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Volume 93

Elements of the Geometry and Topology of Minimal Surfaces in Three-Dimensional Space

A. T. Fomenko
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American Mathematical Society

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ЭЛЕМЕНТЫ ГЕОМЕТРИИ И ТОПОЛОГИИ
МИНИМАЛЬНЫХ ПОВЕРХНОСТЕЙ
В ТРЕХМЕРНОМ ПРОСТРАНСТВЕ

А. Т. Фоменко и А. А. Тужилин

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Abstract. This book arose as a result of writing up the lectures of A. T. Fomenko delivered at the Moscow Mathematical Society (in Moscow University) for students of mathematics, physics and mechanics (in the framework of the cycle of lectures "Readings for students"). The writing up of the lectures for the press has been carried out by A. A. Tuzhilin and A. T. Fomenko.

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Contents

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Introduction

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The reader who wishes to extend his study of the theory of minimal surfaces can turn to the following more special books and articles:

H. Federer, *Geometric measure theory*, Springer-Verlag, New York, 1969.

J. C. C. Nitsche, *Vorlesugen über Minimalflächen*, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

E. Giusti, *Minimal surfaces and functions of bounded variation*, Birkhäuser, Boston-Basel-Stuttgart, 1984.

A. T. Fomenko, *Topological problems in topology. The geometry of length, area and volume*, Moscow University Press, Moscow, 1984; English transl., Gordon and Breach, 1990.

A. T. Fomenko, *Variational principles in topology*, "Nauka", Moscow, 1982; English transl., *Multidimensional minimal surface theory*, Kluwer, 1990.

Dào Trong Thi and A. T. Fomenko, *Minimal surfaces and Plateau's problem*, "Nauka", Moscow, 1987; English transl., Amer. Math. Soc., Providence, RI, 1991.

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R. Osserman, *A survey of minimal surfaces*, Van Nostrand Reinhold, New York-London-Melbourne, 1969.

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APPENDIX

Steiner's Problem for Convex Boundaries

In Chapter 1 we spoke about Steiner's problem, which, we recall, consists in constructing a network of minimal length joining n given points in a plane. In this Appendix we present recent results of A. O. Ivanov and A. A. Tuzhilin [41], in which they obtained a classification of nondegenerate minimal networks without cycles and with convex boundary up to planar equivalence (Theorems 1 and 2). They discovered that all such minimal networks can be described as dual graphs to "planar tree tilings"; for the exact definition of these see below. They also produced an algorithm, realized on a computer, for calculating all networks of the given type for any fixed number of boundary points. Also, the authors found several infinite series of minimal networks with a regular convex boundary and individual interesting examples of such networks not contained in them. In the latter case it is important that the set of boundary points of the network is fixed, which considerably complicates the investigation.

1. General statement of the problem. We shall understand the minimality of a network in the following sense: any small fragment of the network has the shortest length. We recall that soap films have a similar property. When it is a question of shortest length, we need to distinguish the class of admissible variations of the network. First of all, by analogy with the way we defined minimal surfaces by means of the variational principle, we restrict the admissible variations just to those that leave fixed the initial points spanned by the network (henceforth we shall call these points the *fixed points* of the network). Two possibilities arise.

The first possibility: under a deformation of the network its vertices do not split, that is, variations of the type shown in Figure 48a are forbidden. In this case it can be shown (see [33], for example) that the network is minimal if and only if for each moving point the sum of the unit vectors having the directions of the segments going out from it is equal to zero. In this sense the network given in Figure 48a is minimal. In Figure 48b we give an example of a network that is not minimal in this sense.

The second possibility: The vertices of the network are allowed to split. In

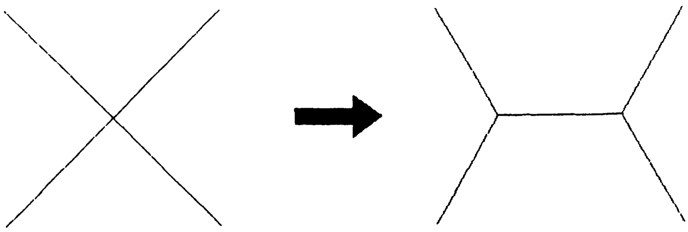

FIGURE 48A

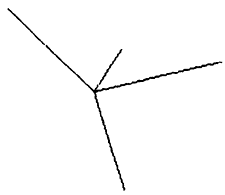

FIGURE 48B

this case under a deformation of the network in the direction of decreasing length its vertices split into points of degree at most three, and if three segments now meet at a vertex, then the angles between them are equal to 120°; if two segments meet at a vertex, then the angle between them is at least 120° and this vertex is fixed; also, each vertex from which exactly one segment goes out is fixed.

All these effects can be observed in the following simple experiment. We take a flat sheet of plexiglass and drill n small holes in it (Figure 49a). These holes will correspond to the fixed points of the network. From a piece of string we cut a set of $n-1$ segments. On one end of $n-2$ of the segments we make a small loose loop. We take the segment without a loop and pass it through an arbitrary number of loops. We can again pass the ends of the resulting configuration through a certain number of loops and so on, continuing this

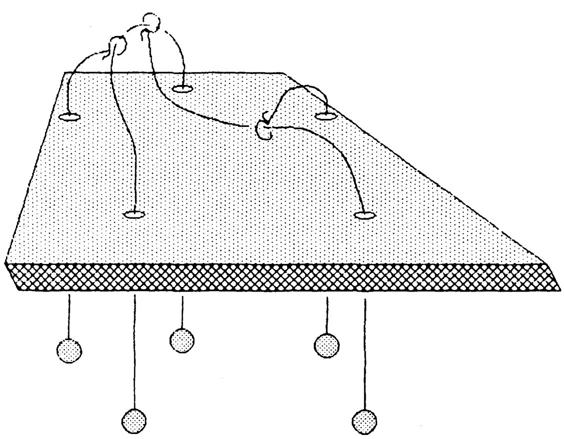

FIGURE 49A

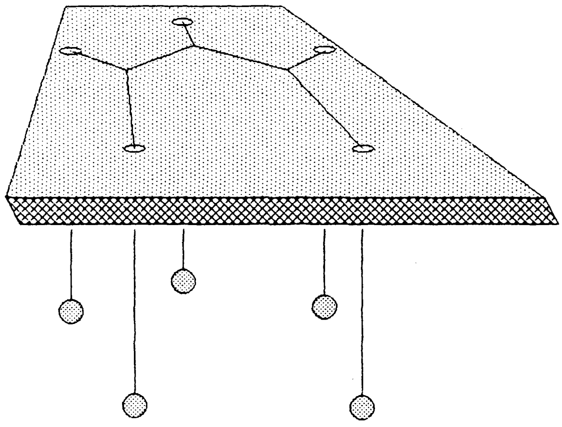

FIGURE 49B

process until all the segments have been used. We note that the number of ends of the resulting configuration is equal to the number of holes.

We place our sheet horizontally and pass all the ends of the resulting configuration upwards through the holes so that through each hole strictly one end passes. To each end we fasten a load, the loads being equal in mass.

After the system arrives at an equilibrium position, the network of string takes the form of a minimal network in one of the senses described above (Figure 49b). More concretely, if all the loops are separated, without interfering with one another, we obtain a minimal network in the second sense. But if at least one pair of loops is coupled, then there is a disallowed vertex in the resulting network, and the network is minimal only in the first sense.

Henceforth we shall study networks that are minimal in the second sense. In order to state our problem more precisely, we give the following definitions.

DEFINITION 1. A *topological Steiner network* is defined as a connected graph for which the degree of the vertices is at most three.

A realization of a topological network in a plane is called a *planar network.* To give a stricter definition we recall that a planar graph is a collection of curves in the plane that intersect only at their ends.

