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Introduction to the Classical Theory of Abelian Functions

A. I. Markushevich

American Mathematical Society

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Introduction to the Classical Theory of Abelian Functions

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Providence, Rhode Island

ВВЕДЕНИЕ В КЛАССИЧЕСКУЮ ТЕОРИЮ
АБЕЛЕВЫХ ФУНКЦИЙ

МОСКВА, «НАУКА», 1979

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ABSTRACT. Introduction to the classical theory of Abelian functions. Moscow, Nauka, Glavnaya redaktsiya fisiko-matematicheskoy literaturi, 1979, 240 pages.

The theory of Abelian functions, which was at the center of mathematics of the XIX century, is attracting a significant amount of attention again. But today it is frequently considered not as a chapter of the general theory of functions but as an area of application of the ideas and methods of commutative algebra. In this book, we present an exposition of the fundamentals of the theory of Abelian functions based on the methods of the classical theory of functions. This theory includes the theory of elliptic functions as a special case. A detailed historical introduction also distinguishes our exposition from others.

This book is directed toward undergraduate and graduate students and toward the mathematicians teaching at universities. The reader is assumed to have acquired only the minimal set of facts about functions of many variables. Such facts are usually presented in a standard course on the theory of analytic functions given at universities. All additional necessary material is given in the Appendices. The book is written on the basis of a course which was taught by the author in the Fall semester of the 1976/77 academic year in the Department of Professional Development (Fakultet Povisheniya Kvalifikatsii) of Moscow State University.

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Preface

"When I was a student, during the First World War," recalls F. Klein, "Abelian functions were considered (under the influence inherited from Jacobi) to be the indisputable pinnacle of mathematics. Each of us, naturally, had some ambition to make an advance in this area on his own. But now? It is doubtful whether the younger generation is familiar with Abelian functions at all."([1])

More than half a century has passed since the time of this quotation and the interest in the "temporarily outdated" (as Klein called it) theory of Abelian functions is again on the rise. However, today it has become quite different from the theory that Klein was discussing. It is considered not as a chapter of the theory of functions, but rather as a chapter or even an area of application of the ideas and methods of commutative algebra, algebraic geometry, and complex analysis. A typical example of such interpretations is the book "Introduction to Algebraic and Abelian Functions" by the algebraist S. Lang.([2]) We may also mention several books, for example: "Introduction à l'étude des variétés kahleriennes" by André Weil([3]) (mainly Chapter VI) and "Abelian Varieties" by D. Mumford([4]) (in collaboration with K. P. Ramanujan).

With all due respect to these excellent monographs, we still would like to offer the reader a different exposition of the subject. Our exposition is based on the ideas and methods of the classical theory of functions. Here the reader is assumed to know only a minimal number of facts about functions of several variables, such as are commonly presented in a standard university course on the theory of analytic functions. All the additional material that will be needed is given in the appropriate parts of this book.

Our exposition is distinguished by a detailed historical introduction (Chapter I), covering the period up to the middle of the nineteenth century during which the initial accumulation of the ideas and facts took place. In the next

([1])F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, Part I, Springer, 1926.

([2])S. Lang, *Introduction to algebraic and Abelian functions*, Addison-Wesley, 1972.

([3])A. Weil, *Introduction à l'étude des variétés kähleriennes*, Actualités Sci. Indust., no. 1267, Hermann, Paris, 1958.

([4])D. Mumford, *Abelian Varieties*, Oxford Univ. Press, 1970.

three chapters we present a systematic construction of the foundations of the theory of Abelian functions as considered as a study of meromorphic functions of p complex variables with the maximum number ($2p$) of independent periods. This theory includes as a special, and in some sense trivial, case the theory of elliptic functions ($p = 1$). In the three main chapters we have been considerably influenced by Fabio Conforto's book "Abelsche Functionen und algebraische Geometry," edited by W. Gröbner, A. Andreotti, and M. Rosati.([5]) This book was published in 1956 in the series "Grundlagen der mathematischen Wissenschaften" in which Klein's lectures also appeared. We note that an exposition which is similar in spirit can be found in C. L. Siegel's book "Analytic functions of several variables"([6]) (Chapters IV-IX). The contents of this book which we now offer to readers (undergraduate and graduate students, as well as to university teachers of mathematics) was the subject of a course given in the fall semester of 1976/77 in the Department of Professional Development of University Instructors at the Moscow State University.

In conclusion I thank my daughter L. A. Markushevich, who assisted me in the preparation of the manuscript for publication.

A. Markushevich

([5])See footnote ([18]) on page 44.

([6])C. L. Siegel, *Analytic functions of several complex variables*, Institute for Advanced Study, Princeton, N. J., 1950; Russian transl., IL, Moscow, 1954.

Appendix

§A. Skew-Symmetric Determinants

A.1. The Pfaffian. A matrix

$$A = (a_{jk}), \qquad j,\ k = 1, \dots, n, \tag{A.1.1}$$

is called *skew-symmetric* if

$$a_{jk} = -a_{kj} \quad \text{(in particular, } a_{jj} = 0). \tag{A.1.2}$$

The corresponding determinant $|A|$ is also called skew-symmetric.

The equalities (A.1.2) can be written in the form

$$\widetilde{A} = -A. \tag{A.1.2$'$}$$

This means that $|\widetilde{A}| = (-1)^n|A|$ for the determinant $|A|$. On the other hand, $|\widetilde{A}| = |A|$. Hence $|A| = (-1)^n|A|$ and therefore when n is odd we have $|A| = 0$.

Let $n = 2m - 1$. Consider the minors M_{jk} and M_{kj} of the determinant. If we transpose M_{kj}, we get a minor of the matrix $\widetilde{A}$ with index j, k. Because of $(A.1.2')$ it follows that $M_{kj} = (-1)^{n-1}M_{jk} = M_{jk}$. Thus

$$M_{kj} = M_{jk}, \qquad j,\ k = 1, \dots, 2m - 1. \tag{A.1.3}$$

We denote by A_{jk} the term corresponding to the entry a_{jk} in the expansion for the determinant $|A|$. Then $A_{jk} = (-1)^{j+k}M_{jk}$, and under the same conditions we have

$$A_{jk} = A_{kj}, \qquad j,\ k = 1, \dots, 2m - 1. \tag{A.1.4}$$

Since $|A| = 0$, it follows in addition that

$$A_{jj}A_{kk} - A_{jk}^2 = 0, \qquad j,\ k = 1, \dots, 2m - 1. \tag{A.1.5}$$

In fact, if the rank of the matrix A is less than $n - 1 = 2m - 2$, then all $A_{jk} = 0$ and (A.1.5) obviously holds. If, on the other hand, this rank is equal to $n - 1$, we conclude, on the basis of the equations $\sum_{k=1}^n a_{jk}A_{jm} = 0$, which hold for each $m = 1, \dots, n$, that the n-dimensional vectors with coordinates $A_{1m}, \dots, A_{nm}$ are collinear so that

$$\frac{A_{1k}}{A_{1j}} = \frac{A_{2k}}{A_{2j}} = \cdots = \frac{A_{nk}}{A_{nj}}; \tag{A.1.5$'$}$$

and in particular

$$\frac{A_{jk}}{A_{jj}} = \frac{A_{kk}}{A_{kj}}.$$

By (A.1.4), it follows from this that (A.1.5) is true.

We now turn to the case $n = 2m$ and prove that in this case *there exists a form* $\mathscr{P}$ *of degree* m *in the elements of the determinant of* A, *all of whose coefficients are integers, and with the property that*

$$|A| = \mathscr{P}^2. \tag{A.1.6}$$

This form can be constructed explicitly in terms of the entries of the matrix A. It is called the *Pfaffian* after Johann Friedrich Pfaff (1765–1825), an older contemporary of Gauss. However, we will not give the actual construction of the Pfaffian here and will limit ourselves to proving the theorem stated above.

It is most easily proved by induction. In the case $n = 2$ we have

$$|A| = \begin{vmatrix} 0 & a_{12} \\ -a_{21} & 0 \end{vmatrix} = a_{12}^2$$

and we can set $\mathscr{P} = a_{12}$. Assume that the theorem has been proved for $n = 2m - 2$ $(m \geq 2)$. We consider a determinant B of order $2m - 1$ which is a minor of the entry $a_{2m,2m}$ in $|A|$:

$$|B| = \begin{vmatrix} a_{1,1} & \cdots & a_{1,2m-1} \\ a_{2m-1,1} & \cdots & a_{2m-1,2m-1} \end{vmatrix}.$$

It is obvious that $|B|$ is skew-symmetric simultaneously with $|A|$. Since the principal minors A_{rr} $(r = 1, \dots, 2m - 1)$ of the latter determinant are also skew-symmetric and their order is $2m - 2$, by the inductive assumption we get

$$A_{rr} = \mathscr{P}_r^2, \qquad r = 1, \dots, 2m - 1, \tag{A.1.7}$$

where $\mathscr{P}_r$ is a form of degree $m - 1$ in the corresponding elements a_{ij} with integral coefficients.

