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Nonlinear Partial Differential Equations of Second Order

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Nonlinear Partial Differential Equations of Second Order

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Providence, Rhode Island

董 光 昌

非线性二阶偏微分方程

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Translated from the Chinese by Kai Seng Chou [Kaising Tso]

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ABSTRACT. This is a treatise on nonlinear partial differential equations of second order. With the exception of the first chapter, all the remaining chapters are based on the published or unpublished work of the author.

A priori estimation is the main theme of this book. Emphasis is placed on elliptic and parabolic equations. Nevertheless, some hyperbolic equations are also discussed. Each chapter of this book has its own physical background.

This book can be used as a text for graduate students in mathematics, or as a reference for researchers, teachers, or university seniors.

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Symbols

Ω	usually a bounded (connected) open set
$\overline{\Omega}$	the closure of Ω
$\partial\Omega$	the boundary of Ω
$\subset\subset$	$A \subset\subset B$ means $\overline{A} \subset B$
Ω'	$\Omega' \subset\subset \Omega$
$d(x, A)$	the distance between x and the set A, i.e., $d(x, A) = \inf_{y\in A} \overline{xy}$
Ω_ρ	$\Omega_\rho \subseteq \Omega$ and $d(\Omega_\rho, \partial\Omega) = \rho$
d_x	$d_x = d(x, \partial\Omega)$
d_{xy}	$d_{xy} = \min\{d_x, d_y\}$
Q	usually the cylinder $\Omega \times (0, T]$
$\partial^* Q$	the parabolic boundary of Q, i.e., $\partial^* Q = \Omega \times \{t = 0\} \cup \partial\Omega \times [0, T]$
Q'	$Q' \subseteq Q$ and $d(Q', \partial^* Q) > 0$
Q_ρ	$Q_\rho \subseteq Q$ and $d(Q_\rho, \partial^* Q) = \rho$
d_P	the distance between the point $P = (x, t)$ and $\partial^* Q$
$d_{P_1P_2}$	$d_{P_1P_2} = \min\{d_{P_1}, d_{P_2}\}$
$K(\rho)$, B_ρ	a ball with radius ρ
$B_\rho(x^0)$	a ball with radius ρ, centered at x^0, i.e., $\{x: \lvert x - x^0\rvert < \rho\}$
$k_R(x^0)$	a square with length $2R$, centered at $x^0 = (x_1^0, \dots, x_n^0)$, i.e., $$\{x: \lvert x_i - x_i^0\rvert < R,\ 1 \le i \le n\}$$
$K_R(x^0, t^0)$	$k_R(0) \times (0, T)$
$\lvert A\rvert$	the measure of the set A, or $\lvert A\rvert = \operatorname{meas} A$
a.e.	almost everywhere
$\varnothing$	empty set
$\partial u/\partial N$	N is the normal at the boundary, $\partial u/\partial N$ in general is the normal derivative of u along the boundary; usually we take N to be the inner one.
ω_n	the area of the unit sphere in $\mathbb{R}^n$
κ_n	the volume of the unit ball in $\mathbb{R}^n$

$\|\|u\|\|_p$	$\|\|u\|\|_p = (\int_\Omega \|u\|^p \, dx)^{1/p}$, Ω given
$C^{k+\alpha}$	$u = u(x_1, \ldots, x_n)$ is continuously differentiable up to kth order and its kth derivatives satisfy a Hölder condition with exponent α $(0 < \alpha \le 1)$
$C^{2,1}$	$u = u(x_1, \ldots, x_n, t)$ twice continuously differentiable in $x_1, \ldots, x_n$ and continuously differentiable in t
$C^{2+\lambda, 1+\lambda/2}$	u_{ij} $(1 \le i, j \le n)$ and u_t are Hölder continuous in x with exponent λ and Hölder continuous in t with exponent $\lambda/2$
$\|u\|_{\alpha,\Omega}$	$u \in C^\alpha(\Omega)$ $(0 < \alpha \le 1)$, $\|u\|_{\alpha,\Omega}$ is the Hölder constant
$\|u\|_{\alpha,Q}$	$u \in C^{\alpha,\alpha/2}(Q)$, $Q = \Omega \times (0, T]$ and $\|u\|_{\alpha,Q} = \sup \dfrac{\|u(x,t) - u(x',t')\|}{(\|x-x'\|^2 + \|t-t'\|)^{\alpha/2}}$
$W_p^k(\Omega)$	The function space consisting of all functions in Ω whose weak derivatives up to kth order belong to $L^p(\Omega)$
$\mathring{W}_p^k(\Omega)$	The closure (under the $W_p^k(\Omega)$-norm) of functions in $W_p^k(\Omega)$ which vanish near $\partial\Omega$. In other words, it is the closure of $C_0^\infty(\Omega)$ under $W_p^k(\Omega)$-norm
$W_p^{k,k/2}(Q)$	$Q = \Omega \times (0, T]$, the function space consisting of all functions in Q whose weak derivatives in x up to kth order and weak derivatives in t up to $k/2$ order (k is even) belong to $L^p(Q)$
a^+	$a^+ = \max\{a, 0\}$
a^-	$a^- = -\min\{a, 0\}$

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Epilogue

Our discussion on quasilinear and fully nonlinear equations is far from complete. Many important aspects, such as the oblique derivative problem for fully nonlinear equations, as well as the applications of fully nonlinear equations, are left out. Hopefully they will be covered in a future publication.

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