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Complements of Discriminants of Smooth Maps: Topology and Applications

Revised Edition

V. A. Vassiliev

American Mathematical Society

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Providence, Rhode Island

В. А. ВАСИЛЬЕВ

ДОПОЛНЕНИЯ К ДИСКРИМИНАНТАМ
ГЛАДКИХ ОТОБРАЖЕНИЙ:
ТОПОЛОГИЯ И ПРИЛОЖЕНИЯ

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ABSTRACT. The discriminant is the subset in a function space consisting of functions or maps having singularities of some distinguished types. Many key objects of mathematics can be described as the complements of suitably defined discriminants.

We show how to investigate the topology of such complements. As applications of our method we get a new series of knot invariants (which includes all known polynomial knot invariants), the Smale-Hirsch homotopy principle for the spaces of functions without complicated singularities, and the stable cohomology rings of spaces of nondiscriminant holomorphic functions. These results imply and improve many known results in this area.

The book also contains an introduction to configuration spaces and braid groups, as well as their applications to complexity theory.

The book is intended for graduate students and professionals interested in algebraic topology and its application to various areas of mathematics.

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Table of Contents

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APPENDIX 1

Classifying Spaces and Universal Bundles. Join

0. For more detailed discussion of these subjects, see the textbooks [Husemoller], [RF], [Steenrod].

1. Principal bundles. Let G be a topological group (i.e. a group with a topology on it such that the group operations are continuous).

A *principal G-bundle* is a locally trivial bundle $E \to B$ with an effective action of the group G on the space E (i.e., for every $x \in E$ and $g \in G$, $g(x) = x \Leftrightarrow g = 1$), such that the orbits of this action coincide with the fibers of the bundle (and hence the base B can be viewed as the quotient space with respect to this action). By definition, the fibers of a principal G-bundle are homeomorphic to G.

EXAMPLE 1. Given an m-fold covering $p: X \to Y$, there is a principal $S(m)$-bundle over Y associated to it with the fiber over $y \in Y$ consisting of all possible orderings of the points in $p^{-1}(y)$.

EXAMPLE 2. Given an m-dimensional vector bundle $\zeta: A \to Y$, there is a principal $GL(m)$-bundle associated to it whose fiber over $y \in Y$ consists of all possible m-frames in the linear space $\zeta^{-1}(y)$. If the bundle ζ is Euclidean, or oriented, or both, then it is possible to associate to it a principal $O(m)$-bundle ($SL(m)$- or $SO(m)$-bundle, respectively) whose fibers consist of orthonormal (positively oriented, orthonormal positively oriented) frames.

This correspondence can obviously be generalized to other G-bundles; it is the main source of principal bundles.

2. Universal G-bundle. In the class of all principal G-bundles there is a special *universal G-bundle* to which all others can be reduced. Here we list its properties.

THEOREM 1. *For any topological group G that has the type of a* CW-*complex there exists a principal G-bundle of* CW-*complexes $EG \to BG$ such that*

(1) *its space EG is homotopically trivial,*

(2) *any principal G-bundle π over any base B' is equivalent to the bundle induced from the bundle $EG \to BG$ by a map of the bases $B' \to BG$. This map is uniquely (up to homotopy) determined by the initial bundle π and is*

denoted by $\mathrm{cl}(\pi)$. *In particular, G-bundles over* B' *(considered up to equivalence) are in one-to-one correspondence with maps* $B' \to BG$ *(considered up to homotopy).*

The bundle $EG \to BG$ is called the *universal G-bundle*, and BG the *classifying space* of the group G.

Conditions (1) and (2) of the theorem are equivalent; each of them alone can serve as a definition of the universal bundle.

COROLLARY. *The classifying space* BG *is uniquely determined up to homotopy equivalence by the group* G.

The most important examples of classifying spaces will be considered in detail in §§3 and 4; their general construction is given in §6 below.

The correspondence $G \to BG$ is functorial with respect to G in the following sense.

THEOREM 2. *Any continuous homomorphism of topological groups* $G_1 \to G_2$ *defines a (unique up to homotopy) continuous map* $BG_1 \to BG_2$.

3. The spaces $K(G, 1)$. Let G be a discrete group. Then the space $K(G, 1)$ is a path-connected CW-complex satisfying the conditions

$$\pi_1(K(G, 1)) = G, \qquad \pi_i(K(G, 1)) = 0 \quad \text{for } i \geq 2.$$

THEOREM 3. *For any discrete group* G, *the space* $K(G, 1)$ *exists, is defined uniquely up to homotopy equivalence, and coincides with its classifying space* BG.