DEFINITION 2. A *planar Steiner network* is a planar graph for which there is a one-to-one correspondence with a topological Steiner network under which vertices correspond to vertices, curves correspond to edges, and the incidence relation is preserved.

A *set of fixed points* of a planar Steiner network is defined as an arbitrary subset of the set of vertices of the corresponding planar graph in which there occur all vertices of degree one or two (which agrees with the description of possible types of vertices of a network that is minimal in the second sense). Clearly, a set of fixed points of a network is not uniquely defined.

DEFINITION 3. A planar Steiner network is said to be *minimal* if it is minimal for some set of fixed points of it.

THE GENERAL STEINER PROBLEM. Describe the class of Steiner networks that can be realized as minimal networks. More precisely, suppose we are given a class $\{M\}$ of finite sets M of points of the plane. It is required to describe all Steiner networks that can be realized as minimal networks with a set of fixed points lying in this class.

REMARK. In Chapter 1 we talked about closed minimal networks on a sphere (we needed to study them to prove Plateau's principles). The problem of describing closed minimal networks on a closed two-dimensional orientable surface of genus g is an interesting generalization of Steiner's problem. A description of special classes of such networks was recently obtained by Shklyanko [34].

To start with, we consider as the class $\{M\}$ all possible subsets of points of the plane.

PROBLEM 1. Describe all Steiner networks that can be realized as minimal networks.

The class $\{M\}$ is the widest of all possible classes. However, it is not possible to realize all Steiner networks even on this. Figure 50a shows an example of a topological Steiner network that cannot be realized as a planar network, and hence as a minimal network. Figure 50b shows a planar Steiner network that cannot be realized as a minimal network. We note that in both cases all the trouble arises because of the presence of cycles. It turns out that this is the only obstacle to the realization of a Steiner network as a minimal network.

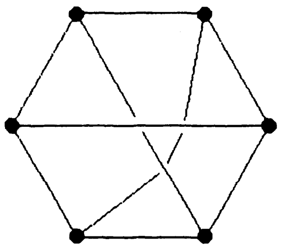

FIGURE 50A

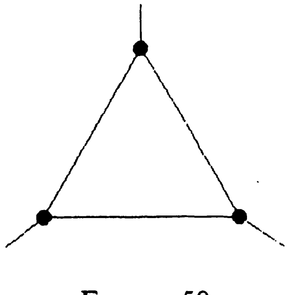

FIGURE 50B

PROPOSITION 1. *Any acyclic topological Steiner network (Steiner tree) can be realized as a planar network. Moreover, any planar Steiner tree can be realized as a minimal network for some set M of fixed points.*

We say that a Steiner network is *degenerate* if it has at least one vertex of degree two. We note that nondegenerate acyclic Steiner networks, by definition, are 2-trees. These networks have vertices of degree one, which we call *boundary vertices*, and vertices of degree three, which we call branch *points*. For such networks it is natural to take the boundary points as fixed, and we shall do this from now on. Starting from here we shall study acyclic nondegenerate minimal Steiner networks, that is, *minimal 2-trees*.

On the set of all planar graphs we can introduce a natural equivalence relation. We say that two planar graphs are *equivalent* if there is a homeomorphism of the plane onto itself (preserving the orientation) that takes one planar graph into the other. It is easy to see that there are only finitely many equivalence classes of 2-trees with a fixed number of boundary points.

The following natural question arises: how many such equivalence classes are there for a given number n of boundary points? From Proposition 1 it follows that there are exactly as many of them as there are equivalence classes of 2-trees. The number of the latter was calculated in 1964 by Brown [35] in implicit form. The numerical results for $n \leq 23$ can be found in [36].

We note that to solve this problem, instead of planar trees we can consider the dual objects, namely triangulations by diagonals of convex n-gons. Let us describe in more detail the correspondence between planar 2-trees and triangulations.

Suppose we are given a 2-tree with n boundary points. Consider a convex n-gon. We number the boundary points of the 2-tree in succession, going around anticlockwise, for example. Similarly we number the sides of the polygon. These numberings generate a natural one-to-one correspondence between the vertices of the 2-tree and the sides of the n-gon.

Obviously, the boundary points incident with the same branch point have consecutive numbers. For each pair of such points we consider the corresponding pair of sides of the n-gon and construct a triangle on these sides, drawing a diagonal of the polygon (this can always be done, since these sides are adjacent).

We cut out all the triangles obtained in this way and simultaneously discard from the 2-tree all the edges going out from the boundary points. Obviously, we again obtain a convex polygon and a 2-tree, and the number of vertices and the number of boundary points, respectively, are equal.

Between the boundary points and sides of the resulting objects there is a natural one-to-one correspondence, which is obtained directly from the correspondence established at the previous stage.

We repeat the procedure just described until the 2-tree is exhausted. (The last stage is a little more delicate, but it does not present any essential difficulty, so we leave the details to the reader.)

As a result we obtain a partition of the convex n-gon into triangles, which is called a *triangulation by diagonals* corresponding to a planar 2-tree.

Conversely, if we are given a triangulation of a convex n-gon by diagonals, it is easy to construct the corresponding 2-tree. As boundary points of such a tree we can take the midpoints of the sides of the n-gon, as branch points the centers of the triangles of the triangulation, and as edges the segments joining the centers of adjacent triangles and also the segments joining the midpoints of the sides of the n-gon to the centers of the triangles constructed on these sides.

Two triangulations are said to be *equivalent* if they are equivalent as planar graphs. In each equivalence class it is convenient to choose as a representative the corresponding triangulation of a regular polygon inscribed in the unit circle. Two triangulations of such regular polygons are equivalent if they are obtained from each other by a motion of the plane. It is easy to see that equivalent planar 2-trees correspond to equivalent triangulations of convex n-gons by diagonals and conversely.

Thus, the following proposition is true.

PROPOSITION 2. *The equivalence classes of planar 2-trees with n boundary points are in one-to-one correspondence with the equivalence classes of triangulations of convex n-gons by diagonals.*

Figure 51 shows all possible triangulations in the cases when $n = 3, 4, 5, 6$. We note that for $n < 6$ the triangulation by diagonals is unique (up to equivalence), but for $n = 6$ there are three different triangulations.

Another natural class $\{M\}$ of boundary points of networks is the class of extremal sets. We recall that a set is called *extremal* if it lies on the boundary of some convex set. If the set of boundary points of a network is extremal, such a network is called a *network with convex boundary*.

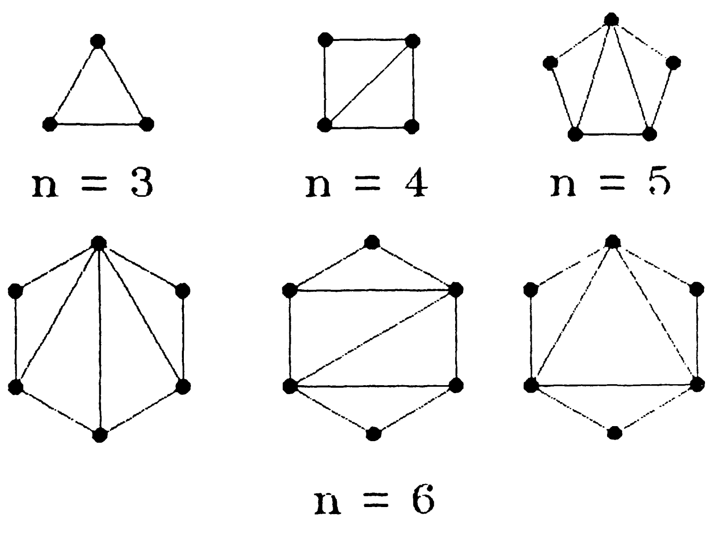

FIGURE 51

PROBLEM 2. Describe all minimal Steiner networks with convex boundary.