Assume that among the forms $\mathscr{P}_r$ there is at least one which is not identically zero. Without loss of generality, we can assume that it is $\mathscr{P}_1$. The equality $A_{11} = \mathscr{P}_1^2$ is satisfied by the two forms $\mathscr{P}_1$ and $-\mathscr{P}_1$. We choose one of them and denote it by $\mathscr{P}_1$. We now choose $\mathscr{P}_2, \dots, \mathscr{P}_{2m-1}$ (at each step we select one of the two forms which differ only by the factor -1) in such a way that we have

$$A_{1r} = \mathscr{P}_1 \cdot \mathscr{P}_r, \qquad r = 1, \dots, 2m - 1,$$

by (A.1.5). Now by (A.1.5′) we have

$$A_{11}A_{rs} - A_{1s}A_{r1} = 0, \qquad \mathscr{P}_1^2 A_{rs} = \mathscr{P}_1^2 \mathscr{P}_r \mathscr{P}_s,$$

and since $\mathscr{P}_1 \not\equiv 0$ we have

$$A_{rs} = \mathscr{P}_r \mathscr{P}_s, \qquad r,\ s = 1, \dots, 2m-1. \tag{A.1.8}$$

We notice, that in the case when $\mathscr{P}_1, \dots, \mathscr{P}_{2m-1}$ are all identically zero, all A_{rr} are also zero and hence all A_{rs} are also zero, as follows from (A.1.5). Therefore the equalities (A.1.8) hold in this case.

Turning now to the determinant $|A|$, we expand it by the entries of the last column and then we expand each of the minors of these elements (except the principal one, which is equal to $|B| = 0$) by the elements of the last row. We get

$$|A| = -\sum_{r,s=1}^{2m-1} a_{r,n} a_{n,s} A_{rs} = \sum_{r,s=1}^{n-1} a_{r,n} a_{s,n} A_{rs},$$

or, by substituting for A_{rs} their expressions (A.1.8),

$$|A| = \sum_{r,s=1}^{n-1} a_{r,n} a_{s,n} \mathscr{P}_r \mathscr{P}_s = (a_{1,n}\mathscr{P}_1 + \cdots + a_{n-1,n}\mathscr{P}_{n-1})^2. \tag{A.1.9}$$

It remains only to set

$$\mathscr{P} = a_{1,n}\mathscr{P}_1 + \cdots + a_{n-1,n}\mathscr{P}_{n-1}. \tag{A.1.10}$$

As an illustration, we consider the case $n = 4$. Here

$$|A| = \begin{vmatrix} 0 & a & b & d \\ -a & 0 & c & e \\ -b & -c & 0 & f \\ -d & -e & f & 0 \end{vmatrix} \quad \text{and} \quad |B| = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}.$$

In our case

$$A_{11} = \begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix} = c^2, \qquad A_{22} = \begin{vmatrix} 0 & b \\ -b & 0 \end{vmatrix} = b^2, \qquad A_{33} = \begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = a^2.$$

We set $\mathscr{P}_1 = c$. Since $A_{12} = bc$ and $A_{13} = ac$, we have $\mathscr{P}_2 = b$ and $\mathscr{P}_3 = a$. Therefore, according to (A.1.9), we have

$$|A| = (dc + eb + fa)^2, \quad \text{i.e.,} \quad \mathscr{P} = cd + be + af.$$

A.2. The Frobenius Theorem. Let P be a skew-symmetric $(2p, 2p)$ matrix, all of whose entries are integers, and with $|P| \neq 0$. For each unimodular $(2p, 2p)$ matrix M with integral entries we can construct a transformed matrix

$$P^* = MP\widetilde{M}. \tag{A.2.1}$$

It is obvious that the entries of the matrix P^* are also integers. Moreover, $\widetilde{P}^* = M\widetilde{P}\widetilde{M} = -MP\widetilde{M} = -P^*$, i.e., P^* is skew-symmetric and (because of (A.2.1)) we have

$$|\widetilde{P}^*| = |M|\,|P|\,|\widetilde{M}| = |P|\,|M|^2 = |P|.$$

It is easy to check that the relation between the skew-symmetric $(2p, 2p)$ matrices P and P^* with integral entries and nonzero determinants, given by an equation of the form (A.2.1), is an *equivalence relation.* In particular, if we start with P and perform, one after another, the transformations of the form (A.2.1) which are determined by the unimodular matrices $M_1, \dots, M_n$, where n is any natural number, we stay in the same equivalence class. It is natural to try to select a matrix of the simplest structure from each class in order to have a representative of the class. This is made possible by the following statement.

THE FROBENIUS THEOREM (1878). *For each skew-symmetric $(2p, 2p)$ matrix P with integral entries and nonzero determinant, there exists a unimodular $(2p, 2p)$ matrix M with integral coefficients for which*

$$P^* = MP\widetilde{M} = \operatorname{Diag}\left\{\beta_1\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \dots, \beta_p\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right\}, \qquad \text{(A.2.2)}$$

where $\beta_1, \dots, \beta_p$ are positive integers such that

$$\beta_1 | \beta_2 | \cdots | \beta_p |. \qquad \text{(A.2.3)}$$

PROOF. The transformation we are looking for can be obtained step by step by applying unimodular matrices of a particular form. We show that it suffices to use three kinds of matrices.

(A) The matrix M is derived from the unit matrix E_{2p} by permuting the jth and kth rows (or by permuting the jth and kth columns). In this case the matrix $P^* = MP\widetilde{M}$ differs from P only by having jth and kth rows permuted and at the same time its jth and kth columns permuted.

(B) M is derived from the unit matrix by multiplying the jth diagonal entry of the unit matrix by -1. As a result, P^* will differ from P by the signs of the entries in its jth row and jth column (note that the entry which belongs both to the jth row and jth column is zero).

(C) M is derived from the unit matrix by substituting an integer l for the zero at the intersection of the jth row and kth column. As a result $P^* = MP\widetilde{M}$ is obtained from P by adding to the entries of the jth row the corresponding entries of the kth row multiplied by l and then performing the same operation on the entries of the jth and kth columns. We note that we then get the following entry at the intersection of jth row and jth column of the product $MP\widetilde{M}$:

$$a_{jj} + l a_{jk} + l a_{kj} + l^2 a_{kk}.$$

But this entry is equal to zero, as it should be in a skew-symmetric matrix, since $a_{jj} = a_{kk} = 0$ and $a_{jk} = -a_{kj}$.

Thus, the equivalence class to which the matrix P belongs contains all matrices which can be obtained from P by combining, in any way, a finite number of the elementary transformations just described. We will show that by performing them in an appropriate order we can arrive at the required result.

Let δ be *the least absolute value of all nonzero entries of all matrices* P which belong to the same equivalence class and let P^* be a matrix containing the entry $\pm\delta$. By using the operations (B) we arrive at the case when the entries equal to δ occur among the entries of P^*. Let δ be at the intersection of the j'th row and k'th column, i.e., $a_{j'k'} = \delta$. We show that the rest of the entries of P^* are integral multiples of δ, i.e., δ is the greatest common divisor of all the entries of P^*.

Indeed, if we assume, for example, that $a_{jk'}$ $(j \neq j')$ is not divisible by δ, then $a_{jk'} = \delta(-l) + \delta'$ where l and δ' are integers and moreover $0 < \delta' < \delta$. By applying an appropriate operation (C) we get an entry δ' in place of the entry $a_{jk'}$ which contradicts the definition of δ as the least possible entry. Thus, first of all, the entries of P^* which are in the same row or in the same column with δ are divisible by it. It follows from this that by applying operations (C) to them again we can make them equal to zero (except the entry δ itself, of course). Assume that *after* this we can find an entry a_{jk} $(j \neq j', k \neq k')$ which is not divisible by δ. Then, by means of an operation (C) we can get a new matrix whose jth row is the sum of the jth and j'th rows of the matrix P^*. Moreover, the former entry a_{jk} will occur in the same row as δ and will have to be divisible by δ. Thus we have again obtained a contradiction.

We now notice that the common divisors of the entries of P are obviously preserved by the operations (A), (B), and (C) and hence the g.c.d. of the entries of P is also preserved by these operations. But the g.c.d. of the entries of the matrix P^* that occur in the Frobenius Theorem is equal to β_1. Thus $\beta_1 = \delta$.

By changing the notation, if necessary, we can assume that the matrix P is, from the very beginning, in the form where at least one of its entries is equal to $\delta = \beta_1$ and all the others in the row and the column to which β_1 belongs are equal to zero. Applying an operation (A) we can make the entry β_1 occur second in the first row. Then $-\beta_1$ will be the first entry on the second row (because of skew-symmetry) and therefore the matrix P must be in the form

$$P = \left(\begin{array}{cc|c} 0 & \beta_1 & \\ & & 0 \\ -\beta_1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & P_1 \end{array}\right). \tag{A.2.4}$$

The matrix denoted by P_1 has the same properties as the matrix P, but its order is the order of P minus two. Since β_1 is the g.c.d of *all* entries of P, each entry of P_1 is divisible by β_1. Hence if we denote by β_2 the g.c.d of all the entries of P_1,

$$\beta_1 | \beta_2.$$

It is clear that those transformations of the matrix P of types (A), (B), and (C) which do not change its first two rows and first two columns play the same role with respect to P_1 as they played with respect to the entire matrix P. Hence, merely by the preceding transformations, we transform P_1 to the form

$$P_1 = \left(\begin{array}{cc|c} 0 & \beta_2 & \\ & & 0 \\ -\beta_2 & 0 & \\ \hline \multicolumn{2}{c|}{0} & P_2 \end{array}\right),$$

and we transform the entire matrix P to the form

$$P = \left(\begin{array}{cccc|c} 0 & \beta_1 & 0 & 0 & \\ -\beta_1 & 0 & 0 & 0 & \\ & & & & 0 \\ 0 & 0 & 0 & \beta_2 & \\ 0 & 0 & -\beta_2 & 0 & \\ \hline \multicolumn{4}{c|}{0} & P_2 \end{array}\right).$$

By repeating these steps p times we get P into the form (A.2.2).

