# Translations of MATHEMATICAL MONOGRAPHS

Volume 95

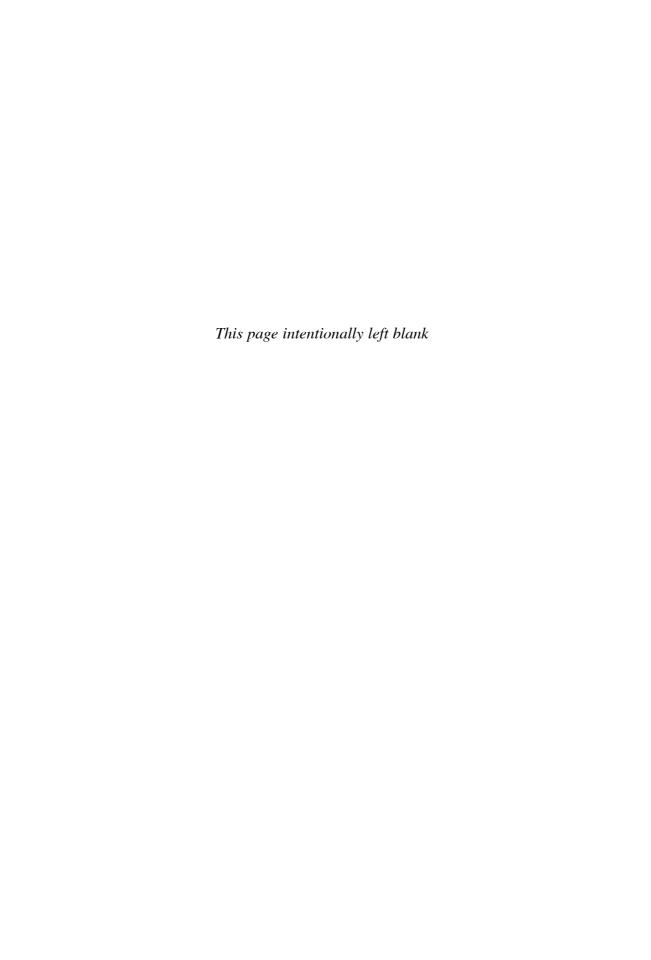
# Nonlinear Partial Differential Equations of Second Order

**Guangchang Dong** 



American Mathematical Society

Nonlinear Partial Differential Equations of Second Order



# Translations of MATHEMATICAL MONOGRAPHS

Volume 95

Nonlinear Partial Differential Equations of Second Order

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# 非线性二阶偏微分方程

### 清华大学出版社

Translated from the Chinese by Kai Seng Chou [Kaising Tso]

2000 Mathematics Subject Classification. Primary 35Kxx; Secondary 35Jxx, 35Lxx.

ABSTRACT. This is a treatise on nonlinear partial differential equations of second order. With the exception of the first chapter, all the remaining chapters are based on the published or unpublished work of the author.

A priori estimation is the main theme of this book. Emphasis is placed on elliptic and parabolic equations. Nevertheless, some hyperbolic equations are also discussed. Each chapter of this book has its own physical background.

This book can be used as a text for graduate students in mathematics, or as a reference for researchers, teachers, or university seniors.

