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Рациональные аппроксимации и ортогональность

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NOTATION. This book presents a wide range of problems connected with rational approximations of numbers and analytic functions. These problems touch on many topics in contemporary mathematical analysis: analytic functions, orthogonal polynomials, spectral theory of operators, potential theory, etc. The development of the theory of Padé approximants, and the current application of this theory in related disciplines, motivate the book, which is an introduction to this circle of problems.

The book is intended for students and future research workers who are interested in function theory and number theory.

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Contents

Preface

This is a book about rational approximation, principally of analytic functions and, to some extent, of real numbers. In both theories (numerical and functional) we encounter many general features that originate from the concept of the Euclidean algorithm. In the functional domain, these lead to the Padé approximants; in the numerical domain, to the theory of continued fractions and Diophantine approximation. In turn, the theory of Padé approximants for classes of functions of Markov type (integrals of Cauchy type for measures on the line) coincides with the theory of orthogonal polynomials and the theory of discrete Sturm-Liouville operators. These, and more general objects (simultaneous approximations of several functions) and their study, require the introduction of potential theory, and of boundary problems for analytic functions. Consequently, the study of rational approximations involves a variety of mathematical concepts and objects. The book aims to introduce the reader to these concepts and objects.

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NOTATION.

$\mathrm{GCD}(p, q)$	greatest common divisor of the integers p and q
$\mathrm{LCM}(p, q)$	least common multiple of the integers p and q
$[\alpha]$	integral part of the real number α
$\deg \alpha$	degree of the algebraic number α
$H(\alpha)$	height of the algebraic number α
L_μ^2	Lebesgue space
l_1	the Banach space of sequences such that $\sum \lvert x_n \rvert < +\infty$
l_2	the Banach space of sequences such that $\sum \lvert x_n \rvert^2 < +\infty$
$\lim_{r \to 1-}$	limit from the left
$\lim_{\varepsilon \to 0+}$	limit from the right
$\stackrel{\text{def}}{=}$	formula of definition
$\triangleright$	end of proof
$\in$	"is a member of"

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