In the main part of this book we have, however, used a somewhat different form for a skew-symmetric matrix. By repeatedly moving the rows and columns of the matrix (A.2.2) (i.e., by applying operations (A) repeatedly) we obtain the following representation of P which lies in the same equivalence class:

$$P = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}, \tag{A.2.5}$$

where B is the diagonal matrix

$$B = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \beta_p \end{pmatrix}. \tag{A.2.6}$$

Recall that β_1, $\beta_1 \geq 1$, is the g.c.d of all entries of a given $(2p, 2p)$ matrix P and β_2 is the g.c.d of all entries of a $(2p-2, 2p-2)$ matrix P_1 which satisfies (A.2.4), and so on. We also note that

$$|P| = |B|^2 = \beta_1^2 \beta_2^2 \cdots \beta_p^2. \tag{A.2.7}$$

This consequence of the Frobenius Theorem would allow us to avoid referring, in the main part of the book, to the possibility of representing a skew-symmetric determinant as a square of the corresponding Pfaffian.

§B. Divisors of analytic functions

B.1. General theorems. In section II.1 we have already discussed the relation of divisibility between analytic functions. Here we show how the main properties of divisibility can be derived from more general algebraic theorems. These can be found in B. L. Van der Waerden's *Algebra.*

Since the notion of divisibility of analytic functions involves the function in a neighborhood of a given point $a \in \mathbf{C}^p$, we will, without loss of generality, take $a = 0$ in this section. We denote by $\mathfrak{M}_p$ the set of functions of u which are meromorphic at the origin, i.e., which are representable in some neighborhood of the origin as a quotient of two analytic functions. We denote by $\mathfrak{H}_p$ the set of functions analytic at the origin. $\mathfrak{M}_{p-1}$ and $\mathfrak{H}_{p-1}$ denote analogous sets of functions of $p-1$ variables $(u_2, \dots, u_p) = 'u$.

We form two rings of polynomials in u_1 with coefficients in $\mathfrak{M}_{p-1}$ and $\mathfrak{H}_{p-1}$ respectively: $\mathfrak{M}_{p-1}[u_1]$ and $\mathfrak{H}_{p-1}[u_1]$. It is obvious that the pseudopolynomials defined in section II.1 belong to $\mathfrak{H}_{p-1}[u_1]$.

The ring $\mathfrak{M}_{p-1}[u_1]$ is a *Euclidean ring* since the following relations hold for any two of its nonzero elements f and g:

(1) the degree of the product fg is not less than the degree of f;

(2) the representation $f = gh + r$ exists when either $r = 0$ or the degree of r is less than the degree of g (division algorithm).

It follows from this that the elements f and g have a g.c.d., and it can be represented in the following form by means of consecutive divisions (Euclidian algorithm):

$$(f, g) = \text{g.c.d. } (f, g) = ff_1 + gg_1,$$

where f_1 and g_1 are elements of the same ring.

Furthermore, each nonzero element of $\mathfrak{M}_{p-1}[u_1]$ can be uniquely decomposed (up to unit factors) into a product of prime elements. In particular, it follows from this that Euclid's theorem holds.

If $f|gh$ *and* $(f, g) = 1$, *then* $f|h$.

We note that, according to the general theory, the units of our ring are exactly the elements ε which belong to $\mathfrak{M}_{p-1}[u_1]$ together with ε^{-1}. It follows from this that only a polynomial of a nonzero degree can be a unit of the ring: the set of all units of $\mathfrak{M}_{p-1}[u_1]$ coincides with the set of meromorphic functions in $\mathfrak{M}_{p-1}$ which are not identically zero.

This means that a theory analogous to the theory of divisibility of polynomials in one variable with rational coefficients (in this case all nonzero rational numbers form the set of units) holds for the ring $\mathfrak{M}_{p-1}[u_1]$.

In order to move from the ring $\mathfrak{M}_{p-1}[u_1]$ to the ring $\mathfrak{H}_{p-1}[u_1]$ and then to the larger set $\mathfrak{H}_p$ of functions of p variables which are analytic at the origin, we use induction on the number of variables. Note that in the case of one variable the prime elements are reduced (up to multiplication by units) to a single function $\varphi(u_1) = u_1$ and hence the decomposition of $f(u_1) \in \mathfrak{H}_1$

into prime factors has the form $f(u_1) = u_1^{\nu} g(u_1)$ where $\nu \geq 0$ and $g(0) \neq 0$ (compare with section II.1).

Assume now that the theorem about the uniqueness of decomposition into prime factors and all its consequences is true for $\mathfrak{H}_{p-1}$. Let $\varphi(u_1)$ be a nonzero polynomial in $\mathfrak{H}_{p-1}[u_1]$. It is called *primitive* if the g.c.d of the set of its coefficients is equal to 1 (it is enough that the coefficient of the highest power is a nonzero constant and therefore all pseudopolynomials are primitive). If $f(u_1)$ is any nonzero polynomial in $\mathfrak{H}_{p-1}[u_1]$ and $d = d('u)$ is the g.c.d of its coefficients, then $f(u_1)$ can be represented in the form

$$f(u_1) = d('u)\varphi(u_1), \tag{B.1.1}$$

where $\varphi(u_1)$ is a primitive polynomial.

We have the following general facts.

(I) The product of two primitive polynomials is a primitive polynomial (an analog of the *Gauss lemma*).

(II) If $f(u_1) \in \mathfrak{M}_{p-1}[u_1]$, there exists a primitive polynomial $\varphi(u_1) \in \mathfrak{H}_{p-1}[u_1]$ such that

$$f(u_1) = \frac{d('u)}{g('u)}\varphi(u_1), \qquad g, d \in \mathfrak{H}_{p-1}. \tag{B.1.2}$$

In order to establish this under our conditions it is enough to find a common denominator $g('u)$ of the meromorphic coefficients of the polynomial $f(u_1)$ and then to apply a representation of the form (B.1.1) to the numerators.

(III) The polynomial $\varphi(u_1)$ in the formula (B.1.2) is uniquely determined by $f(u_1)$ up to unit factors from $\mathfrak{H}_{p-1}$. Analogously, $f(u_1)$ is uniquely determined by the primitive polynomial $\varphi(u_1)$ up to unit factors from $\mathfrak{M}_{p-1}$.

(IV) The relation between the polynomials in $\mathfrak{M}_{p-1}[u_1]$ and the primitive polynomials in $\mathfrak{H}_{p-1}[u_1]$ associates with a product of two primitive polynomials $\varphi_1(u_1)$ and $\varphi_2(u_1)$ the product of the polynomials $f_1(u_1)$ and $f_2(u_1)$ corresponding to them (and conversely). Hence if the polynomial $f(u_1)$ is prime in $\mathfrak{M}_{p-1}[u_1]$, then $\varphi(u_1)$ is also prime in $\mathfrak{H}_{p-1}[u_1]$ (and conversely).

The following important corollary follows from the facts listed above: *under the inductive assumption the theorem about the uniqueness of the decomposition into prime factors is valid for the ring* $\mathfrak{H}_{p-1}[u_1]$.

In particular, any pseudopolynomial $P(u_1) \in \mathfrak{H}_{p-1}[u_1]$ decomposes into prime factors in a unique way. It follows from this, in view of the Weierstrass preparation theorem, that *the theorem about uniqueness is valid in* $\mathfrak{H}_p$ *as well, i.e., it is valid for functions of* p *variables that are analytic at the origin.* This completes the construction of the theory of divisibility for analytic functions of several variables.

We will discuss a consequence of this theory. The derivation of this consequence requires the use of the simplest properties of the resultant of two polynomials.

Consider two pseudopolynomials

$$P(u_1) = u_1^n + A_1 u_1^{n-1} + \cdots + A_n,$$
$$Q(u_1) = u_1^m + B_1 u_1^{m-1} + \cdots + B_m.$$

We will not use the restriction on their lower orders (see the definition of a pseudopolynomial in section II.1). The fact that $P(u_1)$ and $Q(u_1)$ are primitive polynomials in $\mathfrak{H}_{p-1}[u_1]$ is sufficient. Eliminating u_1 from P and Q, we obtain their *resultant* d in the form of a determinant whose entries are coefficients of the polynomials P and Q. It is obvious that, under our conditions, d belongs to the set $\mathfrak{H}_{p-1}$. The main property of the resultant states that it is identically equal to zero if and only P and Q are not relatively prime, i.e., they have a common divisor different from a unit of $\mathfrak{H}_{p-1}[u_1]$. Since the polynomials P and Q are primitive, this common divisor cannot be a divisor of their coefficients and therefore is a pseudopolynomial of degree not less than 1.

