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A Genetic Introduction to the Dynamics of Systems with Impacts

Valeriĭ V. Kozlov
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Providence, Rhode Island

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Preface

> "Le jeu de billard, tel qu'il est devenu aujourd'hui, par l'usage des queues propres à donner aux billes d'assez forts mouvements de rotation, offre divers problèmes de dynamique que l'on trouvera résolus dans cet ouvrage. Je pense que les personnes qui ont des connaissances de mécanique rationnelle, comme les élèves de l'École Polytechnique, verront avec intérèt l'explication de tous les effets singuliers qu'on observe dans le mouvement des billes."
>
> G. Coriolis, *Theórie mathématique des effets du jeu de billard*

The book is devoted to mathematical aspects of the theory of dynamical systems of billiard type. Starting with the works of G. D. Birkhoff, billiards have been a popular topic of investigation where various subjects of ergodic theory, Morse theory, KAM theory, etc. are intertwined. On the other hand, billiard systems are further remarkable in that they arise naturally in a number of important problems of mechanics and physics (vibro-impact systems, the diffraction of shortwaves, etc.).

As the subtitle of this book indicates, the genetic approach has been chosen as the basic method of exposition. The authors strive to clarify the genesis of the basic ideas and concepts of the theory of dynamical systems with impact interactions and also to demonstrate that they are natural and effective. The limit theorems recently found, which justify various mathematical models of impact theory, form a key feature. They consist essentially in the following. A single-sided constraint imposed on a system is replaced by a field of elastic and dissipative forces. The elastic and dissipative coefficients are then allowed to extend to infinity in some coordinated manner. It is proved that the motion of such a "free" system with fixed initial values tends on each finite time interval to a motion with impacts. In the absence of dissipation of energy we obtain an elastic impact, while for a suitable choice of the dissipative Rayleigh function (which depends on the structure of frictional forces) we obtain in the limit the Newton model and a more general impact with viscous friction. The idea of realizing constraints by means of passage to the limit in the full dynamical equations goes back to the works of Klein, Prandtl, Carathéodory, and Courant. In particular, these results make it possible to solve a number of new problems on stability of periodic motions with impacts and also to investigate the evolution of billiard systems with inelastic collisions when weak dissipation of the energy occurs.

Questions of the existence and stability of periodic trajectories of elastic billiards occupy a special place in the book. A variational proof is given of the familiar Birkhoff theorem on the estimation of the number of distinct periodic trajectories of a convex billiard. Interesting questions, going back to Poincaré, are discussed regarding the connection of the stability of a periodic trajectory with properties of the corresponding critical point of the action functional. It turns out, for example, that the Morse index of a nondegenerate, even-link periodic trajectory of elliptic type is always odd. Results of this sort are obtained from a general formula connecting the characteristic exponents of a periodic solution with the Hesse determinant of the perimeter function at the corresponding critical point.

Considerable attention is devoted to integrable billiards. Aside from the known integrable problems (the elliptic billiard, the billiard in affine Weyl cells), new case are indicated: a harmonic oscillator inside an ellipse, some billiards on surfaces of constant curvature, and a number of others. The problem of integrability of systems of billiard type is discussed. A brief survey is given of works on billiards with ergodic behavior.

Conversations with S. V. Bolotin have been of invaluable assistance to us in writing the last chapter. We amicably thank him.

V. Kozlov
D. Treshchëv

APPENDIX I

Systems with Elastic Reflections and KAM Theory

1. The simplest system with elastic reflections is a billiard in Euclidean space. The trajectories of such a billiard can obviously be identified with light rays reflected from a rectilinear mirror. From geometric optics there is the simple observation that light rays can be "rectified" by considering after reflection not the ray itself but its symmetric image relative to the mirror. In the work [17] Zhuravlev proposed "rectification", in a similar way of the trajectory of an arbitrary mechanical system with elastic reflections. He showed that with the help of a suitable change of coordinates it is possible to make the generalized velocities (or momenta) continuous, where the accelerations, generally speaking, remain discontinuous. This approach in combination with the method of averaging made it possible to solve a number of new interesting problems in the dynamics of vibro-elastic systems (see [18]).

Ivanov and Markeev [19] represented the equations of motion of systems with elastic impacts in Hamiltonian form. We shall briefly present their construction.

Suppose a natural mechanical system with Lagrange function $L = T - V$ is subject to an ideal one-sided constraint. Then there exist local coordinates $q_0, \ldots, q_n$ on configuration space in which the constraint has the form $q_0 > 0$, while the kinetic energy is equal to

$$T = \frac{1}{2} \sum_{i,j=0}^{n} a_{ij}(q)\dot{q}_i\dot{q}_j; \qquad a_{0m} \equiv 0, \qquad m = 1, \ldots, n.$$

The existence of such coordinates was established in §3, Chapter I.

The equations of motion in intervals between impacts on the constraint have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_0} - \frac{\partial L}{\partial q_0} = F, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0, \qquad k \geq 1,$$

where F is the reaction of the stressed constraint: $F = 0$ for $q_0 \neq 0$. We write the first of these equations for $q_0 = \dot{q}_0 = 0$:

$$a_{00}\ddot{q}_0 - \left.\frac{\partial V}{\partial q_0}\right|_{q_0=0} = F.$$

Since $a_{00} > 0$, following [33], we obtain

$$F = \max\left\{0, -\left.\frac{\partial V}{\partial q_0}\right|_{q_0=0}\right\}. \tag{1}$$

We define an auxiliary system by means of the Lagrange function $L'(q, \dot{q}, t) = L(|q_0|, q_1, \dots, q_n, \dot{q}, t)$ and the generalized force (1). It is not hard to verify that the trajectories $q(t)$ of the initial system and $q'(t)$ of the auxiliary system with the same initial conditions are connected by the simple relations

$$q_0(t) = |q_0'(t)|, \qquad q_k(t) = q_k'(t), \qquad k \geq 1.$$

The auxiliary system is no longer subject to constraints.