REMARK. This problem can be generalized. For this we need the following definition.

DEFINITION 4. Suppose we are given an arbitrary finite set M of points of the plane. We split it into classes, which we call *levels of convexity*.

In the first level of convexity we put all points lying on the boundary of the convex hull of M. Consider the set M' obtained from M by discarding all points of the first level.

The second level of convexity contains the points of the first level of convexity for the set M' (if M' is not empty).

Continuing this operation until all the original set M is exhausted, we obtain the necessary partition.

We observe that extremal sets, and only they, have exactly one level of convexity.

PROBLEM $2'$. Describe all minimal Steiner networks for which the number of levels of convexity of sets of boundary points does not exceed some fixed number.

Below we give a complete solution of Problem 2 for 2-trees.

One more important variant of the general Steiner problem is obtained if for the class $\{M\}$ of sets of boundary points of networks we consider the class consisting of exactly one set.

PROBLEM 3 (the classical Steiner problem). Describe all minimal Steiner networks whose set of boundary points is fixed.

An interesting variant of this problem is the following.

PROBLEM $3'$. Describe all minimal Steiner networks whose set of boundary points consists of the vertices of a regular polygon.

Below we give some results of A. O. Ivanov and A. A. Tuzhilin, devoted to investigations of Problem $3'$ again for 2-trees.

In connection with the statement of the general Steiner problem, the following interesting question arises: is there a set M consisting of n points on which all equivalence classes of planar 2-trees with n boundary points can be realized as minimal networks? For $n = 3, 4, 5$ we can take as such a set the vertices of the corresponding regular n-gon. For $n > 5$ this is not so. From Proposition 3 (see below) it follows that, generally speaking, such a set M must have quite a complicated structure (for example, it cannot be extremal).

2. Classification of minimal 2-trees with convex boundary. An important role in the classification of minimal 2-trees with a convex set of boundary points is played by the so-called twisting number; we begin this section with a definition of it.

For each branch point of a planar 2-tree there is a circular neighborhood whose intersection with the tree consists of three smooth nonclosed curves going from its center, not having any other points of intersection, and going out to the boundary of this neighborhood. Clearly, the intersection of the boundary of the neighborhood (the circle) with the tree consists of three points. We choose one of these curves and call the edge, of which it is a part, *incoming*. We call the remaining two edges *outgoing*. We call the partition of the edges incident with a point into incoming and outgoing an *orientation* of the neighborhood of the branch point.

Now suppose that the plane is oriented. Then we are given a positive direction for motion along each circle lying in this plane. This gives the possibility of naturally ordering a pair of outgoing edges in such a way that a motion along the circle from the first edge to the second along an arc that does not intersect an incoming edge takes place in the positive direction. We assign the number -1 to the first outgoing edge, and $+1$ to the second. An oriented neighborhood of a branch point together with these numbers is said to be *clothed*, and the numbers themselves are called *clothings* of the corresponding edges of the tree.

We recall that a path joining a pair of edges of a planar 2-tree is defined as a minimal connected subtree containing these edges. We now give a definition of the twisting number between a pair of edges of a planar 2-tree.

Let a and b be a pair of edges of a planar 2-tree. We choose a path γ joining a and b. We orient the path γ from a to b. Consider all branch points lying inside γ. The orientation of γ canonically specifies orientations of small neighborhoods of these branch points; for each point we shall take as incoming an edge for which the point is an end. We fix a certain orientation of the plane and clothe the neighborhoods of all the branch points under consideration. A path γ together with clothings of all its edges will be called *clothed*.

DEFINITION 5. The *twisting number* $\mathrm{tw}(a, b)$ of an ordered pair (a, b) of distinct edges of a 2-tree is defined as the sum of the clothings of all outgoing edges of the oriented path γ going from a to b. We take $\mathrm{tw}(a, a)$ to be zero.

For the edges a and b of the 2-tree shown in Figure 52a the twisting number $\mathrm{tw}(a, b)$ is five, while for the tree in Figure 52b it is zero.

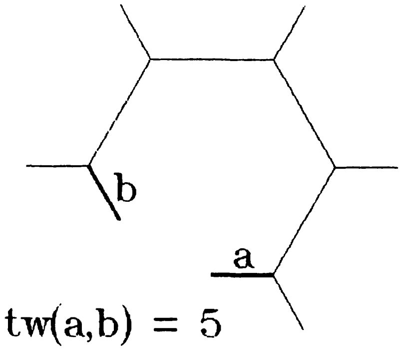

FIGURE 52A

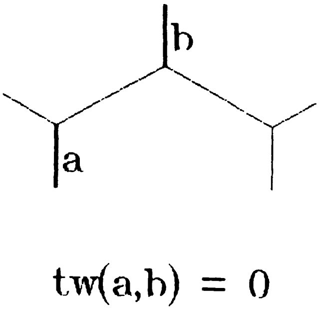

FIGURE 52B

Let us mention a property that the twisting number has (we leave the proof to the reader as a useful exercise):

SKEW-SYMMETRY: $\mathrm{tw}(a, b) = -\mathrm{tw}(b, a)$.

DEFINITION 6. The *twisting number* $\mathrm{tw}(D)$ of a planar 2-tree D is defined as the largest twisting number of all possible ordered pairs of edges of this tree:

$$\mathrm{tw}(D) = \max \mathrm{tw}(a, b).$$

The next proposition is a key result in obtaining a complete classification of minimal 2-trees with convex boundary.

PROPOSITION 3. *The twisting number of a minimal Steiner 2-tree with convex boundary is not greater than five.*

REMARK. From the classification theorems (see below) it follows that this bound is exact: any planar 2-tree with twisting number not greater than five can be realized as a minimal tree with an extremal set of boundary points.

It is convenient to state the classification theorem in the language of tilings. We define a (triangular) tiling of the plane as a canonical partition of the plane into regular congruent triangles, which we call the *cells* of the tiling. This partition can be obtained as follows.

Let A and B be families of equally spaced parallel lines, and suppose that the angle between the directions of the lines of A and B is $60°$. Through the points of intersection of lines of A and B we can uniquely draw a third family C of parallel lines so that together the families A, B, and C give a partition of the plane into regular congruent triangles. We call these lines the *directrices* of the tiling of the plane, and the six possible directions of these lines the *directions* of the tiling.

DEFINITION 7. We define a *tiling* as an arbitrary collection of cells of a tiling of the plane.

In exactly the same way as from a triangulation of a convex polygon by diagonals, from the tiling we can construct a planar graph, which we call the *dual graph* of this tiling. We call a tiling *connected* if its dual graph is connected. The tilings corresponding to the connected components of the dual graph are called the *components* of the tiling. Henceforth we shall almost always be dealing with connected tilings, so we shall omit the word "connected" provided it does not lead to misunderstanding.

We note that the dual graph of an arbitrary (connected) tiling is actually a minimal Steiner network.

DEFINITION 8. A tiling whose dual graph is a 2-tree is called a *tree tiling.*

In fact, not every equivalence class of planar 2-trees has as its representative the dual graph of some tree tiling. Nevertheless, the following proposition is true.

PROPOSITION 4 (on a tiling realization). *Any planar 2-tree with twisting number no greater than five can be realized as the dual graph of some tree tiling.*

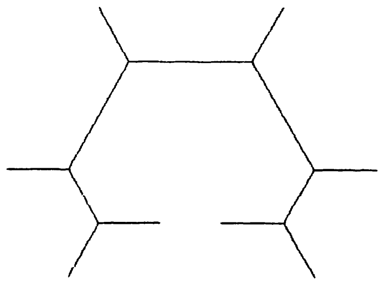

FIGURE 53

REMARK. Although some 2-trees with twisting number greater than five can be realized as the dual graph of a tree tiling (construct an example), the bound on the twisting number given in Proposition 4 is exact. Figure 53 shows an example of a planar 2-tree with twisting number equal to six that cannot be realized as the dual graph of a tree tiling.