The last assertion follows from the fact that the universal covering space over the space $K(G, 1)$ is homotopically trivial, as one can see from the exact homotopy sequence of the covering.

EXAMPLES. (1) $K(\mathbb{Z}, 1) = S^1$.

(2) $K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty$.

(3) Any two-dimensional compact surface except for S^2 and $\mathbb{R}P^2$ is a space $K(G, 1)$ for some group; indeed, its universal covering is contractible.

(4) The spaces $K(G, 1)$ for the braid groups and symmetric groups are considered in detail in Chapter I.

4. Universal $O(m)$**- and** $SO(m)$**-bundles.** The *Grassmann manifold* G_N^m, $N \geq m$, is defined as the set of all m-dimensional subspaces in $\mathbb{R}^N$ with the natural topology. The *tautological bundle* $T_N(m) \to G_N^m$ is a vector bundle whose space consists of pairs (point of the Grassmannian, point of the corresponding m-plane in $\mathbb{R}^N$).

The sequence of standard imbeddings $\mathbb{R}^N \to \mathbb{R}^{N+1} \to \mathbb{R}^{N+2} \to \cdots$ defines imbeddings of Grassmannians $G_N^m \hookrightarrow G_{N+1}^m \hookrightarrow G_{N+2}^m \hookrightarrow \cdots$; the union of these spaces equipped with the direct limit topology is called the *stable m-Grassmannian* and is denoted by G^m. The tautological bundles over Grassmannians G_*^m are compatible with these imbeddings and define a limit bundle $T(m) \to G^m$.

THEOREM 4. *The space* G^m *is the classifying space* $BO(m)$ *of the orthogonal group* $O(m)$. *The universal principal bundle over* G^m *is the bundle whose fiber over* $\zeta \in G^m$ *consists of various orthonormal frames in the fiber of the tautological bundle* $T(m) \to G^m$.

The *oriented Grassmann manifold* $\widetilde{G}_N^m$ is defined as the set of all oriented m-subspaces in $\mathbb{R}^N$; forgetting the orientations makes $\widetilde{G}_N^m$ a double covering over G_N^m. Similarly to the above, we can define the limit space $\widetilde{G}^m = \bigcup_{N=m}^{\infty} \widetilde{G}_N^m$ and the tautological oriented bundle $\widetilde{T}(m) \to \widetilde{G}^m$.

THEOREM $\tilde{4}$. *The space* $\widetilde{G}^m$ *is the classifying space* $BSO(m)$; *the corresponding universal* $SO(m)$*-bundle over* $\widetilde{G}^m$ *is the bundle of positively oriented orthonormal m-frames associated to the tautological bundle* $\widetilde{T}(m) \to \widetilde{G}^m$.

THEOREM 5. *The ring* $H^*(G^m, \mathbb{Z}_2)$ *is a polynomial algebra freely generated by* m *classes* $w_1, \dots, w_m$, $\dim w_i = i$. *The map* $H^*(G^m, \mathbb{Z}_2) \to H^*(\widetilde{G}^m, \mathbb{Z}_2)$ *induced by the double covering* $\widetilde{G}^m \to G^m$ *is an epimorphism whose kernel is the ideal generated by* w_1.

The classes $w_i \in H^i(G^m, \mathbb{Z}_2)$ are called *universal Stiefel-Whitney classes.*

Any m-dimensional vector bundle ζ over a CW-complex X canonically defines m classes $w_i(\zeta) \in H^i(X, \mathbb{Z}_2)$, $i = 1, 2, \dots, m$. In fact, supply the bundle X with arbitrary smooth Euclidean structure and consider the principal $O(m)$-bundle over X associated to the bundle ζ (see Example 2 above). The classifying map of this bundle acts from X into $BO(m) = G^m$; the classes $w_i(\zeta)$ are the classes induced by this map from the universal classes $w_i \in H^i(G^m, \mathbb{Z}_2)$. They are called the *Stiefel-Whitney classes* of the bundle ζ. A vector bundle ζ is orientable if and only if $w_1(\zeta) = 0$ or, equivalently, the classifying map $X \to G^m$ lifts to a map into $\widetilde{G}^m$.