Hence in the case when P and Q are relatively prime, i.e., when $(P, Q) = 1$, the resultant is a function of the $p-1$ variables $(u_2, \dots, u_p) = {}'u$, which is analytic at the origin and is not identically zero. Then it follows from a simple calculation that there exist polynomials P' and Q' in the ring $\mathfrak{H}_{p-1}[u_1]$ such that the condition $PP' + QQ' = d$ holds. Thus *the condition $(P, Q) = 1$ for pseudopolynomials is equivalent to the existence of polynomials P' and Q' in $\mathfrak{H}_{p-1}[u_1]$ such that*

$$PP' + QQ' = d, \tag{B.1.3}$$

where $d = d({}'u) \in \mathfrak{H}_{p-1}$ and $d \not\equiv 0$.

B.2. Continuation of the divisibility relation from a point to a region. The notion of a divisor of an analytic function depends on the point $a \in \mathbf{C}^p$ under consideration. If, for example, a function $f(u)$ is a unit at a, i.e., $f(a) \neq 0$, then it may be prime or may decompose into a product of several prime functions at a neighboring point b where $f(b) = 0$. However, of course, in a sufficiently small neighborhood U of a in which $f(u) \neq 0$ it will remain a unit for any point $b \in U$.

Consider two functions $f(u)$ and $g(u)$ that are analytic at a point a. If they have a common divisor $h(u)$ $(h(a) = 0)$, then in any sufficiently small neighborhood of a there is a point b for which $h(b) \neq 0$. Therefore $h(u)$ is not a common (*proper*) divisor of $f(u)$ and $g(u)$ at b. At this point $f(u)$ and $g(u)$ may be relatively prime. However, *the relation $(f, g) = 1$ which holds at some points holds also in a sufficiently small neighborhood of b*. This is the content of the following theorem.

THEOREM B.2.1. *If $f(u)$ and $g(u)$ are two functions that are analytic at a and if $(f, g)_a = 1$, then there is a neighborhood $U(a)$ of a such that the condition $(f, g)_b = 1$ holds for each $b \in U(a)$.*

Proof. Apply the Weierstrass preparation theorem to the functions f and g at the point a (if necessary, we begin by performing an appropriate linear transformation of variables). We get

$$f = P_a \varphi_a, \qquad g = Q_a \psi_a. \tag{B.2.1}$$

Here

$$P_a = (u_1 - a_1)^k + A_1(u_1 - a_1)^{k-1} + \cdots + A_k,$$
$$Q_a = (u_1 - a_1)^l + B_1(u_1 - a_1)^{l-1} + \cdots + B_l,$$

and moreover in some neighborhood $U(a)$ of the point a the functions A_i, B_j are analytic functions of $p-1$ variables $u_2, \ldots, u_p$ and φ_a, ψ_a are analytic functions of p variables $u_1, \ldots, u_p$ such that $\varphi_a(u) \neq 0$ and $\psi_a(u) \neq 0$, i.e., φ_a and ψ_a are units at each point $u \in U(a)$.

Because $(f, g)_a = 1$ we have $(P_a, Q_a) = 1$ and therefore (see B.1.3) there exist functions P'_a, Q'_a, and d_a analytic in the same neighborhood $U(a)$ and such that

$$P_a P'_a + Q_a Q'_a = d_a('u). \tag{B.2.2}$$

Let b be any point in $U(a)$. Assume, contrary to the conclusion of the theorem, that f and g are relatively prime at this point. Then there exist functions $\delta_b(u)$, $\alpha_b(u)$, and $\beta_b(u)$ analytic at the point b and such that

$$f(u) = \delta_b(u)\alpha_b(u), \qquad g(u) = \delta_b(u)\beta_b(u), \tag{B.2.3}$$

where moreover $\delta_b(b) \neq 0$.

From (B.2.1) and (B.2.3) we derive analytic representations for the pseudopolynomials $P_a(u)$ and $Q_a(u)$ such that the same representations can also be used in a neighborhood of a point b:

$$P_a(u) = \varphi_a^{-1}(u) f(u) = \varphi_a^{-1}\alpha_i\delta_b, \qquad Q_a(u) = \psi_a^{-1}\beta_a\delta_b. \tag{B.2.4}$$

Hence the pseudopolynomials $P_a(u)$ and $Q_a(u)$ (like $f(u)$ and $g(u)$) must have a common divisor δ_b at b. But it follows from (B.2.2) that each common divisor of $P_a(u)$ and $Q_a(u)$ at b must be a divisor of the function $d_a('u)$ at the same point, i.e., the common divisor mentioned above does not depend on u_1. However, the existence of such a common divisor for the pseudopolynomials which become zero at b contradicts the fact that they remain primitive at b. Hence theorem (B.2.1) is proved.

The following important corollary follows from this theorem: *the fraction $f(u)/g(u)$, where $f(u)$ and $g(u)$ are analytic at a, remains irreducible in some neighborhood of the point a if it is irreducible at the point a itself.*

We will prove another theorem of a similar character, which we used in section IV.3.

Theorem B.2.2. *If $\varphi(u) \not\equiv 0$ is a Jacobian function which corresponds to some Riemann matrix Ω, then there exist points $c \in \mathbf{C}^p$ such that the* *following condition holds for the functions* $\varphi(u)$ *and* $\varphi(u+c)$ *at all points* $u \in \mathbf{C}^p$:

$$(\varphi(u), \varphi(u+c)) = 1.$$

PROOF. Let us verify that it is enough to prove the theorem for points which belong to the parallelotope Π_0 of periods determined by the columns of Ω. Indeed, let $(\varphi(u), \varphi(u+c)) = 1$ for each $u \in \Pi_0$. If $u_0^* \notin \Pi_0$, then there are a period ω and a point $u_0 \in \Pi_0$ such that $u_0^* = u_0 + \omega$. By the definition of a Jacobian function,

$$\varphi(u+\omega) = \exp[l(u)+b]\varphi(u),$$
$$\varphi(u+c+\omega) = \exp[l(u+c)+b]\varphi(u+c),$$

where $l(u)$ is a linear function and b is a constant p-vector. It follows that the values $\varphi(u)$ and $\varphi(u+c)$ reproduce, in a neighborhood of u_0^*, the values of the same functions up to unit (exponential) factors in a neighborhood of u_0. Hence there is no common divisor at u_0^* if there is none at u_0.

Let $a = (a_1, \dots, a_p) \in \Pi_0$. We introduce two types of local coordinates $v = u - a$ and $z = \mathscr{L}_a(v)$ in a neighborhood of a. Here $\mathscr{L}_a$ is a linear transformation needed for an application of the Weierstrass preparation theorem. As a result of this transformation, the terms of lowest order occuring in the expansion of the function

$$\varphi^*(z) = \varphi[a + \mathscr{L}_a^{-1}(z)] = \varphi(a+\nu) = \varphi(u)$$

in a power series in a neighborhood of the origin must contain a term of the form z_1^k, where $k = k(a)$ is the degree of the homogeneous polynomial formed by these terms. Then, according to the Weierstrass theorem, we have the representation

$$\varphi(u) = \varphi(a+v) = P_a(v)f_a(v) = P_a^*(z)f_a^*(z), \qquad \text{(B.2.5)}$$

where

$$P_a^*(z) = z_1^k + A_1('z)z_1^{k-1} + \cdots + A_k('z) = P_a(v), \qquad \text{(B.2.6)}$$

and moreover $A_j('z)$ $(j = 1, \dots, k)$ are functions analytic at the point $z = 0$ and independent of z_1, and $f_a^*(z)$ is a unit at $z = 0$ (i.e., $f_a^*(0) \neq 0$).

Let $P_a(v)$ and $f_a(v)$ remain analytic at all points of a neighborhood $W(a): \|v\| < r_a$ and moreover let $f_a(v)$ be nonzero at these points. Denote by $U(a)$ a neighborhood of the same point with half the radius of the preceding one: $\|v\| < (1/2)r_a$. Thus with each point $a \in \Pi_0$ we associate its neighborhoods $W(a)$, $U(a)$ and the linear transformation $z = \mathscr{L}_a(v) = \mathscr{L}_a(u-a)$ as well as a representation (B.2.5), where all the functions occurring in it are analytic at $W(a)$.

We select a set of U_a forming a finite cover of the parallelotope Π_0 and let the cover be composed of the neighborhoods $U(a^j)$, $j = 1, \dots, n$, which for brevity we denote by U_j. The corresponding pseudopolynomials $P_{a^{(j)}}(v) = P_{a^{(j)}}^*(z)$ will be distinguished just by their indices j without mentioning the point $a^{(j)}$: $P_j(v) = P_j^*(z)$, $j = 1, \ldots, n$. They are all analytic in the same hyperball

$$W\colon \|v\| < r = \min\{r_{a^{(1)}}, \ldots, r_{a^{(m)}}\}.$$

Since no P_j is identically zero, one can find points c in any neighborhood of the origin such that all these pseudopolynomials are nonzero at these points. To do this it suffices to first take a point $c^{(1)}$ such that $P_1(c^{(1)}) \neq 0$. Then, we take another point $c^{(2)}$ in a neighborhood V_1 of the first, such that V_1 is small enough to be in W ($V_1 \subset W$) and such that $P_1(v) \neq 0$ for all $v \in V_1$. The second point is chosen so that $P_2(c^{(2)}) \neq 0$ and then a neighborhood $V_2 \subset V_1$ of it is chosen so that $P_2(v) \neq 0$ for all $v \in V_2$. Continuing this process by choosing a third point, and so on, we get a neighborhood V_n of some point $c^{(n)} = c \in \Pi_0$ at which none of the functions $P_1(v), \ldots, P_n(v)$ is zero. We put one more restriction on c: $\|c\| < r/2$. Such a point satisfies the condition of the theorem, as we now prove.