Making the *Legendre transformation*

$$p = \frac{\partial L'}{\partial \dot{q}}, \qquad H = p\dot{q} - L',$$

we write the equations of motion in the canonical form

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q},$$

where (in accordance with (1)) it is necessary to set

$$\left.\frac{\partial H}{\partial q_0}\right|_{q_0=0} = \min\left\{0, \left.\frac{\partial H}{\partial q_0}\right|_{q_0=0}\right\}.$$

The Hamiltonian system obtained uniquely determines the motion of the original system (with consideration of the identification $q_0 \mapsto |q_0|$). In particular, the motion for the stressed constraint occurs under the condition $\partial H/\partial q_0|_{q_0=0} \geq 0$.

2. Extension of the canonical formalism to the case of systems with elastic reflections makes it possible to pose the question of the applicability of KAM theory (Kolmogorov-Arnold-Moser). The basic difficulty is that the Hamiltonian functions obtained, generally speaking, are not differentiable on the hypersurfaces $q_0 = 0$ in phase space. Ivanov and Markeev [19] showed with a specific example that nondifferentiability of the Hamiltonian may not impede application of KAM theorems. With the help of the constructions presented below we extend this conclusion to more general cases.

The basic idea consists in reducing the problem of investigating a nonsmooth Hamiltonian phase flow to the investigation of a corresponding symplectic mapping which, as a rule, is smooth (infinitely differentiable).

Let M be a smooth manifold.

DEFINITION 1. A submanifold $\Pi \subset M$ is called a smooth hypersurface in M if in a neighborhood of Π there exists a function $P \in C^\infty$ such that $\Pi = \{x \in M : P(x) = 0\}$ and dP is nonzero on Π.

DEFINITION 2. A function $H: M \to \mathbb{R}$ is called piecewise smooth on M if H is continuous on M and there exist smooth hypersurfaces $\Pi_j = \{P_j = 0\}$, $j = 1, \dots, k$, such that

(a) the function H is smooth on the set $M \setminus \bigcup_1^k \Pi_k$;
(b) for any point $x \in \Pi_j$, $j \in \{1, \dots, k\}$, there exists a neighborhood U_x of the point x in M such that the functions $H|_{U_x^+}$ and $H|_{U_x^-}$, where $U_x^{\pm} = U_x \cap \{\pm P_j > 0\}$, extend to smooth (possibly distinct) functions on U_x.

Problem

Show that the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is piecewise smooth on $\mathbb{R}$, while the function $f(x) = x \sin(x^{-1})$ is not.

DEFINITION 3. Two manifolds $N_1, N_2 \subset M$ intersect transversally if at any point $x \in N_1 \cap N_2$ the direct sum of the tangent spaces $T_x N_1 \oplus T_x N_2$ coincides with $T_x M$.

We say that a vector field $\mathbf{v}$ on M is transversal to a manifold $N \subset M$ if at any point $x \in N$ the vector $\mathbf{v}(x)$ does not lie in the tangent plane $T_x N$.

3. We identify a smooth $2n$-dimensional manifold with the phase space of a mechanical system. Let $\omega = d\Omega$ be a nondegenerate, closed 2-form (an exact *symplectic structure*) on M, and let $H: M \to \mathbb{R}$ be a piecewise smooth function. Away from the surfaces Π_j the function H defines a smooth *Hamiltonian vector field* $\operatorname{sgrad} H$ satisfying the equality

$$dH(\cdot) = \omega(\cdot, \operatorname{sgrad} H).$$

Recall that in canonical local coordinates (p, q) on M the form ω takes the form $dp \wedge dq$, while the field $\operatorname{sgrad} H = (-\partial H/\partial q, \partial H/\partial p)$. The Hamiltonian equations $\dot{x} = \operatorname{sgrad} H$ generate a one-parameter group of diffeomorphisms of the manifold M which preserve the symplectic structure ω—a *phase flow* g_H^t.

LEMMA. *Suppose the surfaces Π_l intersect transversally with the level surface of the energy $\{H = h\} \subset M$; let D_1 and D_2 be two connected $(2n-2)$-dimensional domains in the intersection $\{H = h\} \cap (\bigcup \Pi_l)$ such that*

(a) $D_1 \cap D_2 = \varnothing$;
(b) *the vector field* $\operatorname{sgrad} H$ *is transversal to both domains*;
(c) *the phase flow g_H^t induces the first return mapping $G: D_1 \to D_2$, where the phase trajectories do not intersect the surfaces Π_l outside the domains D_1 and D_2.*

Then

i) *the restrictions of the form ω to the domains D_1 and D_2 are nondegenerate*;
ii) *the mapping G is smooth*;

iii) *the mapping* G *is a local diffeomorphism*;

iv) *the diffeomorphism* G *is exact symplectic, that is, for any closed contour* $\gamma \subset D_1$ *the equality* $\oint_\gamma \Omega = \oint_{G\gamma} \Omega$ *holds.*

PROOF. i) Let D be one of the domains D_1, D_2. We suppose that $\mathbf{v}$ is a tangent vector to the surface of D at the point $x \in D$ which annihilates the form $\omega|_D$, i.e., $\omega(\mathbf{u}, \mathbf{v}) = 0$ for any $\mathbf{u} \in T_x D$.

The vector $\mathbf{w} = \operatorname{sgrad}_x H$ is an annihilator of the form $\omega|_{\{H=h\}}$. This follows from the formula $\omega(\mathbf{u}, \operatorname{sgrad} H) = \operatorname{grad} H(\mathbf{u}) = 0$ for any tangent vector field $\mathbf{u}$ on M_h. If $\mathbf{v} \neq 0$, then by the condition that the intersection of the level surface of the energy is transversal to the surfaces Π_l the vectors $\mathbf{w}$ and $\mathbf{v}$ are linearly independent. Since $\omega(\mathbf{v}, \mathbf{w}) = 0$, the vector $\mathbf{v}$ is also an annihilator of the form $\omega|_{\{H=h\}}$. Hence, the skew-orthogonal complement to the $(2n-1)$-dimensional tangent plane at the point x to the level surface of the energy $\{H = h\}$ in the $2n$-dimensional symplectic space $T_x M$ is at least two-dimensional, which cannot be. Thus, $\mathbf{v} = 0$ and the form $\omega|_D$ is nondegenerate.

ii) By condition (c) of the lemma there exists a continuous function $f\colon D_1 \to \mathbb{R}$ such that $G(x_1) = g_H^{f(x_1)}(x_1) \in D_2$, $x_1 \in D_1$, where points of the form $g_H^t(x_1)$, $0 < t < f(x_1)$, lie off the surfaces Π_l.

