equations*, *Uspekhi Mat. Nauk* **35** (1980), no. 5, 213. (Russian)
52. —, *On an estimate of the number of real zeros of fewnomials*, *Uspekhi Mat. Nauk* **37** (1982), no. 1, 165. (Russian)
53. —, *Newton polyhedra (resolution of singularities)*, *Current problems in mathematics*, Vol. 22, *Itogi Nauki i Tekhniki*, Akad. Nauk SSSR, VINITI, Moscow, 1983, pp. 207–239; English transl., *J. Soviet Math.* **27** (1984), no. 3.
54. —, *Analogues of the Aleksandrov-Fenchel inequalities for hyperbolic forms*, *Dokl. Akad. Nauk SSSR* **276** (1984), no. 6, 1332–1334; English transl. in *Soviet Math. Dokl.* **29** (1984), no. 3.
55. —, *Real analytic manifolds with the property of finiteness, and complex abelian integrals*, *Funktsional. Anal. i Prilozhen.* **18** (1984), no. 2, 40–50; English transl. in *Functional Anal. Appl.* **18** (1984), no. 2.
56. —, *Cycles of dynamical systems on a plane and Rolle's theorem*, *Sibirsk Mat. Zh.* **25** (1984), no. 3, 198–203; English transl. in *Siberian Math. J.* **25** (1984), no. 3.
57. —, *Newton polyhedra and the index of a vector field*, *Uspekhi Mat. Nauk* **39** (1984), no. 4, 234. (Russian)
58. —, *Mixed volumes*, *Uspekhi Mat. Nauk* **39** (1984), no. 5, 227–228. (Russian)
59. —, *Linear programming and the geometry of convex polyhedra*, *Collection of Papers, All-Union Sci. Res. Systems Inst.* **1985**, no. 7, 73–80.
60. —, *Linear programming and a generalization of Nikulin's theorem*, *Collection of Papers, All-Union Sci. Res. Systems Inst.* **1985**, no. 7, 81–86.
61. —, *Hyperplane sections of polyhedra, toric varieties and discrete groups in Lobachevsky space*, *Funktsional. Anal. i Prilozhen.* **20** (1986), no. 1, 50–61, 96; English transl. in *Functional Anal. Appl.* **20** (1986), no. 1.
62. N. G. Chebotarev, *The Newton polyhedron and its role in the current development of mathematics*, *Collected Works*, vol. 3, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1950.
63. I. R. Shafarevich, *Basic algebraic geometry*, “Nauka”, Moscow, 1972; English transl., Springer-Verlag, Berlin and New York, 1974.
64. V. I. Arnol'd, A. N. Varchenko, A. B. Givental', and A. G. Khovanskii, *Singularities of functions, wave fronts, caustics and multidimensional integrals*, *Soviet Sci. Rev. Sect. C: Math. Phys. Rev.*, vol. 4, Harwood Academic Publ., Chur, 1984, pp. 1–92.
65. M. F. Atiyah, *Convexity and commuting Hamiltonians*, *Bull. London Math. Soc.* **14** (1982), no. 1, 1–15.