Thus, it follows from Propositions 3 and 4 that for a classification of minimal 2-trees with convex boundary it is sufficient to describe all tree tilings whose dual graphs have twisting number not exceeding five (henceforth for brevity we shall call the twisting number of the dual graph of a tree tiling the twisting number of the tiling itself).

To obtain such a classification we must first of all choose the building blocks from which all possible tree tilings are formed. We choose three types of building blocks, which we shall call *linear parts, branch points*, and *growths*. Roughly speaking, every tree tiling is a collection of linear parts joined to one another by means of branch points and equipped with growths. We now give more formal definitions.

DEFINITION 9. We define a *snake* as a tiling placed between two adjacent directrices of a tiling of the plane (Figure 54).

An *extreme cell* is defined as a cell of a tiling, two sides of which do not lie inside the tiling. An *interior cell* is defined as a cell, all of whose sides lie inside the tiling.

DEFINITION 10. We call an extreme cell of a tiling a *growth* if the only cell

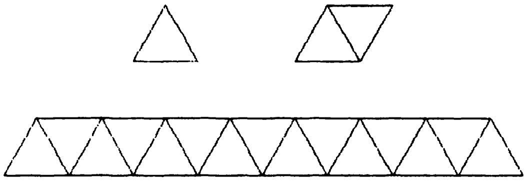

FIGURE 54

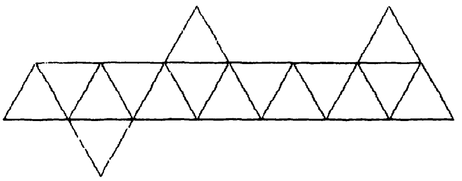

FIGURE 55A

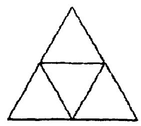

FIGURE 55B

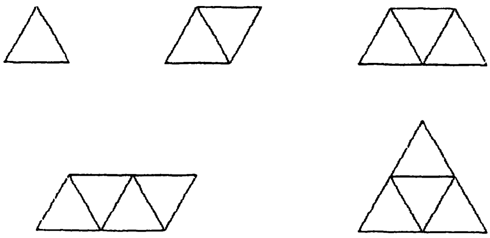

FIGURE 56

of the tiling adjacent to it is internal. We call the tilings a *skeleton* if it does not have growths.

Figure 55 shows a snake with growths.

In order to obtain a skeleton, for each interior cell of the tiling we discard one of the growths adjoining it (if there are any).

Next, we consider a tree skeleton and cut out from it all internal cells. If there is at least one internal cell, the skeleton splits into components.

DEFINITION 11. The components into which a tree skeleton splits after discarding internal cells are called the *linear parts* of the skeleton. The *branch points* are the components into which a tree skeleton splits after discarding the linear parts.

Let us give a complete list of possible branch points of tree skeletons.

PROPOSITION 5. *In tree skeletons there can occur exactly five types of branch points, shown in Figure* 56.

REMARK. We should mention that the linear parts can be fastened to each branch point in different ways. In all there are 18 ways of fastening (list them), which we shall call *forks*. Figure 57 shows the two most important types of forks, which we shall call *T-joints*.

We now describe the structure of the linear parts. For this we give the more general definitions of a linear 2-tree and a linear tiling.

DEFINITION 12. A planar 2-tree is called *linear* if the triangulation of a convex polygon corresponding to it has exactly two extreme triangles (an extreme triangle of a triangulation is a triangle of which two sides coincide with sides of the polygon).

We note that the triangulation corresponding to a linear 2-tree does not have internal triangles. Therefore, for such a triangulation there is a natural linear ordering of its triangles so that the extreme triangles are the first and last in this order.

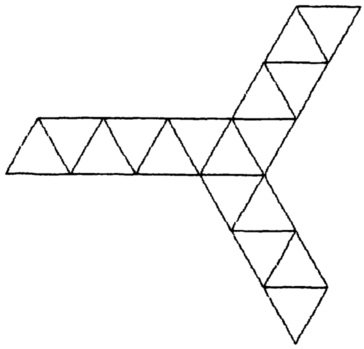

FIGURE 57A

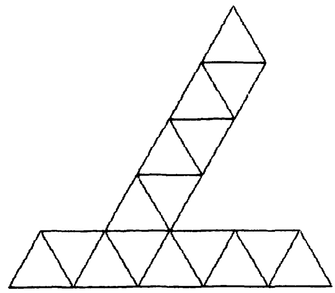

FIGURE 57B

Clearly, there are exactly two possibilities for ordering the triangles of a triangulation, depending on which of the two extreme triangles is taken as the first. The choice of one of these two orders is called an *orientation* of the linear 2-tree and the triangulation corresponding to it.

Now suppose that the dual tree of some tree tiling is linear. In this case the tiling is also said to be linear. We note that the linear parts are actually linear tilings.

We now consider an arbitrary linear tiling and orient it. We then show how we can split such a tiling into a number of snakes, which we call segments of the linear tiling.

Similarly we define all the later segments. Thus, we can state the following result.

PROPOSITION 6. *Every linear tiling, in particular, the linear part of an arbitrary tree tiling, is the union of a linearly ordered family of distinct disjoint snakes (segments of the linear tiling), and the initial cell of each subsequent snake is adjacent to the end cell of the previous one.*

In each nonextreme cell of a skeleton we join by segments the midpoints of its sides inside the skeleton. In each extreme cell we draw a midline parallel to the midline of the adjacent cell already constructed.

DEFINITION 13. The *spine* (*vertebra*) of a linear part (cell) is the part of the graph constructed above that is contained in this linear part (cell).

If we allow the twisting number of the skeleton to take only values not exceeding five, then there arise essential restrictions on the structure of the linear parts of such a skeleton. Namely, the following proposition is true.

PROPOSITION 7. *For each linear part of a tree skelelton with twisting number not exceeding five there is a directrix of the tiling of the plane on which the spine of this linear part projects one-to-one. Such a directrix is called a directrix of the linear part.*

REMARK. Generally speaking, a directrix of a linear part is not unique. For a snake, for example, there are three such directrices. If the twisting number of the linear part is greater than five, then such a part does not have directrices.

DEFINITION 14. A snake is defined as a linear part for which there are three directrices. A *stairs* is defined as a linear part for which there are exactly two directrices. A linear part that has exactly one directrix is called a *broken snake* (Figure 58a).

REMARK. A linear part that is a snake in the sense of Definition 9 may not be a snake in the sense of the last definition. Figure 58b shows an example of such a part. Of course, it all depends on how the given linear part is fastened to the branch points.

We have thus described all the building blocks from which all possible tree tilings are formed. Now, in order to state the classification theorems, it remains to define the operation of reduction for skeletons of tree tilings with twisting number not exceeding five. This operation consists in cutting out certain fragments of the tiling.

Firstly, we can cut from the skeleton any part of a linear part containing an extreme cell.

Secondly, inside the skeleton we can discard any snake Z consisting of an even number of cells and occurring in some linear part. We observe that the snake Z is a parallelogram. Let us consider the pair of sides of this parallelogram not parallel to the spine of this snake. We shall call these sides the *bounding edges* of the snake Z. It turns out that the following assertion is true.