The function f satisfies the equation

$$P_2(g^{f(x_1)}(x_1)) = 0, \tag{2}$$

where P_2 is a smooth function defining the smooth hypersurface Π_l inside which there lies the domain D_2. The derivative

$$\frac{\partial}{\partial f} P_2(g^f(x)) = dP_2 \operatorname{sgrad} H$$

is nonzero by the transversality, following from (b), of the field $\operatorname{sgrad} H$ to the surface $\{P_2 = 0\}$. Thus, by the implicit function theorem from equations (2) it is possible to find f, where the function f is smooth. But then the mapping $G(x) = g_H^{f(x)}(x)\colon D_1 \to D_2$ is also smooth, which was required to prove.

iii) For any $x_1 \in D_1$ we have $G(x_1) = g_H^{f(x_1)}(x_1)$, and hence

$$dG(x_1) = d(g^{t_0})(x_1) + \operatorname{sgrad} H(G(x_1)) \cdot df(x_1), \qquad t_0 = f(x_1).$$

Let $\xi \neq 0$ be a tangent vector to the surface D_1 at the point x_1. By proposition (b) the vectors ξ and $\operatorname{sgrad} H(x_1)$ are linearly independent. Since the mapping $d_x(g^t)$ is nondegenerate, it takes these vectors into linearly independent vectors, and $\xi \mapsto \xi' \neq 0$, $\operatorname{sgrad} H(x_1) \mapsto \operatorname{sgrad} H(G(x_1))$. The image of the vector ξ under the mapping dG is the vector $\xi' + \operatorname{sgrad} H(G(x_1)) \cdot df(x_1) \neq 0$. Hence, the mapping dG is nondegenerate and by the inverse function theorem G is a local diffeomorphism.

iv) Consider the "tube" formed by the phase trajectories of the Hamiltonian equations passing through points of the closed curve $\gamma \subset D_1$. We apply *Stokes' formula* to the piece of the tube bounded by the curves γ and $G\gamma$:

$$\oint_\gamma \Omega - \oint_{G\gamma} \Omega = \int_{\partial\Sigma} \Omega = \int_\Sigma \omega,$$

where Σ is the lateral surface of the tube.

The value of the form ω on any pair of vectors tangent to the tube is equal to zero. Indeed, the tube lies on a level surface of the energy, and the tangent planes to it are two-dimensional and contain the annihilator $\operatorname{sgrad} H$. Hence

$$\int_\Sigma \omega = 0 \quad \text{and} \quad \oint_\gamma \Omega = \oint_{G\gamma} \Omega.$$

The lemma is proved.

4. Corollary 1. *Suppose the Hamiltonian H and the surfaces of loss of smoothness $\Pi_l = \{P_l = 0\}$ depend smoothly on the parameter $\varepsilon : H = H^\varepsilon$, $\Pi_l = \Pi_l^\varepsilon$; for $\varepsilon = 0$ there exists a sequence of pairwise disjoint $(n-2)$-dimensional domains $D_1, \dots, D_m \subset \{H^0 = h\} \cap (\bigcup \Pi_l^0)$ such that each pair $D_1, D_2; D_2, D_3; \dots; D_m, D_1$ satisfies the conditions of the lemma, where $G_1, \dots, G_m$ are diffeomorphisms. We suppose also that the mapping $G_m \circ \cdots \circ G_1 : D_1 \to D_1$ is integrable and nondegenerate, i.e., in some coordinates I, $\varphi \bmod 2\pi$ on D_1 it has the form*

$$I \mapsto I, \qquad \varphi \mapsto \varphi + \omega(I); \qquad \det(\partial\omega/\partial I) \neq 0.$$

Then for small ε the initial Hamiltonian system has invariant n-dimensional tori close to the invariant tori of the unperturbed system whose total measure tends to the full measure as $\varepsilon \to 0$.

Indeed, for small ε the domains $D_1, \dots, D_m$ are slightly deformed together with the surfaces Π_l and go over into domains $D_1^\varepsilon, \dots, D_m^\varepsilon$ which pairwise satisfy the conditions of the lemma (only the mapping G_m^ε is possibly not a diffeomorphism of D_m^ε onto D_1^ε but onto some domain $\widehat{D}_1^\varepsilon$ such that the difference $\widehat{D}_1^\varepsilon \setminus D_1^\varepsilon$ is small together with ε). The composite mapping $G_m^\varepsilon \circ \cdots \circ G_1^\varepsilon : D_1 \to \widehat{D}_1$ is smooth, exactly symplectic, and is close to integrable, i.e., in some coordinates I, φ on $D_1 \cup \widehat{D}_1$ it has the form

$$I \mapsto I + \varepsilon f(I, \varphi), \qquad \varphi \mapsto \varphi + \omega(I) + \varepsilon g(I, \varphi).$$

Thus, by the theorem on invariant tori of exact symplectic mappings [49] the transformation $G_m^\varepsilon \circ \cdots \circ G_1^\varepsilon$ has a large number of nonresonant $(n-1)$-dimensional tori. The phase flow of the original Hamiltonian equations converts them into n-dimensional invariant tori of the perturbed system.

We note that these tori, generally speaking, are not smooth; smoothness is lost at points of their intersection with the surfaces D_j; see Figure 51.