For any point $b \in \Pi_0$, there is a neighborhood U_ν $(1 \leq \nu \leq n)$ among those considered above such that $b \in U_\nu$. We fix this neighborhood. Then the point $u = b + c$ will lie in a larger neighborhood W_ν with the same center, since $v = u - a^{(\nu)}$ for this point and

$$\|v\| = \|(b+c) - a^{(\nu)}\| \leq \|b - a^{(\nu)}\| + \|c\| < \frac{1}{2}r + \frac{1}{2}r = r.$$

Hence one can use the same representation of the form (B.2.5) for both of the functions $\varphi(u)$ and $\varphi(u+c)$. This gives us the right to drop the indices which show the point at which a representation is formed and to which the linear transformation of local coordinates $z = \mathscr{L}(v)$ corresponds.

For $u \in U_\nu$ we have

$$\varphi(u) = \varphi(a+v) = P(v)f(v) = P^*(z)f^*(z), \tag{B.2.5$'$}$$

$$\begin{aligned}\varphi(u+c) &= \varphi(a+v+c) = P(v+c)f(v+c)\\ &= P^*(z+\gamma)f^*(z+\gamma),\end{aligned} \tag{B.2.5$''$}$$

where $\gamma = \mathscr{L}(c)$. Moreover,

$$P^*(z) = z_1^k + A_1 z_1^{k-1} + \cdots + A_k = P(v) \tag{B.2.6$'$}$$

is a primitive pseudopolynomial of degree k. It is obvious that

$$P^*(z+\gamma) = z_1^k + B_1 z_1^{k-1} + \cdots + B_k = P(v+c) \tag{B.2.6$''$}$$

is also a primitive pseudopolynomial (it is obtained by substituting $z_1 + \gamma_1, \ldots, z_p + \gamma_p$ for $z_1, \ldots, z_p$ in $(B.2.6')$ and then grouping the terms with the same power of z_1). We have $z = \mathscr{L}(v) = 0$ when $v = 0$, from which it follows that

$$P^*(0) = P(0) \quad \text{and} \quad P^*(\gamma) = P(c) \neq 0;$$

and therefore the pseudopolynomials $P^*(z)$ and $P^*(z+\gamma)$ are relatively prime at the point $z=0$. It follows from this that there are a function $d('z)$ (which does not depend on z_1) and pseudopolynomials $R(z)$ and $S(z)$ such that the following identities of the form (B.1.3) hold:

$$R(z)P^*(z)+S(z)P^*(z+\gamma)=d('z). \tag{B.2.7}$$

Since $R(z)$, $S(z)$, and $d('z)$ are obtained from $P^*(z)$ and $P^*(z+\gamma)$ by entire rational operations (see the end of the section B.1), they are (like $P^*(z)=P(v)$ and $P^*(z+\gamma)=P(v+c)$) analytic at all points of the neighborhood of the origin that is the image of the hyperball W under $z=\mathscr{L}(v)$.

Assume that, contrary to the statement of the theorem, $\varphi(u)$ and $\varphi(u+c)$ have a common (proper) divisor at a point $b\in U_\nu$. Since these functions differ from the corresponding pseudopolynomials $P^*(z)=P(v)$ and $P^*(z+\gamma)$ $=P(v+c)$ only by factors which remain units over the entire neighborhood U_ν, it follows from the last assumption that the pseudopolynomials $P^*(z)$ and $P^*(z+\gamma)$ must have a common (proper) divisor $\delta(z)$ at the point β when β corresponds to the point b under the transformation $z=\mathscr{L}(v)=\mathscr{L}(u-a)$, i.e., when $\beta=\mathscr{L}(b-a)$. But it follows from (B.2.7) that $\delta(z)$ is also a divisor of $d('z)$, i.e., that $\delta(z)$ *does not depend on* z_1. We have arrived at a contradiction since the primitive polynomials $P^*(z)$ and $P^*(z+\gamma)$ cannot have a common divisor $\delta(z)$ which does not depend on z_1 and at the same time is not a constant.

B.3. Poincaré-Cousin theorem. It is well known that a meromorphic function of one complex variable can be defined in two ways. It is, on the one hand, a function which does not have any singularities in the finite plane except poles; and on the other hand, a function which can be expressed as a quotient of two entire functions. The equivalence of these two definitions was established by Weierstrass in 1876. The corresponding assertion for two complex variables was first stated and proved by Poincaré in 1883.

THEOREM OF POINCARÉ. *If a function $f(u_1, u_2)$ of two complex variables can be represented as a quotient of two functions $\mathscr{M}_a(u_1, u_2)$ and $\mathscr{N}_a(u_1, u_2)$ in a neighborhood of each point (a_1, a_2), where a_1 and a_2 are any complex numbers and the functions $\mathscr{M}_a(u_1, u_2)$, $\mathscr{N}_a(u_1, u_2)$ are analytic at the point, there exist two entire functions $G_1(u_1, u_2)$ and $G_2(u_1, u_2)$ such that $f(u_1, u_2)$ is their quotient for any u_1 and u_2.*

Of course this theorem is quite analogous to the theorem of Weierstrass. Indeed, if a function of one variable $f(u)$ can be represented in the form of a quotient $\mathscr{M}_a(u)/\mathscr{N}_a(u)$ in a neighborhood of a and if $\mathscr{M}_a(u)$ and $\mathscr{N}_a(u)$ are analytic at a ($\mathscr{N}_a(u)\not\equiv 0$), then a is either a regular point or a pole of $f(u)$. The converse is also true. Hence in the case of one variable the question is resolved in a simple manner. It is enough to construct an entire function $G_2(u)$ which has zeros at the poles of $f(u)$ and moreover whose zeros are of the same multiplicity as the corresponding poles of $f(u)$. Then the function $f(u)G_2(u) = G_1(u)$ is also entire, from which we get $f(u) = G_1(u)/G_2(u)$. The construction of $G_2(u)$ is made by means of an infinite product. However, this method is not applicable to a larger number of variables since the denominator of the fraction $\mathscr{M}_a(u_1, u_2)/\mathscr{N}_a(u_1, u_2)$ in the statement of the theorem of Poincaré is zero not at isolated points but on a manifold $N_a(u_1, u_2) = 0$ containing a.

The proof of the theorem of Poincaré was quite complicated and did not allow a natural generalization when the number of variables is more than 2. A simple and at the same time general (for any p) proof was given by the French mathematician P. Cousin in his doctoral thesis (1894). This proof serves as the basis for our exposition.

Everything is reduced to the following statement.

THEOREM OF COUSIN. *Associate with each point of the space $\mathbf{C}^p$ a function $g_a(u)$ which is analytic in some neighborhood $U(a)$ of this point. Also let the following coherence conditions hold for each point $b \in U(a^{(1)}) \cap U(a^{(2)})$, where $a^{(1)}$ and $a^{(2)}$ are different points of $\mathbf{C}^p$: $g_{a^{(1)}}(u)$ and $g_{a^{(2)}}(u)$ differ at this point only by a unit factor (i.e., $g_{a^{(2)}} | g_{a^{(1)}}$ and $g_{a^{(1)}} | g_{a^{(2)}}$ at the point b). Then there exists an entire function $G(u)$ such that*

$$G(u) = g_a(u)e_a(u), \tag{B.3.1}$$

where $e_a(u)$ is a unit at the point a (a is any point of $\mathbf{C}^p$).

The following theorem is a corollary.

THE POINCARÉ-COUSIN THEOREM. *If a function $f(u)$ can be represented in the form*

$$f(u) = \frac{\mathscr{M}_a(u)}{\mathscr{N}_a(u)},$$

in a neighborhood of each point $a \in \mathbf{C}^p$, where $\mathscr{M}_a(u)$ and $\mathscr{N}_a(u)$ are functions which are analytic at the point a, then there exist entire functions $G_1(u)$ and $G_2(u)$ such that

$$f(u) = \frac{G_1(u)}{G_2(u)}$$

at all points of $\mathbf{C}^p$. Moreover, we can require that the condition $(G_1(u), G_2(u)) = 1$ holds for each u.

Exactly this form of the theorem was used in Chapters II–IV of this book.

We will show that it indeed follows from the Cousin theorem. We start by finding for each point $a \in \mathbf{C}^p$ the g.c.d of the functions $\mathscr{M}_n(a)$ and $\mathscr{N}_a(u)$ at that point and then we divide the fraction $\mathscr{M}_a(u)/\mathscr{N}_a(u)$ by this g.c.d. We obtain a local representation of $f(u)$ in the form of an irreducible fraction $f(u) = \mu_a(u)/\nu_a(u)$. Since it is irreducible at a point a, it is also irreducible at each point of some neighborhood $V(a)$ of the point a (Theorem B.2.1).

