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**Topology of
Lie Groups,
I and II**

Mamoru Mimura
Hirosi Toda



American Mathematical Society

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Hiroshi Toda

Topology of Lie Groups, I and II



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Lie groups are very general mathematical objects that appear in numerous areas such as topology, functional analysis, and algebra, as well as differential geometry and differential topology. The purpose of these two parts is to provide a guide to the topology of Lie groups and homogeneous spaces by bringing together a wide range of results relating to them. The first part thoroughly studies topological properties of the classical groups as typical examples of Lie groups. In the second part, the authors study general properties of compact Lie groups, particularly the exceptional groups.

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