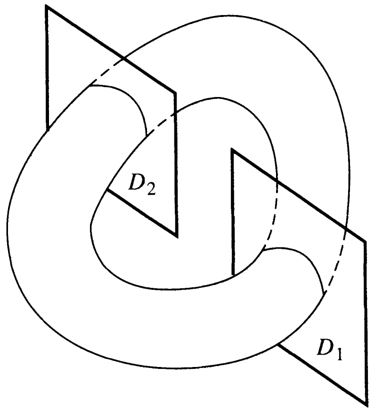


FIGURE 51

5. It is known that elliptic periodic trajectories in general position in a smooth Hamiltonian system with two degrees of freedom are orbitally stable on a level surface of the energy [3].

COROLLARY 2. *This result remains true in the case of piecewise smooth Hamiltonians if it is additionally required that the periodic trajectory transversally intersect the surfaces of loss of smoothness* (which, by the way, is also a condition for general position).

In the case of three or more degrees of freedom it is necessary to speak of orbital stability for a majority of initial conditions.

Indeed, we consider the first return mappings $G_j : D_j \to D_{j+1}$, where D_j is a small neighborhood on $\Pi_j \cap \{H = h\}$ of a point of intersection of the trajectory with this surface (assuming that the surfaces Π_j are enumerated in suitable order). Here to a periodic trajectory there corresponds a fixed point of the smooth, exact symplectic diffeomorphism $G_m \circ \cdots \circ G_1 : D_1 \to \widehat{D}_1$. The assertion of Corollary 2 thus follows from a well-known result concerning fixed points of symplectic diffeomorphisms.

6. As an example we consider the billiard in the Euclidean cylinder

$$C_\varepsilon = \{x\,,\ \varphi \bmod 2\pi : \varepsilon f_1(\varphi) \le x \le \pi + \varepsilon f_2(\varphi)\}\,.$$

As usual, we assume that in the region $C_\varepsilon \setminus \partial C_\varepsilon$ the trajectories are geodesics in the natural metric $dx^2 + d\varphi^2$, while the law of reflection from the boundary ∂C is elastic (see Figure 52).

PROPOSITION 1. *For small ε the billiard in the cylinder C_ε possesses a large number of quasiperiodic trajectories close to the trajectories of the integrable billiard in C_0.*

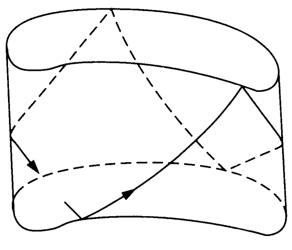


FIGURE 52

Following the constructions of [19] expounded in part 1, we assign to the billiard a Hamiltonian system with piecewise smooth Hamiltonian H_ε in the following manner. As configuration space we take the torus $\mathbb{T}^2$ comprised of two copies $C_\varepsilon^\pm$ of the cylinder C_ε, glued together, where the "upper" ("lower") boundary of the cylinder C^+ is glued to the "upper" ("lower") boundary of the cylinder C^-. As local coordinates giving on $\mathbb{T}^2$ the structure of a smooth manifold in a neighborhood of any point lying in $C^+\backslash\partial C^+$ $(C^-\backslash \partial C^-)$ it is possible to take the coordinates x, φ; in a small neighborhood U on $\mathbb{T}^2$ intersecting the set $\partial C^\pm$ such coordinates are $\hat{x}$, $\hat{\varphi}$, where $|\hat{x}| = y$, and y, $\hat{\varphi}$ are semigeodesic coordinates on the set $U \cap C^+(U \cap C^-)$.

The submanifold $\partial C^+ = \partial C^- \subset \mathbb{T}^2$ is obviously smooth, and in some "global" coordinates $(z, \psi) \bmod 2\pi$ on $\mathbb{T}^2$ it has the form $\{(z, \psi) \in \mathbb{T}^2 : z = 0 \text{ or } z = \pi\}$. We introduce the phase space of this system:

$$M = \{(p_2, p_\psi, z, \psi) : (p_z, p_\psi) \in \mathbb{R}^2, (z, \psi) \in \mathbb{T}^2\}.$$

The Hamiltonian coincides with the kinetic energy. It is a piecewise smooth function on the whole phase space and belongs to the class C^∞ off the set $\Pi_0 \cup \Pi_\pi$, where

$$\Pi_0 = \{(p_z, p_\psi, z, \psi) \in M : z = 0\},$$
$$\Pi_\pi = \{(p_z, p_\psi, z, \psi) \in M : z = \pi\}.$$

For $\varepsilon = 0$ it may be assumed that the coordinates (z, ψ) coincide with the coordinates (x, φ), where $C^+ = \{(x, \varphi) \in \mathbb{T}^2 : 0 \le x \le \pi\}$, $C^- = \{(x, \varphi) \in \mathbb{T}^2 : -\pi \le x \le 0\}$ and the diffeomorphisms $\pi_\pm : C_0 \to C^\pm$ have the form $\pi_\pm(x, \varphi) = (\pm x, \varphi)$ (see Figure 53).

The Hamiltonian $H_0 = \frac{1}{2}(p_\varphi^2 + p_x^2)$ is analytic on M. We shall verify that the conditions of Corollary 1 are satisfied.