5. Join. The join is an associative and commutative binary operation on topological spaces. First we define the join of two finite simplicial polyhedra X, Y. Imbed X and Y in a linear space $\mathbb{R}^N$ of sufficiently large dimension so that for no pairs of points $x_1, x_2 \in X$, $y_1, y_2 \in Y$ do the segments $[x_1, y_1]$ and $[x_2, y_2]$ intersect in their interior points. Then the *join* $X * Y$ is defined as the union of all segments $[x, y]$ over all pairs $x \in X$, $y \in Y$ equipped with the topology induced from $\mathbb{R}^N$.

For arbitrary topological spaces X, Y the join $X * Y$ is defined as the quotient space of the product $X \times Y \times [0, 1]$ with respect to the following equivalence relation. For each point $x \in X$ we identify all points of the set $x \times Y \times \{0\} \sim Y$, and for each point $y \in Y$ we identify all points of the set

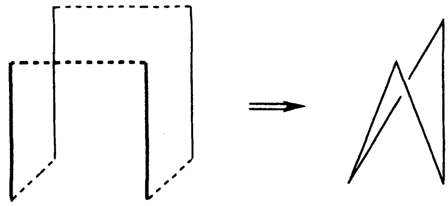

FIGURE 60

$X \times y \times \{1\} \sim X$. The resulting quotient space with the standard quotient topology is $X * Y$.

EXAMPLE 1. Let X, Y be two pairs of points. Then $X * Y \sim S^1$; see Figure 60.

EXAMPLE 2. Let X be a pair of points. Then for any space Y, $X * Y$ is homeomorphic to the (unreduced) suspension ΣY (i.e. the quotient space of the product $Y \times [0, 1]$ with respect to the equivalence that contracts its bases $Y \times 0$ and $Y \times 1$ to a pair of points).

EXAMPLE 3. The join of any two spheres $S^n * S^m$ is homotopy equivalent to S^{n+m+1}.

Sometimes it is more convenient to use the following (equivalent to the first one) definition of join (see [Schwarz]). Consider the cones CX (i.e. the quotient space $(X \times [0, 1])/(X \times \{0\})$) and $CY = (Y \times [0, 1])/(Y \times \{0\})$. Let $\dot{C}X$, $\dot{C}Y$ be the subsets obtained from these cones by removing their bases $X \times \{1\}$, $Y \times \{1\}$. Then the join $X * Y$ is defined as the difference $(CX \times CY)\backslash(\dot{C}X \times \dot{C}Y)$.

The homology of the join $X * Y$ can be easily expressed in terms of the homology of X and Y. Namely, denoting by $\tilde{H}_*$ the homology reduced modulo a point we have

$$\tilde{H}_i(X * Y, F) \cong \bigoplus_{p+q=i-1} \tilde{H}_p(X, F) \otimes \tilde{H}_q(Y, F), \tag{1}$$

for any field F.

6. The Milnor construction of the universal G-bundle. Suppose the topological group G is a CW-complex. Consider the sequence of spaces G, $G * G$, $G * G * G$, etc. Since any space X is canonically imbedded in $X * Y$, the union of spaces $G^{*k} = G * \cdots * G$ (k times) over all k is well defined. Equip it with the direct limit topology. The resulting topological space is contractible. (This follows from the fact that each space X^{*k} is $(k-2)$-connected; moreover, if the initial space X is i-connected, $i \geq 0$, then X^{*k} is $((i+2)k-2)$-connected.) The group G acts effectively on this homotopically trivial space $G^{*\infty}$ in the obvious way, and hence its fibration by the orbits of this action is the universal principal G-bundle.

APPENDIX 2

Hopf Algebras and H-Spaces

The notion of the Hopf algebra is the formalization of a natural algebraic structure on homology and cohomology of topological groups (and many other objects).

Let G be a topological group, i.e. a topological space that is also a group with both group operations being continuous; let A be a field, and suppose that all groups $H^i(G, A)$ are finitely generated. Then, in addition to the usual A-algebra structure on the ring $h^* = H^*(G, A)$, there is an A-coalgebra structure, that is, a homomorphism $\delta: h^* \to h^* \otimes h^*$ ("comultiplication"): it is induced by the multiplication operation $G \times G \to G$. This homomorphism is associative, i.e. two maps $h^* \to h^* \otimes h^* \otimes h^*$ defined by the compositions $(\delta \otimes 1) \circ \delta$ and $(1 \otimes \delta) \circ \delta$ coincide. Some obvious properties of the resulting structure on $H^*(G, A)$ are formalized in the following definition.

Let A be an associative and commutative ring with identity.