Now we apply Cousin's theorem to the case $g_a(u) = \nu_a(u)$. We check that the coherence condition holds here. Indeed, if $b \in V_{a^{(1)}} \cap V_{a^{(2)}}$, then in a sufficiently small neighborhood of this point we have

$$f(u) = \frac{\mu_{a^{(1)}}(u)}{\nu_{a^{(1)}}(u)} = \frac{\mu_{a^{(2)}}(u)}{\nu_{a^{(2)}}(u)},$$

from which we get

$$\mu_{a^{(1)}}(u)\nu_{a^{(2)}}(u) = \mu_{a^{(2)}}(u)\nu_{a^{(1)}}(u), \tag{B.3.2}$$

and moreover

$$(\mu_{a^{(1)}}(u), \nu_{a^{(1)}}(u))_b = 1 \quad \text{and} \quad (\mu_{a^{(2)}}(u), \nu_{a^{(2)}}(u))_b = 1. \tag{B.3.3}$$

Because of (B.3.2), by Euclid's theorem (section B.1) we have

$$\nu_{a^{(1)}}(u)|\nu_{a^{(2)}}(u) \quad \text{and} \quad \nu_{a^{(2)}}(u)|\nu_{a^{(1)}}(u),$$

at the point b and this is then the coherence condition. Hence there exists an entire function $G_2(u)$ which differs from $\nu_a(u)$ only by a unit factor $e_a(u) : G_2(u) = \nu_a(u)e_a(u)$ in a neighborhood of each point a. It follows from this that the function $f(u)G_2(u) = e_a(u)\mu_a(u)$ is an analytic function in a neighborhood of each point a and therefore it is an entire function $G_1(u)$.

It only remains to note that it follows from the equalities

$$G_1(u) = \mu_a(u)e_a(u), \qquad G_2(u) = \nu_a(u)e_a(u)$$

and from the condition $(\mu_a(u), \nu_a(u)) = 1$ that $(G_1, G_2) = 1$.

Thus everyting is reduced to the proof of Cousin's theorem. We precede it with two simple lemmas.

LEMMA 1. *Let C_n be the p-disk $|u_j| < n$, $j = 1, \dots, p$. In order to establish the truth of the Cousin theorem it suffices to prove that for each natural number n there exists a function $G_n(u)$ that is analytic on C_{n+1} and satisfies the conditions* (B.3.1) *at every point $a \in C_{n+1}$.*

PROOF. Assume that the conditions of the lemma hold. Then the function $G_{n+1}(u)/G_n(u)$ can be represented at each point $a \in C_{n+1}$ (by (B.3.1)) in the form of a ratio of two units and therefore is an analytic function, different from zero.

We define a single-valued analytic function $H_n(u)$ on C_{n+1} by

$$H_n(u) = \mathrm{Ln}\,\frac{G_{n+1}(u)}{G_n(u)}, \tag{B.3.4}$$

which is subject, for example, to the following condition at the point $u = 0$:

$$H_n(0) = \ln\frac{G_{n+1}(0)}{G_n(0)}.$$

This function has, in a neighborhood of the point $u = 0$, a power series expansion that converges uniformly on $\overline{C}_n$. Hence there exists a polynomial $P_n(u)$ (a partial sum of the series) such that

$$|H_n(u) - P_n(u)| < \frac{1}{n^2}, \quad u \in \overline{C}_n. \tag{B.3.5}$$

We show that the function

$$G(u) = G_1(u) \exp \sum_1^\infty [H_n(u) - P_n(u)] \tag{B.3.6}$$

is one of the entire functions mentioned in Cousin's theorem.

First, the series $\sum_N^\infty [H_n(u) - P_n(u)]$ converges uniformly in the p-disk C_N (by the conditions (B.3.5)) and therefore represents an analytic function on C_N (the theorem is proved by using Cauchy's integral formula for a p-disk in the same way as for one variable). Therefore the function (B.3.6) is at least analytic on C_1. But it can be continued analytically to C_N for any natural number N since (because of (B.3.4)) we have

$$\begin{aligned} G(u) &= \exp\left\{\left[L_n G_1(u) + \operatorname{Ln}\frac{G_2(u)}{G_1(u)} + \cdots + \operatorname{Ln}\frac{G_N(u)}{G_{N-1}(u)}\right] \right. \\ &\qquad\qquad \left. + \sum_N^\infty [H_n(u) - P_n(u)] - \sum_1^{N-1} P_n\right\} \\ &= G_N(u) \exp\left\{\sum_N^\infty [H_n(u) - P_n(u)] - \sum_1^{N-1} P_n(u)\right\}. \end{aligned} \tag{B.3.7}$$

Hence $G(u)$ is an entire function. Finally, the last formula shows that $G(u)$ differs from the function $G_N(u)$ only by a unit factor at each point $u \in C_N$ and therefore $G(u)$ also satisfies the conditions (B.3.1).

LEMMA 2. *Let l be a directed line segment with initial point a and end point b, let g be a region containing l, and let $h(z)$ be a function that is single-valued and analytic in this region. Then the integral*

$$I_h(z) = \frac{1}{2\pi i}\int_l \frac{h(\zeta)\,d\zeta}{\zeta - z} \tag{B.3.8}$$

of Cauchy type defines an analytic function on the z-plane, with the two branch points a and b. Moreover, $I_h(z)$ has the representation

$$I_h(z) = -\frac{1}{2\pi i}h(z)\operatorname{Ln}(a - z) + \varphi_1(z) \tag{B.3.9}$$

in a neighborhood of a and the representation

$$I_h(z) = \frac{1}{2\pi i}h(z)\operatorname{Ln}(b - z) + \varphi_2(z), \tag{B.3.10}$$

in a neighborhood of b, *where* $\varphi_1(z)$ *and* $\varphi_2(z)$ *are analytic at the corresponding points.*

PROOF. It is obvious that (B.3.8) is a single-valued analytic branch of some function on the plane with a cut along l. In a neighborhood of l, i.e., for $z \in g$, $I_h(z)$ can be represented in the form

$$I_h(z) = \frac{1}{2\pi i} h(z) \operatorname{Ln} \frac{b-z}{a-z} + \frac{1}{2\pi i} \int_l \frac{h(\zeta) - h(z)}{\zeta - z}\, d\zeta, \qquad \text{(B.3.11)}$$

from which the assertion of the lemma follows, since the function defined by the integral on the right-hand side is regular at each point of l. It also follows from this that the single-valued branches which are obtained by analytic continuation of the function across the cut differ from each other at the points of g by integral multiples of $h(z)$. In fact, when z moves from the left edge to the right edge (i.e., when z goes around the point a in the positive direction) the function increases by $h(z)$.

Now we can establish Cousin's theorem. We replace the p-disks C_n of Lemma 1 by p-disks of the form $D_n : |x_j| < n$, $|y_j| < n$, $j = 1, \dots, p$, where x_j and y_j are the real and imaginary parts of u_j. Since $C_n \subset D_n$ for any n, the conclusion of the lemma still holds.

PROOF OF COUSIN'S THEOREM. We associate with each point, not only the neighborhood $U(a)$: $\{|u_j - a_j| < r_a, j = 1, \dots, p\}$ which appeared in the hypotheses of the theorem, but also a concentric neighborhood $V_a(u)$: $\{|u_j - a_j| < \frac{1}{2} r_a\}$ of radius $r_a/2$. With n fixed, we cover $\overline{D}_{n+1}$ by a finite number of neighborhoods $V_1, \dots, V_n$ with centers at the points $a^{(1)}, \dots, a^{(N)}$. We denote the corresponding analytic functions $g_a(u)$ by $g_1(u), \dots, g_N(u)$. If $r = r_n = \min\{r_{a^{(1)}}, \dots, r_{a^{(N)}}\}$, we dissect the large squares T_j : $\{|x_j| < n + 1, |y_j| < n + 1\}$, $j = 1, \dots, p$, into equal squares, with sides parallel to the coordinate axes, so small that the diagonal of each one is less than r. Then the region D_{n+1} is broken into a finite number of elementary parallelotopes each of which is the topological product of p small squares taken from the p partitions T_j $(j = 1, \dots, p)$. If at least one point of a parallelotope τ belongs to a neighborhood V_ν, then the entire parallelotope will be inside U_ν : $|u_j - a_j^{(\nu)}| < r_{a^{(\nu)}}$, $j = 1, \dots, p$, and therefore the corresponding function $g_\nu(u)$ will be defined and analytic inside τ and on its boundary. Of course, τ can be contained in some other neighborhood U_μ related to the function $g_\mu(u)$. However, because of the coherence condition we have $g_\nu(u) | g_\mu(u)$ and $g_\mu(u) | g_\nu(u)$ at all points of $\bar{\tau}$ (i.e., inside τ and on its boundary).

We also notice that if σ and τ are different parallelotopes which have at least one common boundary point b, and if the point b belongs to the neighborhood V_ν, then it follows that σ and τ are contained entirely in the same U_ν and hence the same function $g_\nu(u)$ is defined at σ and τ.