ASSERTION. *Let Z be an arbitrary snake consisting of an even number of cells and occurring in some linear part of the skeleton D with twisting number*

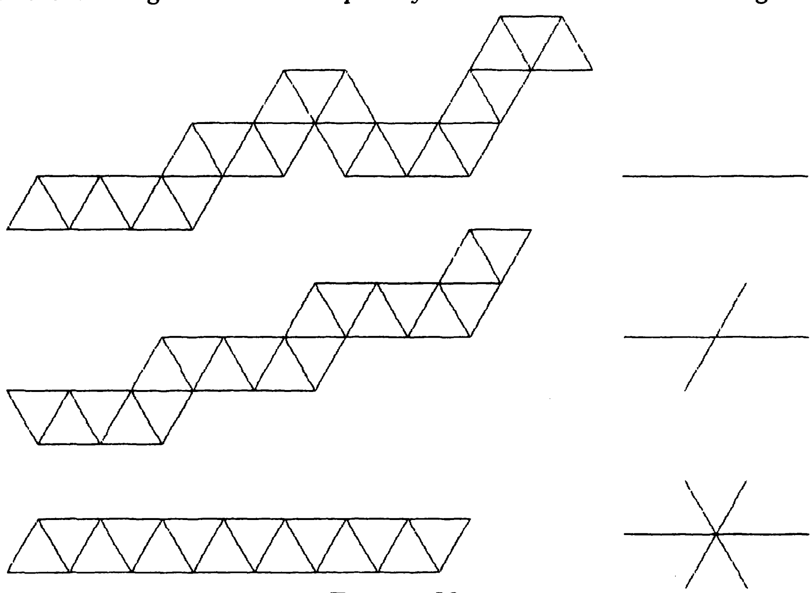

FIGURE 58A

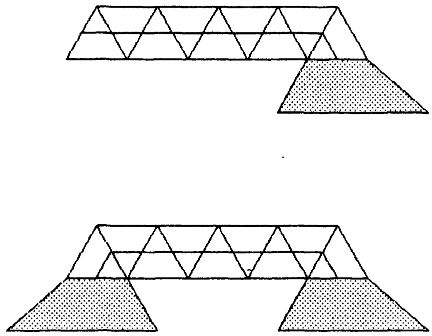

FIGURE 58B

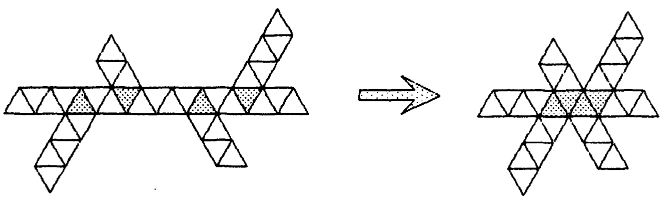

FIGURE 59

no greater than five. Let l_1 *and* l_2 *be the bounding edges of* Z. *Let* D_1 *and* D_2 *be the connected components into which* D *splits after discarding* Z *from it. Then there is a translation* τ *such that the intersection of* D_1 *and* $\tau(D_2)$ *is* $l_1 = \tau(l_2)$. *Moreover,* $D_1 \cup \tau(D_2)$ *is a tree skeleton, whose twisting number is not greater than the twisting number of* D.

We say that the skeleton $D_1 \cup \tau(D_2)$ is *reduced* from the skeleton D by cutting out the snake Z.

REMARK. By means of the reduction operation we can obtain a stairs from a broken snake and a snake from a stairs. By reduction we can turn several branch points into one point (of a different type). This also happens with forks. Figure 59 shows the reduction of several forks of T-joint type to a fork of a more complicated form. The reduction operation also enables us to discard forks (we need to apply reduction several times).

We are now in a position to state a theorem that classifies skeletons of tree tilings with twisting number not exceeding five.

THEOREM 1 (classification of skeletons) (Ivanov, Tuzhilin). *All skeletons with twisting number not exceeding five are obtained by reduction from the three canonical types of skeletons given in Figure* 60.

A broken snake is represented by three dashes, and the dashes are parallel to its directrix.

A stairs is represented by two intersecting dashes, and the dashes are parallel to its two directrices.

A snake is represented by one dash, and the dash is parallel to the spine of the snake.

Forks of T-joint type correspond to points (see Figure 57).

REMARK. If we consider the diagrams of the canonical types as planar 2-trees, it is easy to observe that they represent all possible planar 2-trees with six endpoints.

We now describe the possible positions of growths on a skeleton. For this we need the concept of a profile of a skeleton.

The *contour* of a tiling is defined as the boundary of the tiling regarded as a closed subdomain of the plane. Consider an extreme cell of the skeleton and discard from the contour of the skeleton that edge of it that intersects the vertebra of this cell. We go through all the extreme cells of the skeleton,

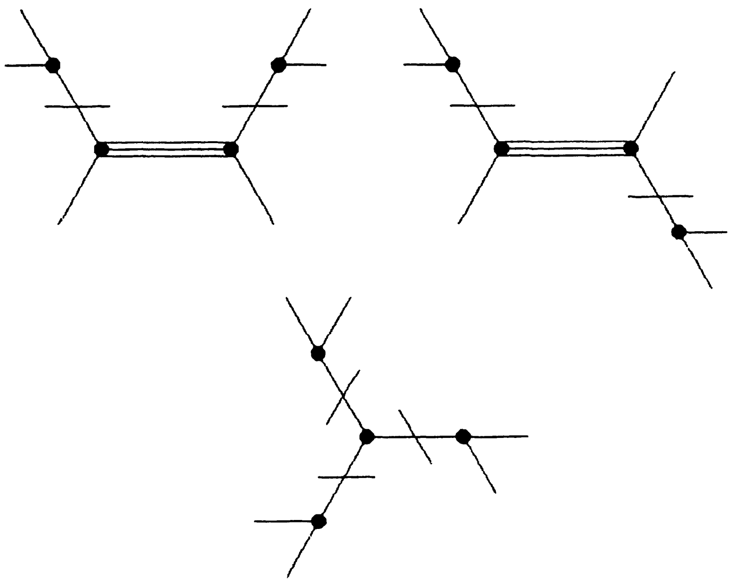

FIGURE 60

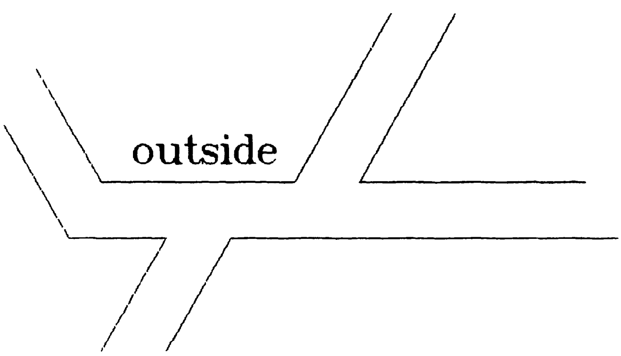

FIGURE 61

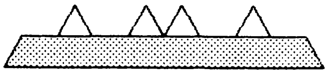

FIGURE 62A

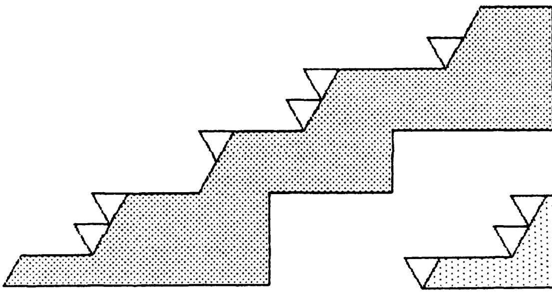

FIGURE 62B

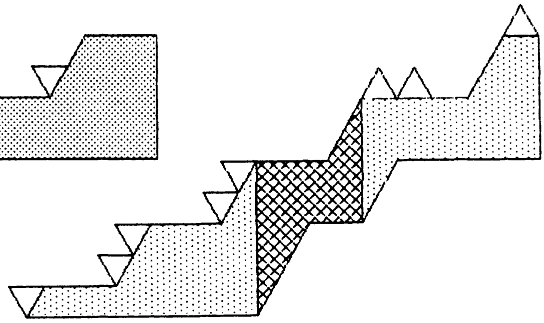

FIGURE 62C

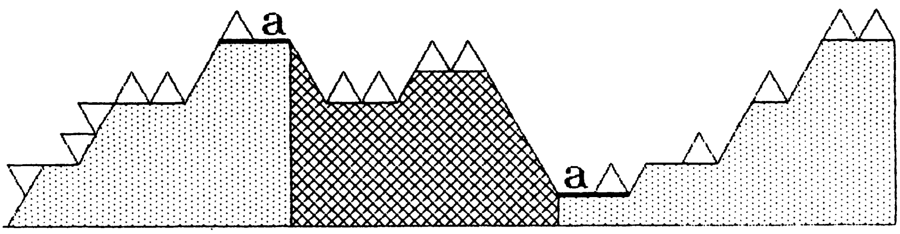

FIGURE 62D FIGURE 62E

performing the same operation. The contour of the skeleton splits into broken lines, which we call the *profiles* of the skeleton. An outer side of a profile is called an outer side of it with respect to the skeleton (Figure 61).