DEFINITION. A *Hopf algebra* over A is a graded associative A-algebra h^* with identity $i: A \hookrightarrow h^*$, $i(A) \subset h^0$, which is also an associative graded coalgebra (with respect to the same grading) with coidentity $\varepsilon: h^* \to A$, $h^{>0} \subset \operatorname{Ker} \varepsilon$ such that:

(1) i is a homomorphism of graded algebras;

(2) ε is a homomorphism of graded algebras;

(3) the comultiplication operation is a homomorphism of graded algebras (or, equivalently, the multiplication is a homomorphism of graded coalgebras).

In the model case when $h^* = H^*(G, A)$, the identity associates to each element $a \in A$ a 0-cocycle in G that takes the value a on any connected component of G; the coidentity is the restriction homomorphism $H^*(G, A) \to H^*(e, A) \cong A$, where e is the identity element in G.

For every field A the group $H_*(G, A)$ has a dual Hopf algebra structure: multiplication is induced by multiplication in G (to a pair of cycles $a, b \subset G$ corresponds the image of the cycle $a \times b \subset G \times G$ under the map $G \times G \to G$), and comultiplication is the dual of multiplication in the algebra $H^*(G, A) = [H_*(G, A)]^*$.

In general, to any Hopf algebra structure over a field corresponds a dual

Hopf algebra structure on the dual space.

Instead of the topological groups above, we could consider arbitrary H-spaces, defined as follows.

DEFINITION. A topological space X with a base point x_0 is an *H-space* if continuous maps $m: X \times X \to X$ ("multiplication") and $\mathrm{inv}: X \to X$ ("taking the inverse") are defined such that

(1) both compositions $X \xrightarrow{i_1} X \times X \xrightarrow{m} X$ and $X \xrightarrow{i_2} X \times X \xrightarrow{m} X$, where $i_1(x) = (x, x_0)$, $i_2(x) = (x_0, x)$, are homotopic to the identity map $X \to X$;

(2) the compositions $X \times (X \times X) \xrightarrow{\mathrm{Id} \times m} X \times X \xrightarrow{m} X$ and $(X \times X) \times X \xrightarrow{m \times \mathrm{Id}} X \times X \xrightarrow{m} X$ are homotopic to each other;

(3) both compositions $X \xrightarrow{\mathrm{inv} \times \mathrm{Id}} X \times X \xrightarrow{m} X$ and $X \xrightarrow{\mathrm{Id} \times \mathrm{inv}} X \times X \xrightarrow{m} X$ are homotopic to a constant map.

If X is a topological group, and x_0 is its identity element, then all these conditions are satisfied precisely (not up to homotopy). An important example of H-spaces that are not topological groups is given by loop spaces; see the next appendix.

THEOREM. *For any H-space X and any field A both modules $H_*(X, A)$ and $H^*(X, A)$ are (dual to each other) Hopf algebras.*

This theorem provides very strong topological obstructions to introducing an H-space structure on a given topological space; see [Borel], [MM].

APPENDIX 3

Loop Spaces

DEFINITION. The *loop space* of a topological space X with base point x_0 is the set of continuous maps $[0, 1] \to X$ sending the endpoints 0 and 1 to x_0, with the standard (compact-open) topology for the space of maps.

The *space of free loops* of X is the set of continuous maps $[0, 1] \to X$ sending the points 0 and 1 to the same point (not necessarily the base point).

These spaces are denoted by ΩX and $\Omega_f X$, respectively.

The homotopy groups of these spaces are intimately related:

PROPOSITION 1. *For each $i \geq 1$ and any path-connected space X there is a short exact sequence*

$$0 \to \pi_i(\Omega X) \to \pi_i(\Omega_f X) \to \pi_i(X) \to 0. \tag{1}$$

Indeed, the space $\Omega_f X$ is naturally fibered over X: to a free loop $[0, 1] \to X$ corresponds the image of the points 0 and 1. In the corresponding homotopy exact sequence all maps $\pi_i(\Omega_f X) \to \pi_i(X)$ are epimorphisms, since this fibration has an obvious section (to a point $x \in X$ corresponds the loop $[0, 1] \to x$). □

Usually the loop sending the interval $[0, 1]$ to the base point in X is declared to be the base point in ΩX.

The definition immediately implies that the set of connected components of the space ΩX is the group $\pi_1(X)$; it is easy to see that the set of connected components of $\Omega_f X$ is the group $H_1(X)$.