Consider the domains

$$D_0 = (\Pi_0 \setminus \{p_x = 0\}) \cap \{H_0 = h\}, \qquad D_\pi = (\Pi_\pi \setminus \{p_x = 0\}) \cap \{H_0 = h\}.$$

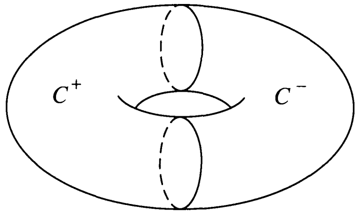


FIGURE 53

It is obvious that the intersections indicated are transversal and the Hamiltonian vector field $\operatorname{sgrad} H_0 = (0, 0, p_x, p_\varphi)$ is transversal to the two-dimensional surfaces D_0 and D_π. As coordinates on D_0 and D_π it is possible to take p_φ and φ. The phase flow induces the mappings $G_1 : D_0 \to D_\pi$ and $G_2 : D_\pi \to D_0$ which in the coordinates p_φ, φ have the form

$$G_i(p_\varphi, \varphi) = (p_\varphi, \varphi + \pi p_\varphi (2h - p_\varphi^2)^{-1/2}), \qquad i = 1, 2.$$

Hence, the mapping $G_2 \circ G_1$ has the form

$$G_2 \circ G_1(p_\varphi, \varphi) = (p_\varphi, \varphi + 2\pi p_\varphi (2h - p_\varphi^2)^{-1/2}).$$

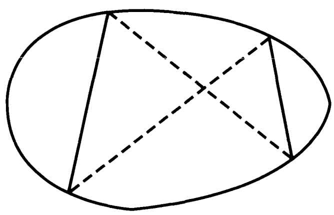
It is integrable and nondegenerate, and the variables p_φ, φ are action-angle variables for it. The conclusion of the proposition thus follows from Corollary 1.

APPENDIX II

On the Connection of Dynamic and Geometric Properties of Periodic Trajectories

It turns out that Theorem 2 of Chapter II is a discrete analogue of a general theorem relating multipliers of periodic trajectories of the Lagrange equations (in particular, geodesic equations) and properties of the corresponding critical point of the action.

In order to illustrate the connection of the billiard with the problem of geodesics on a Riemannian manifold we consider a smooth, closed, two-dimensional surface in $\mathbb{R}^3$ and deform it so that it remains smooth and tends to a plane domain (cf. [42], Chapter 6). The geodesics on the surface then tend to trajectories of the corresponding billiard. A closed geodesic on the surface goes over to a periodic trajectory of the billiard having the same Morse index and an even number of links (see Figure 54).



It is natural to expect that the type (elliptic, hyperbolic) in a typical case also does not change in passing to the limit.

It should, however, be noted that these considerations are of heuristic character. It is apparently not simple to realize the Birkhoff propositions by passing to the limit. In any case there are no rigorous proofs of this in the literature. Moreover, rigorous proofs are also required of the preservation of the Morse index, and of the type of a periodic solution in passing to limit. Thus, the connection just described between billiard problems and geodesics on Riemannian manifolds is only a clue to understanding deeper results.

1. It is known that the dynamics of Hamiltonian systems (including systems with elastic reflections) are subject to variational principles. In connection with this circumstance the characteristics of periodic trajectories of Hamiltonian systems can be subdivided into two classes: dynamic and geometric. The first are determined by the Poincaré mapping corresponding to a given periodic solution of the equations of motion. They contain the characteristic exponents, the properties of nondegeneracy (in the sense of Poincaré), and orbital stability. The second are characteristics of the periodic trajectory as a critical point of the action functional. Among these are the Morse index, nondegeneracy in the sense of Morse, and also the Hill determinant introduced below.

Poincaré first called attention to the connection of orbital stability of a closed geodesic on a Riemannian manifold with properties of the corresponding critical point of the action functional. He proved that a nondegenerate closed geodesic of local minimum length on a two-dimensional orientable Riemannian manifold is hyperbolic and hence unstable [66]. This result was subsequently generalized through efforts of a number of authors. It turns out that the Morse index of a nondegenerate, elliptic closed geodesic on a two-dimensional Riemannian manifold is odd if the geodesic is orientable and even otherwise (see, for example, [60]).

With consideration of the heuristic limiting procedure noted above from this assertion we obtain the following result for the Birkhoff billiard: the Morse index of a nondegenerate, even-link, elliptic, periodic trajectory with elastic reflections is always odd. In Chapter II this result was obtained as a corollary of Theorem 2.

On the other hand, in his classical memoir G. Hill obtained a remarkable formula which in the case of Hamiltonian systems with one and a half degrees of freedom establishes a connection between the geometric and dynamic characteristics of a periodic solution. Hill's theory was expounded in §4, Chapter III. It turns out that Hill's results admit generalization to the multidimensional case, whereby Theorem 2 of the second chapter is a discrete analogue of them. A multidimensional generalization of Hill's formula was found in [36] with the use of the method of finite-dimensional approximation (in the spirit of Morse theory). A version of generalized Hill theory proposed by Bolotin [8] is presented below.

2. Let a smooth n-dimensional manifold M be the space of positions of a mechanical system with n degrees of freedom. The Lagrange function L is defined on the extended phase space $TM \times \mathbb{R}$. It is considered to be smooth, strictly convex in the velocities, and τ-periodic in time. According to Hamilton's principle, τ-periodic trajectories corresponding to the Lagrangian system coincide with the critical points of the action functional $S(\gamma) = \int_0^\tau L(\dot{\gamma}, \gamma, t)\,dt$ on the set of τ-periodic curves $\gamma\colon [0, \tau] \to M$, $\gamma(0) = \gamma(\tau)$.

For any value $t \bmod \tau$ the second differential of the Lagrange function with respect to the velocity is a positive definite quadratic form and defines a scalar product $(\cdot\,, \cdot)$ on the tangent space $T_{\gamma(t)}M$. Let $\dot{\xi}$ be the covariant derivative of the vector field $\xi(t) \in T_{\gamma(t)}M$ along γ consistent with the metric: $(\xi, \eta)^{\cdot} = (\dot{\xi}, \eta) + (\xi, \dot{\eta})$. The second variation of the functional S at the critical point γ is a quadratic form on the set of smooth, τ-periodic vector fields ξ along γ:

$$\delta^2 S|_\gamma(\xi) = \int_0^\tau [(\dot{\xi}(t), \dot{\xi}(t)) + 2(W(t)\xi(t), \dot{\xi}(t)) + (V(t)\xi(t), \xi(t))]\, dt,$$

where V, W are linear operators in $T_{\gamma(t)}M$, and it may be assumed that the operator V is selfadjoint: $V(t) = V^*(t)$. We assign to the quadratic form $\delta^2 S|_\gamma$ a bilinear form h such that $\delta^2 S|_\gamma(\xi) = h(\xi, \xi)$. By integration by parts it can be represented in the form