We note that the profiles of the skeleton of a tiling with twisting number not exceeding five have the same properties as the spines of the linear parts of such skeletons. Therefore for the corresponding profiles we keep the names snake, stairs, and broken snake.

THEOREM 2 (on the position of growths). (Ivanov, Tuzhilin) 1. *On a profile that is a snake we can plant any number of growths* (*Figure* 62a).

2. *For a stairs-profile there are two possibilities.*

a) *The growths are placed arbitrarily only on segments in one direction* (*Figure* 62b).

b) *We are given a partition of the stairs into three successive broken lines, the middle one of which may be empty. The middle broken line consists of an even number of links and the angle between the first pair of links, measured from the outer side, is equal to* $120°$.

There are no growths on the middle broken line. On the first broken line the growths can be situated arbitrarily on the segments that have the direction of its last link, and on the last broken line they can be situated on segments having the direction of its first link (*Figure* 62c).

3. *We present a profile that is a broken snake as the union of three parts, the outer ones of which are maximal possible stairs, and the inner one, which may be empty, is all the rest. On the middle part we can plant arbitrarily many growths only on segments parallel to the directrix of the profile. On the outer stairss we can plant growths as follows.*

Consider a segment of the profile adjacent to an outer stairs. If the angle between it and the neighboring segment a *of the stairs, measured from the outer side of the profile, is equal to* 120°, *then growths may be fastened only on segments of the stairs parallel to the segment* a. *If this angle is equal to* 240°, *then we can plant growths on the stairs according to rule* 2 *(Figure* 62d, e).

Now the classification of possible minimal 2-trees with a convex set of boundary points is obtained from Propositions 3 and 4 and Theorems 1 and 2.

We can show that any planar 2-tree that is the dual graph to the tilings described in Theorems 1 and 2 can be realized as a minimal tree with a convex set of boundary points. Thus, the resulting classification is complete.

3. Some results from the investigation of minimal networks that span the vertices of regular polygons. We begin this section with a description of a simple algorithm: for a given finite set M of points of the plane this algorithm enables us to construct a minimal network spanning it by means of compasses and a straight edge (the idea of this algorithm is due to Melzak [42]). For this it is sufficient to know the structure of this minimal network as a planar 2-tree and the correspondence between the endpoints of this 2-tree and points of the set M. Recall, that the vertices of a minimal network that do not belong to M are called *Steiner points.* Let us illustrate the idea behind this algorithm by an example of constructing the minimal network for the set M of vertices of a triangle ABC, none of whose angles exceeds 120°.

We choose any pair of vertices of the triangle, say A and B, and construct an equilateral triangle ABD on the side AB so that C and D lie on opposite sides of AB. We then describe the circle ABD.

Clearly, the only Steiner point V of the minimal network lies on the minor arc d of this circle joining A and B. Moreover, V lies on the ray DC (prove this). Joining V to the vertices of the triangle ABC, we obtain our minimal network (Figure 63a).

If the triangle ABC has an angle greater than or equal to 120°, then the corresponding minimal network is not a 2-tree. In this case we can carry out the same construction, but the angles between the segments joining the point V of intersection of the ray DC and the circle to the vertices of the triangle will not be equal.

For a quadrilateral $ABCD$ the construction consists of two similar steps. Figure 63b shows a minimal network spanning the vertices of a square. We split the vertices of the square into pairs consisting of bounding vertices of

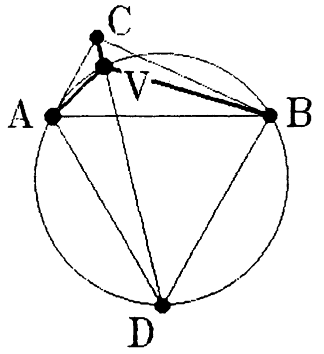

FIGURE 63A

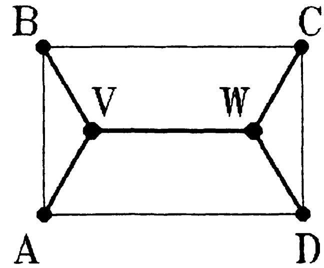

FIGURE 63B

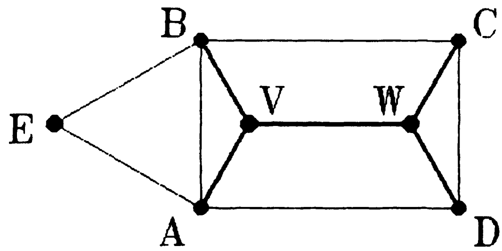

FIGURE 63C

the network for which the edges of the network going out from them meet at one Steiner point, and choose one of these pairs, say A and B. We denote by V the Steiner point at which the edges of the minimal network going out from A and B meet, and the other Steiner point by W.

On AB we construct an equilateral triangle ABE. We place the vertex E of this triangle in such a way that E and V lie on opposite sides of AB (Figure 63c).

We now consider the triangle CDE and describe the minimal network for it in the way described above. The Steiner point of this network coincides with W.

A minimal network for the vertices of the square $ABCD$ is obtained as follows. We describe the circle ABE. The point of intersection of this circle with the minimal network we have constructed is a Steiner point V of the required network (prove this). It remains to join V to A and B.

These ideas are the basis of the algorithm for constructing minimal networks with a given set of boundary points. This algorithm has been realized on a computer. For lack of space we do not give a detailed description of this algorithm here. Figure 64 gives minimal trees constructed by the computer.

A computer experiment has enabled us to formulate a number of conjectures about the structure of minimal 2-trees spanning the vertices of regular n-gons. Some of these conjectures have been proved. We give here a small part of the results we have obtained.