We now need to construct a function $G_n(u)$ that is analytic in the p-disk D_{n+1} and satisfies the conditions (B.3.1) at all points of D_{n+1}. This construction starts from a finite number of functions $g_\nu(u)$, $\nu = 1, \dots, N$; a significant role in this construction is played by Lemma 2. The easiest way to make this construction is by induction on the number of variables.

For each elementary parallelotope τ we select one of the functions $g_\nu(u)$ which are defined on it, and denote this function by $g_\tau(u)$. Then we will have a locally analytic function $\Phi(u)$ on the region D_{n+1}. This function is single-valued and analytic inside each τ, where $\Phi(u) = g_\tau(u)$. As for boundary points common to two or more parallelotopes, the function is multiple-valued at these points. However, if, for example, a point b is common to τ and σ, then there is a neighborhood of this point in which the coherence condition $g_\tau(u)|g_\sigma(u)$ and $g_\sigma(u)|g_\tau(u)$ holds. For brevity, we denote such a relation by $g_\tau(u) \sim g_\sigma(u)$ and say that $g_\tau(u)$ and $g_\sigma(u)$ are *equivalent* (at the point b).

We separate the variables $u_1, \dots, u_p$ into two groups A and B: $u_{i_1}, \dots, u_{i_k}$ and $u_{j_1}, \dots, u_{j_l}$ $(k + l = p)$ in such a way that each variable belongs to one and only one group. The inductive assumption will be that the *locally analytic* function $\Phi(u)$ is single-valued and analytic at each point $u \in D_{n+1}$ with respect to all variables of the group A when arbitrary fixed values are assigned to the variables of the group B (the group A may initially be empty). We will show that we can move from $\Phi(u)$ to another function Φ_1 which is equivalent to Φ at all points D_{n+1}, in such a way that we add another variable, for instance $u_{j_1} = v$, of group B to group A.

With this goal in mind, we arbitrarily fix the values of all variables of the group A, as well as of group B with the exception of v. Then we obtain a function of one complex variable v which we still denote by $\Phi(v)$ and which is locally analytic on the square $T_{j_1} = T$. This square was decomposed into small squares t. In each of these squares, $\Phi(v)$ coincides with the corresponding single-valued analytic function $g_t(v)$. Let t_α and t_β be two squares which have a common side $l_{\alpha\beta}$. We will orient $l_{\alpha\beta}$ so that the square t_α lies to the left of it and the square t_β lies to the right of it. Then at all of its points the functions $g_\alpha(v) = g_{t_\alpha}(v)$ and $g_\beta(v) = g_{t_\beta}(v)$ are analytic and equivalent. Hence $g_\beta(v)/g_\alpha(v)$ is also an analytic zero-free function of v. Therefore in some region $g_{\alpha\beta}$ which contains $l_{\alpha\beta}$ the function

$$h_{\alpha\beta} = \operatorname{Ln} \frac{g_\beta(v)}{g_\alpha(v)} \tag{B.3.12}$$

is determined up to an integral multiple of $2\pi i$, and is single-valued and analytic.

By Lemma 2 the function

$$I_{\alpha\beta}(v) = \frac{1}{2\pi i} \int_{l_{\alpha\beta}} \frac{h_{\alpha\beta}(\zeta)\, d\zeta}{\zeta - v} \tag{B.3.13}$$

is then single-valued and analytic on T with a cut along $l_{\alpha\beta}$. In general, it has singularities α and β. We form the sum

$$\Lambda(v) = \sum I_{\alpha\beta}(v), \tag{B.3.14}$$

by extending it to all pairs of squares t_α and t_β which have a common side. The function $\Lambda(v)$ is analytic inside each square t_α. When we extend it analytically from t_α to t_β, which has a common side $l_{\alpha\beta}$ with t_α, the function gains a summand $h_{\alpha\beta}$, by the lemma. We now select the multiples of $2\pi i$ which are included in the definition (B.3.12) in such a way that at each of the common vertices b of the four squares t_α, t_β, t_γ, and t_δ

$$\begin{aligned} &h_{\alpha\beta}(b) + h_{\beta\gamma}(b) + h_{\gamma\delta}(b) + h_{\delta\alpha}(b) \\ &= \ln \frac{g_\beta(b) g_\gamma(b) g_\delta(b) g_\alpha(b)}{g_\alpha(b) g_\beta(b) g_\gamma(b) g_\delta(b)} + 2\pi i (n_{\alpha\beta} + n_{\beta\gamma} + n_{\gamma\delta} + n_{\delta\alpha}) \\ &= 0. \end{aligned} \tag{B.3.15}$$

It is obvious that all this is possible if, for example, we traverse the squares which belong to T sequentially along the columns from the bottom up and move from a completed column to the next, from left to right. The choice of the multiples $2\pi i$ for the squares of the first column and for the bottom square of the second column can be arbitrary. However, the choice for further steps is forced uniquely by (B.3.15).

Under (B.1.15) the following representation (Lemma 2) exists in a neighborhood of each common vertex b of the four squares :

$$I_{\alpha\beta}(v) + I_{\beta\gamma}(v) + I_{\gamma\delta}(v) + I_{\delta\alpha}(v) = \varphi(v),$$

where $\varphi(v)$ is analytic at b.

This means that the function $\Lambda(v)$ is analytic at such a vertex. We will verify that the function

$$\Phi_1(v) = \Phi(v) \exp[\Lambda(v)] \tag{B.3.16}$$

has the inductive properties needed to complete the proof. Indeed, at the points of each closed square $\bar{t}_\alpha \subset T$ we have $\Phi_1(v) \sim \Phi(v) = g_\alpha(v)$. Furthermore, it is single-valued and analytic at each boundary point of the square t_α since the exponent $\Lambda(v)$, i.e., the function (B.3.14), gains $h_{\alpha\beta}(v) = \ln(g_\beta(v)/g_\alpha(v))$ from analytic continuation across $l_{\alpha\beta}$ to the adjacent square t_β. Consequently $\Phi_1(v)$ gains the factor $g_\beta(v)/g_\alpha(v)$ and thus, in the square t_β,

$$\Phi_1(v) = g_\beta(v) \exp[\Lambda(v)] = \Phi(v) \exp[\Lambda(v)], \tag{B.3.16$'$}$$

which coincides with the definition of $\Phi_1(v)$ on this square.

Since the inductive assumption is trivially true when the group of variables A is empty, the proof of Cousin's theorem is complete. We proved by induction that there exists a function $G_n(u)$, analytic in D_{n+1} and satisfying at each point of this region the condition $G_n(u) \sim g_\alpha(u)$. From this, Cousin's theorem follows by Lemma 1. Hence the Poincaré-Cousin theorem is established.

§C. A summary of the most important formulas

Jacobi function.

$$\vartheta_1(x) = \sum_0^\infty (-1)^n 2q^{(2n+1)^2/4} \sin(2n+1)\pi x,$$

$$\vartheta_2(x) = \sum_0^\infty 2q^{(2n+1)^2/4} \cos(2n+1)\pi x,$$

$$\vartheta_3(x) = 1 + \sum_1^\infty 2q^{n^2} \cos 2\pi x,$$

$$\vartheta_4(x) = 1 + \sum_1^\infty (-1)^n 2q^{n^2} \cos 2n\pi x.$$

$$\operatorname{sn} u = \frac{1}{\sqrt{k}} \frac{\vartheta_1(u/(2K))}{\vartheta_4(u/(2K))}, \qquad \operatorname{cn} u = \sqrt{\frac{k'}{k}} \frac{\vartheta_2(u/(2K))}{\vartheta_4(u/(2K))},$$

$$\operatorname{dn} u = \sqrt{k'} \frac{\vartheta_3(u/(2K))}{\vartheta_4(u/(2K))}.$$

$$k^2 + k'^2 = 1, \qquad K = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}, \qquad K' = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k'^2 \sin^2\varphi}}.$$

The Weierstrass preparation theorem. If $f(u) = \Sigma_{n=\nu}^\infty p_n(u)$, where $p_n(u)$ are homogeneous polynomials of degree n and $p_\nu(u) = \pi_0 u^\nu + \cdots + \pi_\nu$, $\pi_0 \neq 0$, $\nu \geq 1$, then

$$f(u) = (\Pi_0 u_1^\nu + \cdots + \Pi_\nu) f_1(u);$$

here $\Pi_0 = \text{const} \neq 0$, $\Pi_j = \Pi_j(u')$, the lower order of Π_j is not less than j $(j = 1, \ldots, \nu)$, and $f_1(0) \neq 0$.

Reduced representations. If $f(u) = \varphi(u)/\psi(u) = \varphi^*(u)/\psi^*(u)$ are two reduced representations of a meromorphic function, then there is an entire function $g(u)$ such that $\varphi^*(u) = \exp[g(u)]\varphi(u)$, $\psi^*(u) = \exp[g(u)]\psi(u)$.

A condition for the degeneracy of a function. A function $f(u)$ is degenerate if and only if a p-dimensional vector $\tilde{\gamma} \neq 0$, $\tilde{\gamma} = (\gamma_1, \ldots, \gamma_p)$, exists such that

$$\gamma_1 \frac{\partial f}{\partial u_1} + \cdots + \gamma_p \frac{\partial f}{\partial u_p} = 0$$

at each regular point of the function.