66. H. F. Baker, *Examples of application of Newton's polygon applied to theory of singular points of algebraic functions*, Trans. Cambridge Phil. Soc. **15** (1893), 403–450.
67. L. J. Billera and C. W. Lee, *Sufficiency of McMullen's conditions for f -vectors of simplicial polytopes*, Bull. Amer. Math. Soc. (N.S.), **2** (1980), no. 1, 181–185.
68. M. Boshernitzan, *New "orders of infinity"*, J. Analyse Math. **41** (1982), 130–167.
69. P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, vol. 163, Springer-Verlag, Berlin and New York, 1970.
70. F. Ehlers, *Eine Klasse komplexer Mannigfaltigkeiten und die Auflösung einiger isolierter Singularitäten*, Math. Ann. **218** (1975), 127–156.
71. V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), no. 3, 491–513.
72. W. V. D. Hodge, *The isolated singularities of an algebraic surface*, Proc. London Math. Soc. (2) **30** (1929), 133–143.
73. A. Hovansky, *Sur les racines complexes des systèmes d'équations algébriques comportant peu de termes*, C. R. Acad. Sci. Paris. Sér. I Math. **292** (1981), no. 21, 937–940.
74. G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, *Toroidal Embeddings I*. Lecture Notes in Mathematics, vol. 339, Springer-Verlag, Berlin-New York, 1973.
75. A. Khovansky, *Théorème de Bézout pour les fonctions de Liouville*, Preprint IHES/M/81/45, Bures-sur-Ivette, 1981.
76. A. G. Khovansky, *Fewnomials and Pfaff manifolds*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 549–564, PWN, Warsaw, 1984.
77. A. G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), no. 1, 1–31.
78. L. Kronecker, *Über einige Interpolationformeln für ganze Funktionen mehrerer Variablen*, Monatsber. Königl. Preuss. Akad. Wiss. Berlin 1865, 686–691; reprinted in his Werke, Vol. I, Teubner, Leipzig, 1895, pp. 133–141.
79. S. Lojasiewicz, *Ensembles semi-analytiques*, Preprint, Institut des Hautes Etudes Scientifiques, Bures-sur-Ivette, 1965.
80. P. McMullen, *Metric and combinatorial properties of convex polytopes*, Proceedings of the International Congress of Mathematicians (Vancouver, BC, 1974), vol. 1, pp. 491–495.
81. —, *The number of faces of simplicial polytopes*, Israel J. Math. **9** (1971) 559–570.
82. D. Richardson, *Roots of real exponential functions*, J. London Math. Soc. (2) **28** (1983), no. 1, 46–56.
83. J.-J. Risler, *On the Bézout theorem in the real case*, Preprint, Inst. de Mathem., Madrid, 1983.
84. —, *Complexité et géométrie réelle [d'après A. Khovansky]*, Seminar Bourbaki, Vol. 1984/85; Astérisque **1986**, no. 133–134, 89–100.
85. —, *Additive complexity and zeros of real polynomials*, SIAM J. Comput. **14** (1985), no. 1, 178–183.
86. —, *Hovansky's theorem and complexity theory*, Rocky Mountain J. Math. **14** (1984), no. 4, 851–853.
87. D. M. Y. Sommerville, *The relations connecting the angle-sums and volume of polytope in space of n dimensions*, Proc. Roy. Soc. London Ser. A **115** (1927), 103–119.
88. R. Stanley, *The number of faces of a simplicial convex polytope*, Adv. in Math. **35** (1980), no. 3, 236–238.
89. J. H. M. Steenbrink, *Mixed Hodge structure on the vanishing cohomology*, Real and Complex Singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 525–563, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.
90. B. Teissier, *Variétés toriques et polytopes*, Seminar Bourbaki, vol. 1980/81, pp. 71–84; Lecture Notes in Math., vol. 901, Springer-Verlag, Berlin-New York, 1981.
91. A. N. Varchenko, *Zeta-function of monodromy and Newton's diagram*, Invent. Math. **37** (1976), no. 3, 253–262.

Subject Index

- (A_1, A_2) -map, 114
 - composition, 115
- A -form, 113, 116
- A -function, 113
- A -manifold, 111, 116, 119
- A -region, 113
- A -regular, 111
 - function, 111
 - map, 115
 - region, 112
 - simple region, 112
- A -resolution, 113
- Abelian, 128
 - function, 129
 - integral, 128
 - map, 128
- Abelian integral, 8, 123, 128
 - zeroes of, 133
- Affine, 107, 113
 - manifold, 107, 108, 110
 - region, 107, 108
- Algebraic curves, 24
- Algebraic varieties, 1
- Arnold, 123, 134
- Axiom, 60
 - 1, 60, 72, 77
 - 2, 61, 72, 77
 - 3, 61, 72, 77
 - 4, 61, 72, 77
- Basic variety, 124
- Betti numbers, 1, 96, 132
- Bezout theorem, 1, 6, 9, 12, 17, 26, 31, 75, 81, 97, 128
- Borodin, 132
- Boshernitzan, 135
- Cell complex, 96, 97, 99, 121, 128
- Chain, 54
 - characteristic, 56
 - of preimages, 54
 - of submanifolds, 54
 - separating, 54, 56, 58, 63
- Characteristic sequence, 31, 53, 56, 58, 60, 82
 - zeroes of, 32, 60
- Complete intersection, 75, 104
- Complexity, 3, 109, 110, 113, 114, 119, 131
 - of a realisation, 107
 - of polynomials, 131
- Contact point, 22
- Cook, 132
- Coorientation, 31
 - boundary, 34
 - composite, 33
 - induced, 34
- Darboux's theorem, 23
- Descartes, 87, 131
- Diffeomorphism, 115
- Divisorial sequence, 60, 66, 70, 78
 - complete, 60, 68
 - local ring of, 79
 - nonsingular, 66, 74
 - nonsingular point for, 79
 - singular point, 80
- Domain, 118
- Dulac, 134
- Dynamical system, 4
 - contact point of a curve and a, 22
 - cycles of, 4, 21, 25
 - Rolle's theorem for, 21
 - separating solution of, 4, 21
- Elementary equations, 9
 - number of solutions of, 16
 - solutions of, 120
- Elementary functions, 16
- Elementary singular point, 137
- Equivalent equations, 40, 55
- Estimating spectrum, 89