$$h(\xi, \eta) = \int_0^\tau [(\nabla\xi(t), \nabla\eta(t)) + (U(t)\xi(t), \eta(t))]\, dt, \tag{1}$$

where $\nabla\xi = \dot{\xi} + \frac{1}{2}(W - W^*)\xi$ and $U = V - \frac{1}{2}(\dot{W} + \dot{W}^*)$ is a symmetric linear operator. Since $(\nabla\xi, \eta) + (\xi, \nabla\eta) = (\xi, \eta)^{\cdot}$, the operator ∇ is skew-symmetric relative to the scalar product

$$\langle \xi, \eta \rangle = \int_0^\tau (\xi(t), \eta(t))\, dt.$$

Therefore, the variational equation of the trajectory γ has the form

$$-\nabla^2\xi(t) + U(t)\xi(t) = 0. \tag{2}$$

We extend the bilinear form h to a Hermitian form $h(\xi, \overline{\eta})$ on the space X of complex τ-periodic absolutely continuous vector fields ξ along γ such that

$$\langle \dot{\xi}, \overline{\dot{\xi}} \rangle < \infty. \tag{3}$$

We define on X the structure of a Hilbert space by setting $\langle \xi, \eta \rangle_\nabla = \langle \nabla\xi, \overline{\nabla\eta} \rangle + \langle \xi, \overline{\eta} \rangle$. Then

$$h(\xi, \overline{\eta}) = \langle \nabla\xi, \overline{\nabla\eta} \rangle + \langle U\xi, \overline{\eta} \rangle = \langle H\xi, \eta \rangle_\nabla,$$

where $H = (-\nabla^2 + E)^{-1}(-\nabla^2 + U) = E + (-\nabla^2 + E)^{-1}(U - E)$ is a selfadjoint operator in X.

From the boundedness of the operator $U - E$ and the structure of the spectrum of ∇ discussed below it follows easily that $\operatorname{Tr}(H - E)$ is finite. Thus, the operator $H - E$ is *nuclear* and hence the determinant $\det H$ exists [69]. We call it the *Hill determinant* of the trajectory γ. The infinite Hill matrix is the matrix of a *Sturm-Liouville operator* in some basis.

The following generalization of the Hill determinant is essentially contained in Hill's work [55]. For a given ρ, $|\rho| = 1$, let X_ρ be the space of complex, absolutely continuous vector fields ξ along γ which satisfy (3) and are such that $\xi(t + \tau) \equiv \rho\xi(t)$. We define on X_ρ a ρ-index form [60] of the trajectory γ by the formula of second variation (1). We identify X and X_ρ, assigning to a vector field $\xi \in X$ the vector field $e^{\mu t}\xi(t)$ of X_ρ, π where

$\mu = \tau^{-1} \ln \rho$; $0 \leq \mu\tau/i < 2\pi$. We obtain a Hermitian form h_ρ on X which is a generalization of the second variation and has the form

$$h_\rho(\xi, \overline{\eta}) = h(e^{\mu t}\xi, \overline{e^{\mu t}}\overline{\eta})$$
$$= \langle(\nabla + \mu E)\xi, \overline{(\nabla + \mu E)\eta}\rangle + \langle U\xi, \overline{\eta}\rangle = \langle H_\rho \xi, \eta\rangle_\nabla,$$

where

$$H_\rho = (-\nabla^2 + E)^{-1}(-(\nabla + \mu E)^2 + U) \tag{4}$$

is a bounded operator in X. We henceforth assume that ρ may assume any complex values. Although for $\rho \neq 1$ the operator H_ρ is not of trace class, it is possible to define $\det H_\rho$ by means of finite-dimensional approximation (see formula (6) below).

THEOREM 1. *Let P be the monodromy operator of the variational equation* (2) (*the linear Poincaré mapping of the periodic trajectory γ*), *let Q be the monodromy operator of the equation of parallel transport $\nabla\xi(t) = 0$, and let $\sigma = \pm 1$ depending on whether the trajectory γ preserves or reverses the orientation. Then the determinant of H_ρ can be expressed in terms of the characteristic polynomial* $\det(\rho E - P)$ *of the trajectory γ*:

$$\det H_\rho = \sigma(-1)^n \frac{e^{n\tau} \det(\rho E - P)}{\rho^n \det^2(e^\tau E - Q)}. \tag{5}$$

Since Q is an orthogonal operator, the denominator in (5) does not vanish. For $\rho = 1$ formula (5) expresses the Hill determinant $\det H$ in terms of $\det(E - P)$ (cf. Theorem 2 of Chapter II).

In the one-dimensional case (when $n = 1$) $\sigma = 1$, $\det(e^\tau E - Q) = e^\tau - 1$, and

$$\det(\rho E - P) = (\rho - 1)^2 + \rho \det(E - P).$$

With consideration of these relations from (5) we obtain the formula

$$\det H_\rho = \det H - \frac{e^\tau}{(e^\tau - 1)^2} \frac{(\rho - 1)^2}{\rho}$$

which is equivalent to the Hill formula (4.6) of Chapter III. In order to see this it is necessary to consider the orthogonal basis of functions $\{\exp(2int)\}_{-\infty}^{\infty}$ on the circle $t \bmod \pi$ and represent the operators H_ρ and H in this basis. The verification is left to the reader as an exercise.