Period matrices. If $\Omega = (\omega^{(1)}, \ldots, \omega^{(r)})$ and $\Pi = (\pi^{(1)}, \ldots, \pi^{(\rho)})$ are two matrices of fundamental periods of the function $f(u)$, then $\rho = r$ and there exists an (r, r) matrix M with integer entries such that $|M| = \pm 1$ (is *unimodular*) and $\Pi = \Omega M$.

If the variables $u_1, \dots, u_p$ are subjected to a linear transformation $u^* = Au$, $|A| \neq 0$, then the period matrix Ω transforms into $\Omega^* = A\Omega$.

Fourier series expansion. If each column of the matrix $C = \operatorname{Diag}(c_1, \dots, c_p)$, $c_j \neq 0$, $j = 1, \dots, p$, is a period of an entire function $f(u)$, then

$$f(u) = \sum_{(\gamma)} d_\gamma \exp(2\pi i \tilde{\gamma} C^{-1} u),$$

where the summation is over all p-vectors γ with integral coordinates.

Construction of an entire function from its difference. If $f(u)$ is an entire function of one variable $\tau \neq 0$ and

$$f(u + \tau) - f(u) = h(u),$$

where $h(u) = \sum_0^\infty h_n u^n$ is an entire function, then

$$f(u) = \sum_0^\infty h_n \tau^n \psi_n(u/\tau) + P(u),$$

where

$$\psi_n(u) = \frac{n!}{2\pi i} \int_{|z|=(2n+1)\pi} \frac{e^{uz} - 1}{e^z - 1} \frac{dz}{z^{n+1}}, \qquad n = 0, 1, 2, \dots$$

and $P_n(u)$ is any entire function with period τ.

Riemann matrices. Elementary conditions. If $\Omega = (\omega^{(1)}, \dots, \omega^{(2p)})$ is a Riemann matrix and $\Omega' = \operatorname{Re} \Omega$ and $\Omega'' = \operatorname{Im} \Omega$, then

$$D = \begin{vmatrix} \Omega' \\ \Omega'' \end{vmatrix} \neq 0 \quad \text{and therefore} \quad |F| = \begin{vmatrix} \Omega \\ \bar{\Omega} \end{vmatrix} = (-2i)^p D \neq 0.$$

Each equivalence class of Riemann matrices contains a matrix of the form

$$\Omega = (2\pi i E_p, T).$$

Here $|\operatorname{Re} T| \neq 0$. Furthermore, if $T = (t^{(1)}, \dots, t^{(p)})$, there exists a number $\alpha = \alpha(T) > 0$ such that we have, for each p-vector γ with integral coefficients,

$$\max_{1 \leq k \leq p} |\exp[\tilde{\gamma} t^{(k)}] - 1| \geq \alpha.$$

The first system of difference equations. If $f(u) = \varphi(u)/\psi(u)$ is an Abelian function with matrix $\Omega = (2\pi i E_p, T)$, then a reduced representation

$$f(u) = \frac{\varphi(u) \exp G(u)}{\psi(u) \exp G(u)} = \frac{\varphi^*(u)}{\psi^*(u)}$$

exists such that

$$\varphi(u + 2\pi i e^{(h)}) = \exp[l_h(u)]\varphi(u),$$
$$\psi(u + 2\pi i e^{(h)}) = \exp[l_h(u)]\psi(u),$$
$$h = 1, \dots, p,$$

where $l_h(u)$ are linear functions.

An entire function $G(u)$ satisfies the (first) system of difference equations

$$G(u + 2\pi i e^{(h)}) - G(u) = H_h(u), \qquad h = 1, \dots, p,$$

where $H_h(u)$ are entire functions for which

$$H_h(u + 2\pi i e^{(k)}) - H_h(u) = H_k(u + 2\pi i e^{(h)}) - H_k(u), \qquad h, k = 1, \dots, p.$$

The solution $G(u)$ of the first system is determined up to an additive entire function $P(u)$ with periods $2\pi i e^{(k)}$, $k = 1, \dots, p$.

Jacobian functions. An entire function $\varphi(u)$ is called *Jacobian* with respect to a Riemann matrix Ω if for any period ω generated by Ω we have $\varphi(u + \omega) = \exp[\lambda(u) + c]\varphi(u)$, where $\lambda(u) = \lambda_\omega(u)$ is a linear function and $c = c_\omega$ is a constant.

Let $\Omega = (2\pi i E_p, T)$. In order to represent an Abelian function $f(u) = \varphi^*(u)/\psi^*(u)$ as a quotient of two Jacobian functions we have to choose a periodic function $P(u)$ in the expression $G(u) = G_0(u) + P(u)$, where $G_0(u)$ is some solution of the first system, and choose it in such a way that the *second* system

$$P(u + t^{(h)}) - P(u) = \widetilde{H}_h(u), \qquad h = 1, \dots, p,$$

is satisfied. Here $\widetilde{H}_h(u)$ are entire functions that satisfy the conditions

$$\widetilde{H}_h(u + t^{(k)}) - \widetilde{H}_h(u) = \widetilde{H}_k(u + t^{(h)}) - \widetilde{H}_k(u), \qquad h, k = 1, \dots, p.$$

The function $P(u)$ is defined in the form of a Fourier series, up to an additive constant.

Necessary and sufficient conditions for Ω. A $(p, 2p)$ matrix Ω is a Riemann matrix if and only if there exists a *principal* matrix for it, where the principal matrix P is a $(2p, 2p)$ matrix with integral entries such that

$$\widetilde{P} = -P, \qquad \Omega P \widetilde{\Omega} = 0, \qquad i\Omega P \widetilde{\overline{\Omega}} > 0 \ \text{(or } < 0).$$

Let $\varphi(u)$ be a Jacobian function and let

$$\varphi(u + \omega^{(k)}) = \varphi(u + \widetilde{\varepsilon}^{(k)}\Omega) = \exp[2\pi i \widetilde{\varepsilon}^{(k)}(\widetilde{\Lambda} u + \gamma)], \qquad k = 1, \dots, 2p.$$

where $\varepsilon^{(k)}$ is the kth column of a matrix E_{2p} and γ is a $2p$-vector (*a parametric vector* of the function $\varphi(u)$), and let

$$\Lambda = (\lambda_{jk}), \qquad j, k = 1, \dots, p,$$

be the *second period matrix*. Then

$$N = \tilde{\Omega}\Lambda - \tilde{\Lambda}\Omega$$

is the *characteristic matrix*. Its entries are integers which do not depend on the choice of the function $\varphi(u)$. Then nN^{-1} can be taken as the principal matrix for P, where n is an integer.

Theta function. The matrices Ω, N, and Λ can be expressed in the form

$$\Omega = (\pi i B^{-1}, A), \qquad N = n\begin{pmatrix} 0 & B^{-1} \\ B^{-1} & 0 \end{pmatrix}, \qquad \Lambda = \left(0, -\frac{n}{\pi i}E_p\right),$$

where A is a symmetric (p, p) matrix such that $\operatorname{Re} A = A' < 0$ and $B = \operatorname{Diag}(\beta_1, \dots, \beta_p)$, $nB^{-1} = \operatorname{Diag}(\delta_p, \dots, \delta_1)$. The natural numbers β_j, δ_u, l, and n satisfy the following conditions: $1 = \beta_1|\beta_2|\cdots|\beta_p$, $\delta_k = \beta_p/\beta_{p-k+1}$, $1 = \delta_1|\delta_2|\cdots|\delta_p = \beta_p$, $n = l\beta_p = l\delta_p$, $l = 1, 2, \dots$.

We also choose a parametric vector γ in such a way that

$$\tilde{\gamma} = (0, -\frac{n}{2\pi i}\operatorname{Sp} A),$$

where the p-vector $\operatorname{Sp} A$ is formed by the diagonal entries of A. Then any Jacobian function of the type $\{\Omega, \Lambda, \gamma\}$ can be represented as a linear combination of *theta functions of order* n:

$$\Theta_n[g](u) = \sum_m \exp\left\{\left(\tilde{m} + \frac{1}{n}\tilde{g}B\right) n \left[A\left(m + \frac{1}{n}Bg\right) + 2u\right]\right\}.$$

Here g is an arbitrary vector with integral coordinates which belongs to a parallelotope Π constructed from the vectors $(n/\beta_h)e^{(h)}$, $h = 1, \dots, p$. The summation is taken over all p-vectors m with integral coordinates.

The number of different functions $\Theta_n[g](u)$ is equal to $\sqrt{|N|}$, and they are linearly independent.

Algebraic dependencies. Any $p+2$ Jacobian functions of p variables of the same type are related by an algebraic relation (homogeneous relation).

Any $p+1$ Abelian functions of the same field are related by an algebraic relation.

In any field of Abelian functions there exist p functions which are algebraically, and even analytically, independent.

The book presents an exposition of the fundamentals of the theory of Abelian functions based on the methods of the classical theory of functions. Among the topics covered are theta functions, Jacobians, and Picard varieties. The author has aimed the book primarily at intermediate and advanced graduate students, but it would also be accessible to the beginning graduate student or advanced undergraduate who has a solid background in functions of one complex variable. This book will prove especially useful to those who are not familiar with the analytic roots of the subject. In addition, the detailed historical introduction cultivates a deep understanding of the subject. Thorough and self-contained, the book will provide readers with an excellent complement to the usual algebraic approach.

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