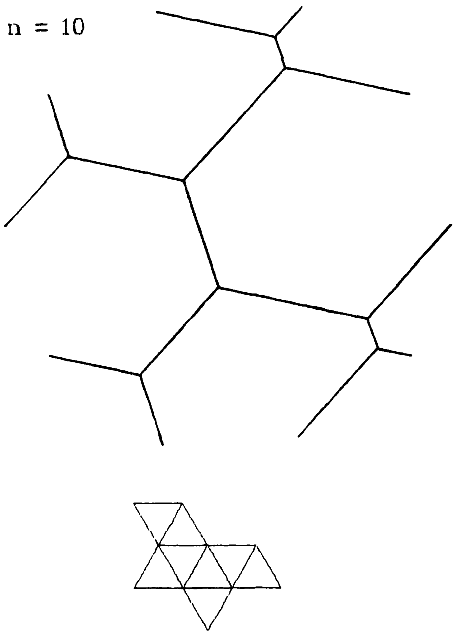

FIGURE 64A

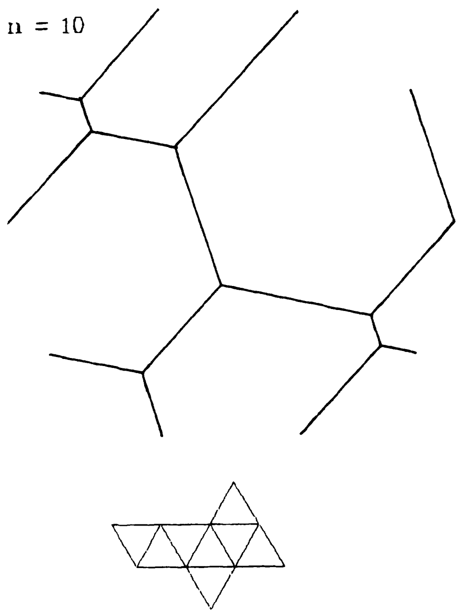

FIGURE 64B

PROPOSITION 8. *For any n, on the vertices of a regular n-gon we can stretch a minimal 2-tree of snake type uniquely up to a motion (Figure 65).*

PROPOSITION 9. *For any $n = 6k + 3$, where $k > 0$, on the vertices of a regular n-gon we can stretch a minimal 2-tree of T-joint type (from Figure 57a) uniquely up to a motion (Figure 66). This network is invariant under rotation about the center of the n-gon through $120°$.*

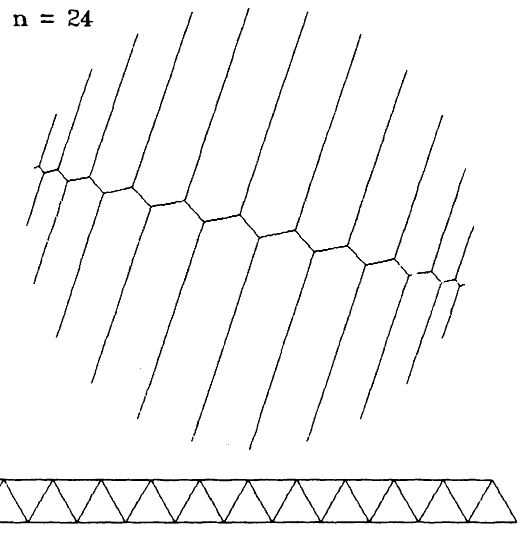

FIGURE 65

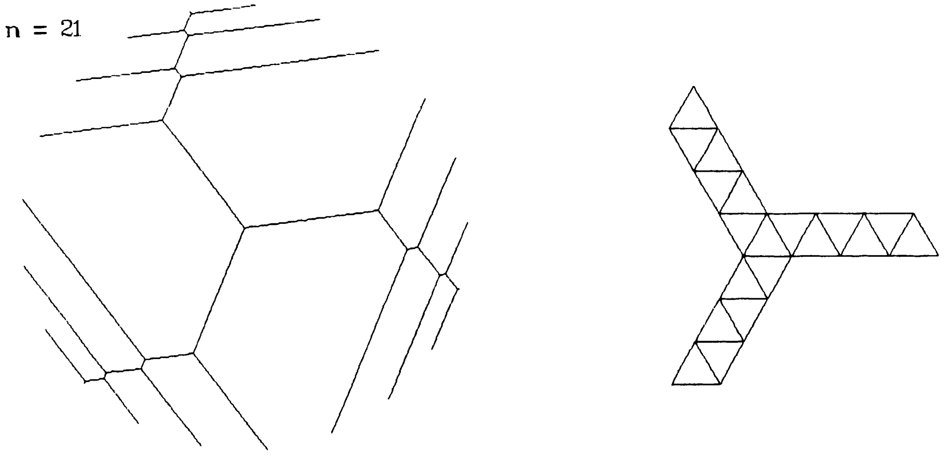

FIGURE 66

There is at least one infinite series of minimal trees—these are snakes with pairs of symmetrical growths situated close to the center of the snake (Figure 67). The authors have obtained estimates for the possible position of these growths, which we cannot give for lack of space.

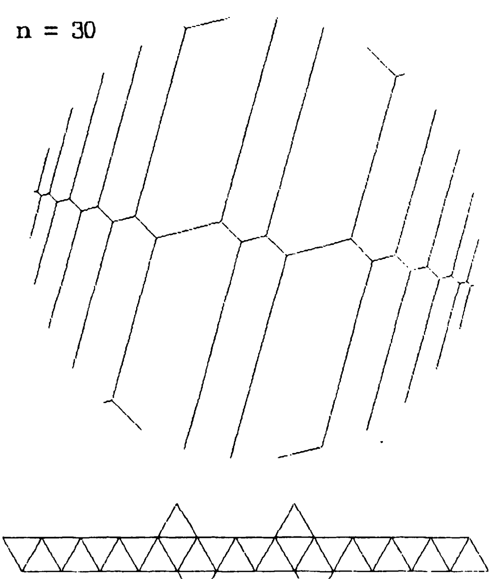

FIGURE 67

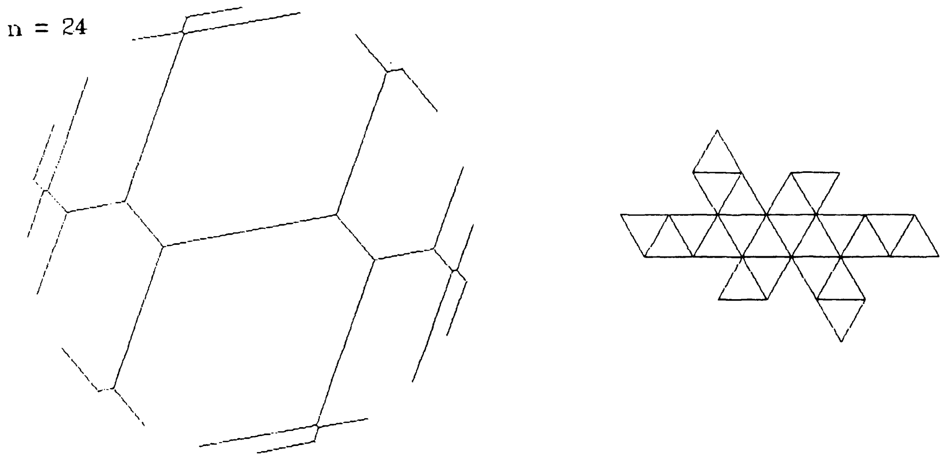

FIGURE 68A

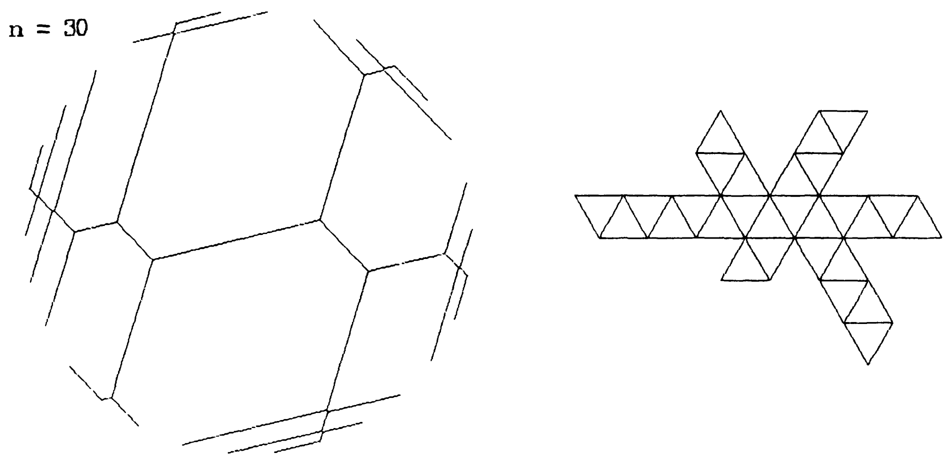

FIGURE 68B

Figure 68 gives representatives of an apparently finite series (as a computer experiment shows) of minimal trees realized on n-gons when $n = 24, 30, 36, 42$. We note that since the corresponding tilings have one branch point and six ends, there can be no growths on these networks.