3. We note a number of corollaries of Theorem 1.

a) Suppose that the Lagrange function which is quadratic in the velocity does not depend explicitly on time and defines a Riemannian metric (,) on M. Then γ is a closed geodesic. In this case γ always has two unit multipliers: $\det H = \det(E - P) = 0$. Let H_ρ^0 be the restriction of the operator H_ρ to the invariant subspace of vector fields $\xi \in X$ orthogonal to $\dot{\gamma}(t)$ $((\dot{\gamma}(t), \xi(t)) \equiv 0)$, let P^0 be the monodromy operator corresponding to the restriction of the variational equation (2) to the set of vector fields orthogonal to $\dot{\gamma}(t)$, and let Q^0 be the mapping of parallel transport along γ of vectors orthogonal to $\dot{\gamma}(0)$. Then

$$\det H_\rho^0 = \sigma(-1)^{n-1} \frac{e^{(n-1)\tau} \det(\rho E - P^0)}{\rho^{n-1} \det^{-2}(e^\tau E - Q^0)}. \tag{6}$$

Indeed, $\det H_\rho$ can be represented in the form of the product of $\det H_\rho^0$ and the determinant D of the restriction of H_ρ to the set of vector fields parallel to $\dot{\gamma}(t)$. We restrict the Lagrangian to the two-dimensional subspace $T\gamma \subset TM$ and apply Theorem 1 to the system obtained. On the left side of formula (5) in this case will be the determinant D. Thus,

$$D = -e^\tau \rho^{-1}(\rho - 1)^2(e^\tau - 1)^{-2}.$$

But

$$\det(\rho E - P) = (\rho - 1)^2 \det(\rho E - P^0),$$
$$\det(e^\tau E - Q) = (e^\tau - 1)\det(e^\tau E - Q^0),$$

which was required to prove.

We note that the investigation of any autonomous natural Lagrangian system in a potential force field reduces to the investigation of the problem of geodesics in the Jacobi metric.

b) Definition A. A periodic trajectory of a Lagrangian system (respectively, a closed geodesic on a Riemannian manifold) is called nondegenerate in the Poincaré sense if the spectrum of the matrix P (respectively, P^0) does not contain one.

Definition B. A periodic trajectory (respectively, a closed geodesic) is called nondegenerate in the sense of Morse if the determinant $\det H$ (respectively, $\det H^0$) is nonzero.

It follows easily from formulas (5), (6) that nondegeneracy in the sense of Poincaré is equivalent to nondegeneracy in the sense of Morse.

c) We define the *ρ-index* $\operatorname{ind}_\rho \gamma$ of a *periodic trajectory* (a closed geodesic) of γ as the index of the Hermitian form h_ρ. As usual, we call the quantity $\operatorname{ind}\gamma = \operatorname{ind}_1 \gamma$ *the Morse index* of γ which is equal to the number of negative eigenvalues of the operator H (or H^0).

Suppose the trajectory γ is nondegenerate. Then

$$(-1)^{\operatorname{ind}_\rho \gamma} = \operatorname{sgn} \det H_\rho = \sigma(-1)^n \operatorname{sgn}(\rho^{-n} \det(\rho E - P)).$$

In the geodesic case we have

$$(-1)^{\operatorname{ind}_\rho \gamma} = \operatorname{sgn} \det H_\rho^0 = \sigma(-1)^{n-1} \operatorname{sgn}(\rho^{1-n} \det(\rho E - P^0)).$$

The argument of sgn is real, since the characteristic polynomial is recurrent.

d) Suppose the periodic trajectory (closed geodesic) is nondegenerate and $\sigma(-1)^{n+\operatorname{ind}\gamma} < 0$ $(\sigma(-1)^{n-1+\operatorname{ind}\gamma} < 0)$. Then by c) $\det(E - P) < 0$

($\det(E - P^0) < 0$), so that there exists a real multiplier $\lambda > 0$ (actually, $\det(\lambda E - P) > 0$ for large real $\lambda > 0$). Therefore, the trajectory γ is orbitally unstable.

In particular, nondegenerate closed geodesics of local minimal length on an even-dimensional orientable manifold are orbitally unstable.

e) Suppose $n = 1$ and the 2τ-periodic trajectory γ^2 is γ traversed twice. If γ^2 is nondegenerate, then the trajectory γ has hyperbolic (elliptic) type if and only if $\operatorname{ind}\gamma^2$ is even (odd).

Indeed, the multipliers $\lambda_1 = \lambda_2^{-1}$ of the trajectory γ^2 are equal to the squares of the multipliers of the trajectory γ. Hence, hyperbolicity of γ^2 is equivalent to the conditions that the quantities λ_1, λ_2 be real and positive or the condition $\operatorname{sgn}(\det(E - P^2)) = (-1)^{1+\operatorname{ind}\gamma^2} = -1$ (γ^2 always preserves orientation). Similarly, ellipticity of γ^2 is equivalent to the condition $(-1)^{1+\operatorname{ind}\gamma^2} = 1$. It remains to use the fact that γ and γ^2 are simultaneously elliptic or hyperbolic.

The corresponding assertion for geodesics looks as follows. Let γ be a closed geodesic on a two-dimensional Riemannian manifold. If γ^2 is nondegenerate, then γ has hyperbolic (elliptic) type if and only if $\operatorname{ind}\gamma^2$ is even (odd).

4. The proof of Theorem 1 follows the proof of Hill's result [55]. The real, skew-Hermitian operator $\nabla = \overline{\nabla} = -\nabla^*$ has compact resolvent $(\nabla + \mu E)^{-1}$. Its spectrum $\Lambda \subset i\mathbb{R}$ coincides with the set of characteristic exponents of the equation $\nabla\xi(t) = 0$, i.e., with the set of $\nu \in \mathbb{C}$ such that $\det(e^{\nu\tau}E - Q) = 0$. If $\nu \in \Lambda$, then $-\nu$ and $\nu + \omega$ belong to Λ where $\omega = 2\pi i/\tau$.

Define $\det H_\rho$ by means of the finite-dimensional approximation

$$\det H_\rho = \lim_{N\to\infty} \det P_N H_\rho P_N^*, \tag{7}$$

where $P_N \colon X \to X$ is the orthogonal projection onto the finite-dimensional eigensubspace of the operator ∇ corresponding to the eigenvalues $\nu \in \Lambda$ such that $|\nu| \le N$.

