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Fewnomials

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Dedicated to the memory of
ALLEN LOWELL SHIELDS
by the author and the translator

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Contents

APPENDIX

Pfaffian Equations and Limit Cycles [1]

The above estimates of the number of solutions of Pfaffian functional systems may be applied to investigations of bifurcations of limit cycles of planar vector fields. This observation is due to S. Yu. Yakovenko; together with Yu. S. Il′yashenko he obtained the theorem, stated below. We begin with some definitions.

A singular point of a planar vector field is called *elementary* if it has at least one nonzero eigenvalue. A *polycycle* of the planar vector field is a separatrix polygon: a connected finite union of singular points and phase curves, coming from and tending to some of these points. A polycycle is called *elementary* if all its singular points are elementary.

THEOREM. *In a typical finite-parameter family of smooth vector fields in the plane the only elementary polycycles which can occur are those which generate a finite number of limit cycles under the bifurcation in this family. Moreover, for any natural number* n *a number* $E(n)$ *exists such that any polycycle in the typical* n*-parameter family generates no more than* $E(n)$ *limit cycles under bifurcation in this family.*

The first step of the proof is to give a list of finitely smooth normal forms of local families, obtained from the perturbations of the elementary singular points in typical finiteparameter families (Yu. S. Il′yashenko and S. Yu. Yakovenko, Russian Math. Surveys **46**(1991), no. 1). These normal forms have two crucial properties: they are polynomial and integrable. The corresponding change of coordinates becomes smoother as the neighbourhood of the critical value of the parameter in the base of the family becomes smaller. As the neighbourhood collapses to a point, the rate of smoothness tends to infinity.

The correspondence map of the hyperbolic sector of the singular point is the map from the entrance semi-interval to the exit one along the phase curves (Figure 1). The integrability of the normal form of the family allows one to calculate the related correspondence maps. They are elementary

[1] This Appendix, written by Yu. S. Il′yashenko, is similar to part of a forthcoming paper of Yu. S. Il′yashenko and S. Yu. Yakovenko, "Bifurcations of elementary polycycles in typical families".

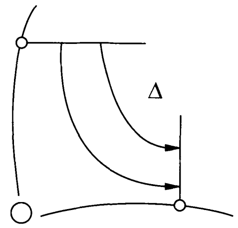

FIGURE 1

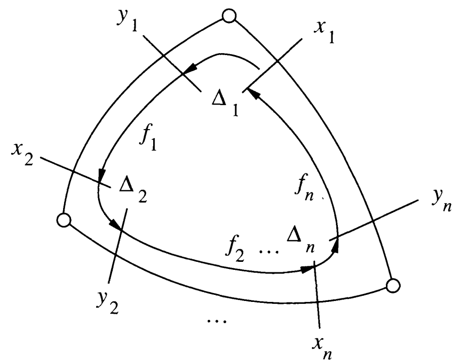

FIGURE 2

transcendental functions, satisfying some polynomial Pfaffian equations. For instance, the perturbation of a nonresonance saddle with ratio of eigenvalues equal to $-\lambda(\varepsilon)$ (ε being the parameter of the family) has the orbital normal form $\dot{x} = x$, $\dot{y} = -y\lambda(\varepsilon)$. Its correspondence map is equal to $y = x^{\lambda(\varepsilon)}$ and satisfies the Pfaffian equation $xdy - \lambda(\varepsilon)ydx = 0$.

The investigation of the number of limit cycles, generated by the perturbation of the polycycle, is reduced to the study of the Pfaffian functional system in the following way. Take the elementary polycycle (Figure 2) and separate each of its vertices by two intervals transversal to the polycycle with coordinates x_i (at the entrance) and y_i (the exit). Denote the correspondence map from the subset of the former to the latter by $y_i = \Delta_i(x_i, \varepsilon)$, and denote the map of the exit interval to the next entrance interval along the phase curves by $x_{i+1} = f_i(y_i, \varepsilon)$. We can consider our family to be normalized near each singular point, x_i and y_i being the restrictions onto the transversal intervals of the normalizing coordinates near the i th singular point. So the first series of equations is given by some standard transcendental functions, and may be replaced by polynomial Pfaffian equations that are consequences of previous ones. For instance, the equation $y_i = x_i^{\lambda(\varepsilon)}$ is replaced by $x_i dy_i - \lambda(\varepsilon)y_i dx_i = 0$.

The x_i, y_i coordinates of the intersections of the limit cycle with the transversals satisfy the system of equations

$$y_i = \Delta_i(x_i, \varepsilon) \tag{1}$$

$$x_{i+1} = f_i(y_i, \varepsilon), \qquad i = 1, \dots, n, \tag{2}$$

where n is the number of vertexes of the polycycle, and the numeration of variables is cyclic modulo n. This system is replaced with the system (3), (2), with (3) equal to

$$\omega_i(x_i, y_i, \varepsilon) = 0, \qquad i = 1, \dots, n. \tag{3}$$

Modulo some technical details, the system (1) gives the manifold which is the dividing solution Γ of the system (3).

It is not difficult to prove that the number of limit cycles (e.g., the number of solutions of the system (1, 2)) is no larger than the upper number of preimages for the map $\Gamma \to \mathbf{R}^n$ given by the functions $x_{i+1} - f_i(y_i, \varepsilon)$ from (2).

One can reduce system (3) to the following:

$$F(x, y, f', f'', \dots, f^{(n)}, \varepsilon) = 0. \tag{4}$$

Here $x = x_1, \dots, x_n$; $y = y_1, \dots, y_n$; $f = f_1, \dots, f_n$; $\varepsilon \in \mathbf{R}^k$; $F : \mathbf{R}^N \to \mathbf{R}^n$, $N = n^2 + 2n + k$, is a polynomial map in all its variables.

The following fact is useful in the end of the proof.

THEOREM. *For any polynomial map $\mathscr{F} : \mathbf{R}^N \to \mathbf{R}^m$ of rank m at a generic point there is a number ε such that the upper number of preimages for the composition $\mathscr{F} \circ g$ of $\mathscr{F}$ with a typical smooth map $g : (B^m, 0) \to (\mathbf{R}^N, 0)$ is no larger than ε. Here B^m is a small ball in $\mathbf{R}^m$, centered at 0.*

The proof of this theorem uses the ideas of A. M. Gabrielov (*The formal relations between analytic functions*, Functional Anal. Appl. **5**(1971), no. 4, 64–65).

In fact, we need a similar fact for some special maps of the kind $\mathscr{F} = (F, \mathrm{id}) : \mathbf{R}^N \times \mathbf{R}^k \to \mathbf{R}^n \times \mathbf{R}^k = \mathbf{R}^m$, $g(x, \varepsilon) = (x, f(x, \varepsilon), \dots, f^{(n)}(x, \varepsilon), \varepsilon) \in \mathbf{R}^N$, $(x, \varepsilon) \in B^m$, with only f being generic. This g is not generic in the space of all maps $B^m \to \mathbf{R}^N$. Yet the transversality arguments in the proof of the previous theorem may be modified so that the theorem would be applicable to the class of maps g described above. This observation is due to O. V. Shelkovnikov and completes the proof.

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