

Translations of
**MATHEMATICAL
MONOGRAPHS**

Volume 94

**Finiteness Theorems
for Limit Cycles**

Yu. S. Il'yashenko



American Mathematical Society



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Providence, Rhode Island

Ю. С. ИЛЬЯШЕНКО

ТЕОРЕМЫ КОНЕЧНОСТИ ДЛЯ ПРЕДЕЛЬНЫХ ЦИКЛОВ

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To my family:

Lena

Serezha

Lizochka

Aleksandr

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