Figure 69 gives an example of a network whose corresponding tiling has one branch point, four ends, and one growth. A computer experiment shows that there may exist an infinite series of such minimal trees.

These examples show that the problem of classifying minimal 2-trees whose sets of boundary points consist of the vertices of regular polygons is nontrivial.

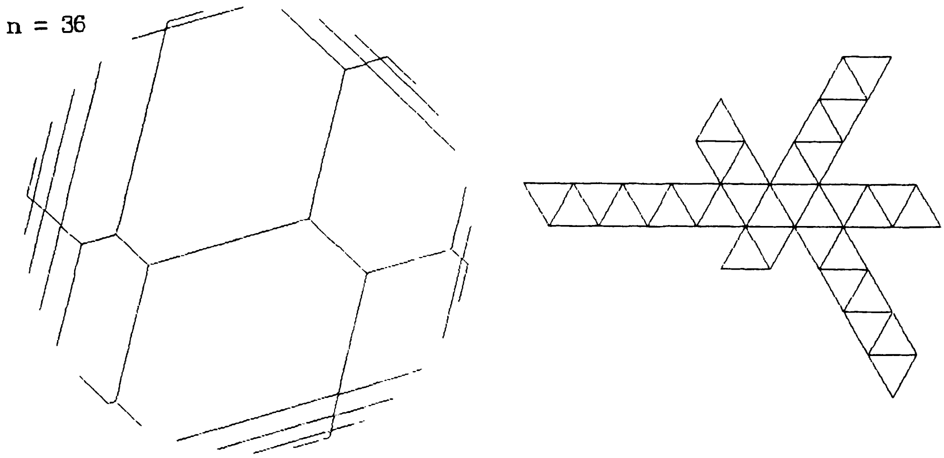

FIGURE 68C

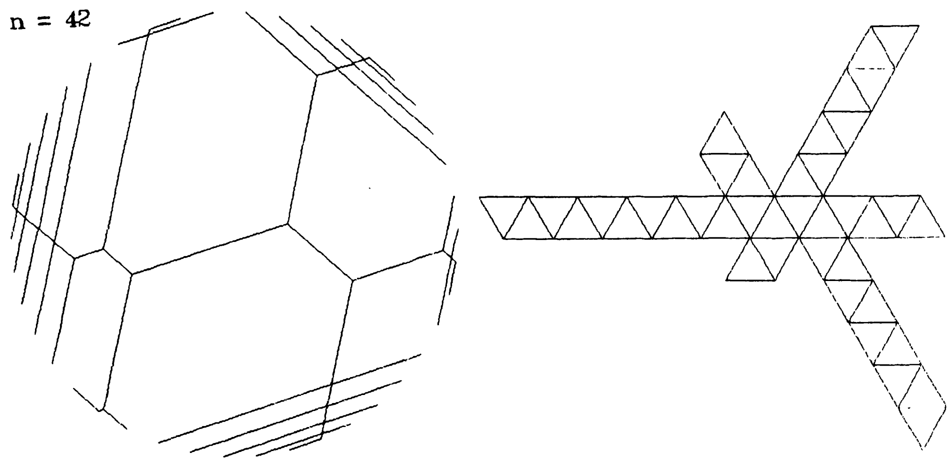

FIGURE 68D

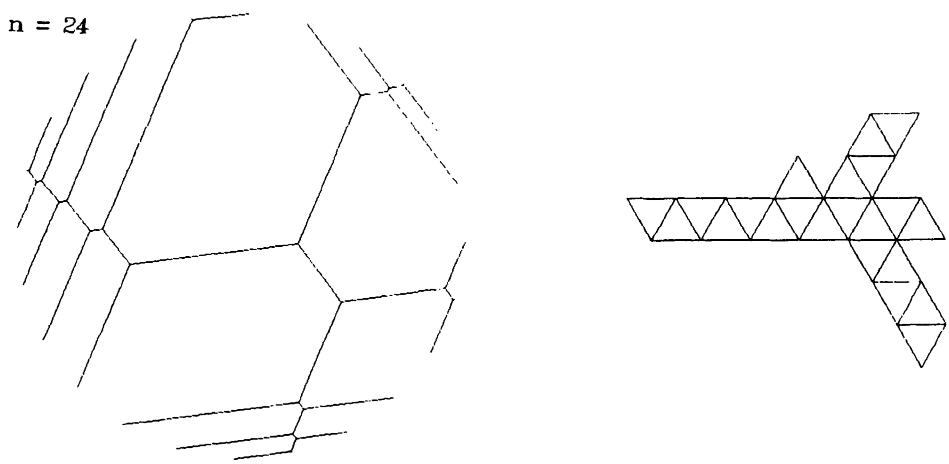

FIGURE 69

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Subject Index

Errata

(-9 means line 9 down from the top, and $+4$ means line 4 up from the bottom; not including chapter headings or footnotes)

Page	Line	Currently	Should Be
vii	-9	Nitsche ref.	add , rev. English transl. of Chaps. I–V, *Lectures on minimal surfaces.* Vol. I, Cambridge Univ. Press, Cambridge, 1989.
vii	+4	Osserman ref.	add (Dover)
1	-9	a soap is left...	a soap film is left...
3	+2	containing to $N(P)$	containing $N(P)$
9	-19	It is well known that critical points...	Critical points...
10	+6	Dougals	Douglas
10	+9	Jesse Douglas	Douglas
13	+1	We note that a system...	A system...
14	+1	is a networks on the sphere S^2	is a network on the sphere S^2
30	-1	original contour and,	original contour and which,
34	+1	$l_0, l_{1/2}n, \ldots, l_1$	$l_0, l_{1/2n}, \ldots, l_1$
49	+1	large surface	larger surface
49	+3	large sur-	larger sur-
49	+5	large connected minimal surface	larger connected minimal surface
50	-2	large connected minimal surface	larger connected minimal surface
54	-7,8	reduces to the solution of differential equation [8].	reduces to the solution of a differential equation [8].
55	-6	a differential inverse	a differentiable inverse
71	-3	is increased by a factor	is expanded* by a factor (footnote added: * "Increased" is poor; better, "expanded", "multiplied" or "changed".)

Page	Line	Currently	Should Be
75	-8,9	Riemann manifold	Riemannian manifold
83	-6	genus h	genus h
90	-5	Continuting this construction	Continuing this construction
90	-10,11	minimal surfaces	minimal surface
101	-7	complex plane c	complex plane $\mathbb{C}$
101	+18	P-function	$\mathcal{P}$-function
101	+15	$P(z) =$	$\mathcal{P}(z) =$
101	+8	P-function becausec	$\mathcal{P}$-function because
101	+5	$a = 2\sqrt{2\pi}P(1/2)$	$a = 2\sqrt{2\pi}\mathcal{P}(1/2)$
101	+4	if P' denotes	if $\mathcal{P}'$ denotes
101	+3	P-function P	$\mathcal{P}$-function $\mathcal{P}$
101	+2	$(Pdz, a/P')$	$(\mathcal{P}dz, a/\mathcal{P}')$
102	-2	$(Pdz, c/P')$	$(\mathcal{P}dz, c/\mathcal{P}')$
102	-7	P-function	$\mathcal{P}$-function
103	+1	diffeomorphism with an image	diffeomorphism onto its image
105	+9	smooth funciton	smooth function
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141	1st col., -19	Convex Hull, 60	Convex hull, 60
141	2nd col., +5	classification, 82	classification (of 2-dimensional), 82
142	2nd col., +3	P-function	$\mathcal{P}$-function

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