- Euler characteristic, 27, 52
- Fewnomials, 1, 30, 87, 123, 131
 - complex, 2, 87
 - real, 2
 - theory of, 21
- Film, 36
 - spanning, 36, 55
- Finiteness theorems, 127
- Flag, 55
 - of linear subspaces, 55
- Form, 104
 - regular, 104
- Forms, 70
 - representative family of, 70
 - volume, 73
- Function, 16, 104
 - abelian, 128
 - admissible, 126
 - complexity of, 16
 - covering, 124
 - of order k , 16
 - regular, 104, 106
 - resolution of, 126
 - separating, 124
- Functional equations, 63
- Gabriellov's theorem, 124
- Gabriellov, 139
- Graph, 109, 119
- Gurevich, 133
- Hardy fields, 135
- Hilbert's 16th problem, 123, 133
- History, 131
- Il'yashenko, 134, 137
- Integral, 128
 - abelian, 128
- Intersection, 35
 - cycle, 36, 50
 - index, 35, 50
 - points, 48, 51
- Jacobian, 15
- Jordan's theorem, 21, 40
- Kushnirenko, 131
- Kushnirenko-Bernstein, 2
 - number, 2
 - theorem, 2, 75
- Level line, 4
- Level set, 123
- Lift intersection, 112, 113
- Limit cycle, 133, 137
- Linking index, 31, 36, 38, 50, 52
- Manifold, 110, 112, 115, 116, 123
 - analytic, 125
 - immersed, 116
 - integral, 31, 40
 - product of, 110, 116
- Mapping, 6
 - analytic, 127
 - implicit, 117
 - regular, 6
- Monodromy operator, 129
- Monomial, 16
 - of order k , 16
- Newton polyhedron, 2, 33, 87, 93
 - concordant edges, 88
- Nonmultiple, 109, 110
- P -chain, 27
- P -curves, 23
 - algebraic properties, 23
 - Bezout theorem for, 26
- P -system, 27
- Perestroikas, 134
- Petrov, 134
- Pfaff, 4, 111
 - A -function, 113
 - A -manifolds, 104, 111
 - A -region, 113
 - affine manifolds, 106
 - analytic manifold, 125
 - atlas, 126
 - chain, 27, 85
 - curve, 4
 - domain, 118
 - equation, 5, 18, 31, 40, 47, 53, 63, 104, 109, 119, 137
 - form, 7
 - functions, 6, 103, 118, 135
 - manifold, 6, 103, 115, 123
 - map, 116
 - region, 6, 118
 - simple affine manifold, 104
 - simple analytic manifold, 125
 - simple analytic submanifold, 125
 - structure, 6
 - submanifold, 6, 104, 106
 - variety, 123
 - vector field, 7
- Poincaré, 22
 - return map, 134
- Point of contact, 5
- Polycycle, 137
 - elementary, 137
- Polynomial, 2
 - support of, 2
- Polynomial equations, 30, 94, 97, 100
- Preimages, 77

- Preimages (*continued*)
 number of, 82
 upper number of, 77
- Quasipolynomials, 12, 14, 30
 solutions of, 12
 system of, 12
 trigonometric, 14
- Real transcendental manifolds, 1
- Realisation, 104, 109, 110, 112, 115, 116, 118
 complexity of, 104
 simple affine Pfaff manifold, 104
- Region, 107, 126
 admissible, 126
 affine, 107
 analytic, 127
 resolution of, 126
- Regular, 104
 form, 104, 107
 function, 104, 107, 109, 111
 projection, 112
- Ring of functions, 108, 111
 basic, 111
- Risler, 132
- Rolle's estimate, 44, 81
 higher-dimensional analogues, 47
- Rolle's theorem, 5, 21, 22, 28, 31
 for dynamical systems, 49
- Roots, 1, 120
 complex, 2
 isolated, 90, 93, 95
 multiplicities of, 93, 95
 nondegenerate, 1, 64, 83, 85, 87, 90, 97, 131
 nonsingular, 86
 number of, 120
- Sard's theorem, 24
- Semi-Pfaff set, 133
- Separating set, 124
- Separating solution, 5, 6, 22, 31, 40, 44, 47, 53, 58, 60, 62, 63, 81, 97, 100, 105, 109, 117, 127
 of ordered systems, 56
- Separatrix polygon, 134, 137
- Sevast'yanov, 131
- Shelkownikov, 139
- Sign change, 45, 61
 component with, 48
- Singularity, 78
 index of, 78
 multiplicity of, 78
- Solutions, 125, 127
 number of nondegenerate, 127
 number of nonsingular, 125
- Span, 35
- Submanifold, 106, 109
 separating, 31, 36, 38, 39, 49, 52
- System of equations, 10, 128
 equivalent, 56
 exponential, 89
 Fuks-type, 129
 nonsingular roots, 128
 polynomial, 94
 roots of, 10
- Tarski's problem, 132
- Tarski-Seidenberg theorem, 132
- Topology, 1
 of geometric objects, 1
- Truncated region, 120
- Varchenko, 123, 134
- Variety, 123
 basic analytic, 124
- Vector fields, 21
 polynomial, 21
- Vector-Functions, 103
- Volume form, 124
- Whitney cusps, 5
- Yakovenko, 137
- Zeroes, 125
 generalised number of, 60, 62, 64, 68, 81
 ideal of, 79
 manifold of, 67
 upper number of, 72, 75
 virtual number of, 60, 66, 68, 73, 75, 77, 78, 125

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