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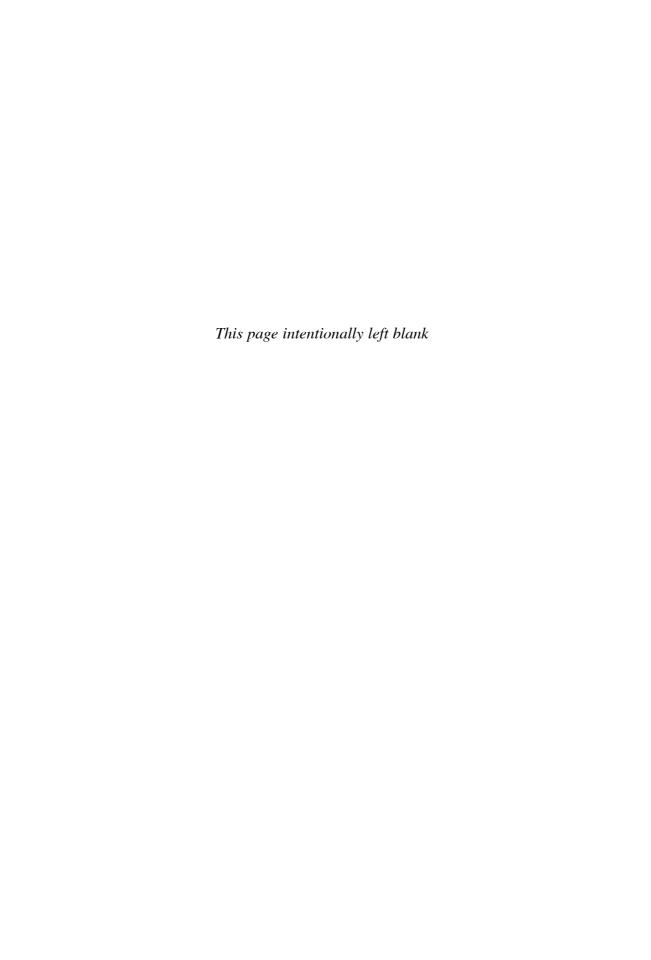
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## **Symbols**

```
Ω
                     usually a bounded (connected) open set
            \overline{\Omega}
                    the closure of \Omega
          \partial \Omega
                    the boundary of \Omega
                     A \subset\subset B means \overline{A}\subset B
          CC
           \Omega'
                     \Omega' \subset\subset \Omega
                    the distance between x and the set A, i.e., d(x, A) =
   d(x, A)
                    \inf_{v \in A} \overline{xy}
                     \Omega_{\rho} \subseteq \Omega and d(\Omega_{\rho}, \partial\Omega) = \rho
                     d_{x} = d(x, \partial \Omega)
                    d_{xy} = \min\{d_x, d_y\}
                    usually the cylinder \Omega \times (0, T]
                    the parabolic boundary of Q, i.e., \partial^* Q = \Omega \times \{t = 0\} \cup \partial \Omega
                  Q' \subseteq Q and d(Q', \partial^*Q) > 0
Q_{\rho} \qquad Q_{\rho} \subseteq Q \text{ and } d(Q_{\rho}, \partial^*Q) = \rho
d_{P} \qquad \text{the distance between the point}
d_{P_{1}P_{2}} \qquad d_{P_{1}P_{2}} = \min\{d_{P_{1}}, d_{P_{2}}\}
K(\rho), B_{\rho} \qquad \text{a ball with radius } \rho
                    the distance between the point P = (x, t) and \partial^* Q
    B_{\rho}(x^0) a ball with radius \rho, centered at x^0, i.e., \{x: |x-x^0| < \rho\}
                    a square with length 2R, centered at x^0 = (x_1^0, \dots, x_n^0), i.e.,
    k_{P}(x^{0})
                                   \{x: |x_i - x_i^0| < R, 1 < i < n\}
K_R(x^0, t^0)
                    k_R(0) \times (0, T)
                    the measure of the set A, or |A| = \text{meas } A
         a.e.
                    almost everywhere
            Ø
                    empty set
   \partial u/\partial N
                     N is the normal at the boundary, \partial u/\partial N in general is
                    the normal derivative of u along the boundary; usually we
                    take N to be the inner one.
                    the area of the unit sphere in \mathbb{R}^n
          \omega_n
           \kappa_n
                    the volume of the unit ball in \mathbb{R}^n
```

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$$||u||_{p}$$
  $||u||_{p} = (\int_{\Omega} |u|^{p} dx)^{1/p}$ ,  $\Omega$  given

 $u = u(x_1, \ldots, x_n)$  is continuously differentiable up to kth order and its kth derivatives satisfy a Hölder condition with exponent  $\alpha$   $(0 < \alpha \le 1)$ 