Since $P_N\nabla = \nabla P_N$, by (4) for $\mu \notin \Lambda = -\Lambda$ we have

$$\det H_\rho = \det(-(-\nabla^2 + E)^{-1}(\nabla + \mu E)^2)\det(E - (\nabla + \mu E)^{-2}U), \tag{8}$$

where the finite-dimensional approximation of the determinant $f(\mu) = \det(E - (\nabla + \mu E)^{-2}U)$ converges absolutely for $\mu \notin \Lambda$, since the operator $(\nabla + \mu E)^{-2}U$ is of trace class. Thus, f is a holomorphic function on $\mathbb{C}\backslash\Lambda$ having at points of Λ poles of multiplicity no greater than double the multiplicity of the corresponding points of the spectrum of ∇ [69].

The function f is periodic with period $\omega = 2\pi i/\tau$: $f(\mu + \omega) \equiv f(\mu)$. Indeed, if $\xi \in X$, then $e^{\omega t}\xi \in X$ and

$$(E - (\nabla + \mu E)^{-2}U)e^{\omega t}\xi = e^{\omega t}(E - (\nabla + (\mu + \omega)E)^{-2}U)\xi.$$

The poles of the function f are contained among the poles of $\det^{-2}(e^{\mu\tau}E - Q)$. It is therefore possible to choose a polynomial $g(\rho)$ of degree no higher than $2n - 1$ such that the functions $f(\mu)$ and $g(\rho)\det^{-2}(\rho E - Q)|_{\rho=e^{\mu\tau}}$ have the same principal parts of the Laurent expansion at each pole. Since $f(\mu) \to 1$ as $\operatorname{Re}\mu \to +\infty$ or $|\rho| \to \infty$, by *Liouville's theorem*

$$f(\mu) = 1 + g(\rho)\det{}^{-2}(\rho E - Q)|_{\rho=e^{\mu\tau}}. \tag{9}$$

The second determinant in (8) converges conditionally, but it can be computed explicitly. By formula (7)

$$\det(-(-\nabla^2 + E)^{-1}(\nabla + \mu E)^2) = \lim_{N\to\infty} \prod_{\substack{\nu\in\Lambda \\ |\nu|\le N}} \frac{(\nu+\mu)^2}{\nu^2 - 1}$$

$$= \lim_{N\to\infty}(-\mu)^k \prod_{\substack{\nu\in\Lambda \\ |\nu|\le N,\, i\nu>0}} \left(\frac{\nu^2-\mu^2}{\nu^2-1}\right)^2 = (-1)^k \prod_{\nu\in\Lambda} \frac{\nu^2-\mu^2}{\nu^2-1},$$

where k is the multiplicity of zero in the spectrum of ∇. We have used the fact that $\Lambda = -\Lambda$.

From the form of the spectrum of the operator ∇ it follows that the last product converges absolutely. To compute it we use the formula (see [13])

$$\prod_{n=-\infty}^{\infty}\left(1 - \frac{\mu^2}{(\nu+\omega n)^2}\right) = \frac{\operatorname{ch}\,\mu\tau - \operatorname{ch}\,\nu\tau}{1 - \operatorname{ch}\,\nu\tau}.$$

Let $\rho_1, \dots, \rho_n$ be the roots of the characteristic polynomial $\det(\rho E - Q)$; $\nu_j = \tau^{-1}\ln\rho_j$, $j = 1, \dots, n$. Then

$$\prod_{\nu\in\Lambda}\frac{\nu^2-\mu^2}{\nu^2-1} = \prod_{\nu\in\Lambda}\left(1-\frac{\mu^2}{\nu^2}\right)\left(1-\frac{1}{\nu^2}\right)^{-1}$$

$$= \prod_{j=1}^{n}\prod_{m=-\infty}^{\infty}\left(1-\frac{\mu^2}{(\nu_j+\omega m)^2}\right)\left(1-\frac{1}{(\nu_j+\omega m)^2}\right)$$

$$= \prod_{j=1}^{n}\frac{\operatorname{ch}\,\mu\tau - \operatorname{ch}\,\nu_j\tau}{\operatorname{ch}\,\tau - \operatorname{ch}\,\nu_j\tau} = \prod_{j=1}^{n}\frac{\rho+\rho^{-1}-\rho_j-\rho_j^{-1}}{e^{\tau}+e^{-\tau}-\rho_j-\rho_j^{-1}}$$

$$= e^{n\tau}\rho^{-n}\prod_{j=1}^{n}\frac{(\rho-\rho_j)(\rho-\rho_j^{-1})}{(e^{\tau}-\rho_j)(e^{\tau}-\rho_j^{-1})} = e^{n\tau}\rho^{-n}\left(\prod_{j=1}^{n}\frac{\rho-\rho_j}{e^{\tau}-\rho_j}\right)^2$$

$$= e^{n\tau}\rho^{-n}\frac{\det^2(\rho E - Q)}{\det^2(e^{\tau}E - Q)}.$$

By (8) and (9)

$$\det H_\rho = (-1)^k\frac{e^{n\tau}(\det^2(\rho E - Q) + g(\rho))}{\rho^n\det^2(e^{\tau}E - Q)}.$$

Thus, $\rho^n \det H_\rho$ is a polynomial of degree $2n$ in ρ with leading coefficient $(-1)^k e^{n\tau} \det^{-2}(e^\tau E - Q)$.

It can be shown that $\det H_\rho = 0$ if and only if the operator H_ρ has a nonzero kernel. The kernel of H_ρ consists of τ-periodic vector fields ξ such that $(-\nabla^2 + U)e^{\mu t}\xi(t) = 0$. Therefore, the roots of the polynomial $\rho^n \det H_\rho$ and of the characteristic polynomial $\det(\rho E - P)$ of the variational equation (2) coincide. Thus,

$$\rho^n \det H_\rho = (-1)^k e^{n\tau} \frac{\det(\rho E - P)}{\det^2(e^\tau E - Q)}.$$

We remark that k is the dimension of the subspace on which the orthogonal operator Q is the identity, while $\sigma = (-1)^l$ where l is the dimension of the subspace on which Q is a reflection. Since the dimension $n - k - l$ of the complementary subspace is even, formula (5) has been proved.

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