- $C^{2,1}$   $u = u(x_1, ..., x_n, t)$  twice continuously differentiable in  $x_1, \ldots, x_n$  and continuously differentiable in t
- $C^{2+\lambda,\,1+\lambda/2}$  $u_{ij}$   $(1 \le i, j \le n)$  and  $u_t$  are Hölder continuous in x with exponent  $\lambda$  and Hölder continuous in t with exponent
  - $|u|_{\alpha,\Omega}$   $u \in C^{\alpha}(\Omega)$   $(0 < \alpha \le 1)$ ,  $|u|_{\alpha,\Omega}$  is the Hölder constant  $|u|_{\alpha,Q}$   $u \in C^{\alpha,\alpha/2}(Q), Q = \Omega \times (0,T]$  and

$$|u|_{\alpha,Q} = \sup \frac{|u(x,t) - u(x',t')|}{(|x - x'|^2 + |t - t'|)^{\alpha/2}}$$

- $W_n^k(\Omega)$  The function space consisting of all functions in  $\Omega$  whose weak derivatives up to kth order belong to  $L^p(\Omega)$
- The closure (under the  $W_n^k(\Omega)$ -norm) of functions in  $W_n^k(\Omega)$  which vanish near  $\partial \Omega$ . In other words, it is the closure of  $C_0^{\infty}(\Omega)$  under  $W_n^k(\Omega)$ -norm
- $W_n^{k,k/2}(Q)$   $Q = \Omega \times (0,T]$ , the function space consisting of all functions in Q whose weak derivatives in x up to kth order and weak derivatives in t up to k/2 order (k is even) belong
  - to  $L^{p}(Q)$   $a^{+}$   $a^{+} = \max\{a, 0\}$   $a^{-}$   $a^{-} = -\min\{a, 0\}$

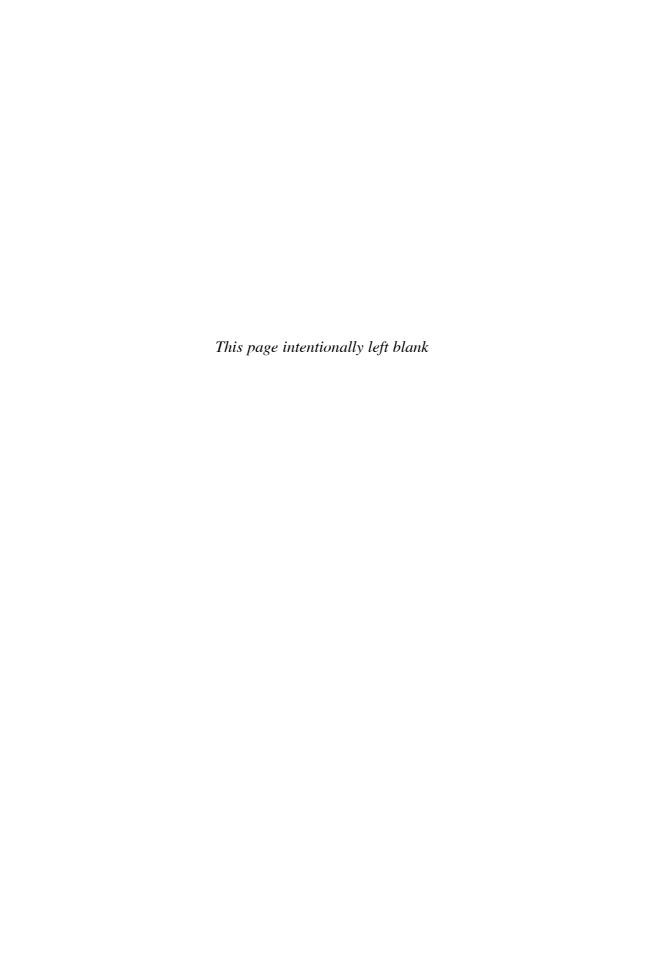
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## **Epilogue**

Our discussion on quasilinear and fully nonlinear equations is far from complete. Many important aspects, such as the oblique derivative problem for fully nonlinear equations, as well as the applications of fully nonlinear equations, are left out. Hopefully they will be covered in a future publication.



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