Moreover, for any topological space Y with base point y_0 there is a natural one-to-one correspondence

$$[Y, \Omega X] \xrightarrow{\sim} [\Sigma Y, X], \tag{2}$$

where $[A, B]$ is the set of homotopy classes of maps $A \to B$ sending the base point to the base point, and ΣY denotes the reduced suspension over Y, i.e. the space obtained from the product $Y \times [0, 1]$ by factoring by the union of the subsets $Y \times 0$, $Y \times 1$, and $y_0 \times [0, 1]$.

For example, if $Y = S^i$ then ΣY is homotopy equivalent to S^{i+1}, and (2) becomes the identity

$$\pi_i(\Omega X) \cong \pi_{i+1}(X). \tag{3}$$

This identity is given by the following construction. We consider an i-dimensional spheroid in B as a map $\mathbb{R}^i \to B$ sending the complement to some ball to the base point, and a loop in X as a map $\mathbb{R}^1 \to X$ sending the complement to the interval $[0, 1]$ to the base point. In particular, an i-spheroid in the space ΩX is an i-parameter family of functions $f_\lambda: \mathbb{R}^1 \to X$, $\lambda \in \mathbb{R}^i$, such that $f_\lambda(t) = x_0$ for sufficiently large $|t| + |\lambda|$. Define a map $F: \mathbb{R}^{i+1} \to X$ by the formula $F(t, \lambda) = f_\lambda(t)$. This map is an $(i+1)$-spheroid, and correspondence (3) is constructed. Correspondence (2) is a trivial generalization of this construction.

Proposition. *The space of k-fold loops $\Omega^k(X) \equiv \Omega(\Omega(\cdots\Omega(X)\cdots))$ is homeomorphic to the space of continuous maps $S^k \to X$ sending the base point to the base point.* □

For any path-connected pointed space X and any positive integer i there is a natural *Freudenthal imbedding*

$$\Omega^i X \to \Omega^{i+1}\Sigma X. \tag{4}$$

Moreover, if Y and X are path-connected spaces with base points then the space $(X, x_0)^{(Y, y_0)}$ of continuous maps $(Y, y_0) \to (X, x_0)$ is naturally imbedded in $(\Sigma X, *)^{(\Sigma Y, *)}$. In fact, to each map $\varphi: (Y, y_0) \to (X, x_0)$ corresponds a map $\varphi \times \mathbf{1}: (Y \times [0, 1]) \to (X \times [0, 1])$ that sends the point $y \times t$ to $\varphi(y) \times t$. This map extends to a map of the quotient space ΣY of the space $Y \times [0, 1]$ to the quotient space ΣX of the space $X \times [0, 1]$.

In particular, the Freudenthal construction defines a homomorphism

$$\pi_i(X) \to \pi_{i+1}(\Sigma X). \tag{5}$$

Freudenthal theorem (see, for example, [Ad], [Fuchs$_1$]). *If X is an $(n-1)$-connected* CW*-complex (for example, the sphere S^n) then map* (5) *is an isomorphism for $i \le 2n-2$ and an epimorphism for $i = 2n-1$.*

Every loop space is an H-space (see Appendix 2): to an ordered pair of loops we associate their composition, and the inverse is the loop traversed in the opposite direction. This structure defines a multiplicative structure on the homology group of the loop space called the *Pontryagin multiplication.*

Proposition. *The group $H_*(\Omega S^k)$ equipped with the Pontryagin multiplication is isomorphic to the polynomial algebra over $\mathbb{Z}$ in one $(n-1)$-dimensional generator.*

If the space X is itself a loop space (or is homotopy equivalent to a loop space) then the H-space ΩX is homotopy commutative (two maps $\Omega X \times \Omega X \to \Omega X$ sending a pair of loops ω_1, ω_2 to their compositions $\omega_1\omega_2$ and $\omega_2\omega_1$ are homotopic).

APPENDIX 4

Germs, Jets, and Transversality Theorems

0. The material of this appendix is taken from [GG].

1. Germs.

Let M, N be smooth manifolds.

DEFINITION. Two maps φ, $\psi: M \to N$ belong to the same germ at a point $x \in M$ if they coincide in some neighborhood of this point.

It is obvious that belonging to one germ is an equivalence relation; the set of equivalence classes is called the *space of germs* of maps $M \to N$ at the point x. The class containing the map φ is called the germ of this map.

If N is a ring (for example, $N = \mathbb{R}$), then the space of germs of maps $M \to N$ at any point of M inherits an obvious ring structure.

2. Jets.

2.1. DEFINITION. Two smooth maps φ, $\psi: M \to N$ belong to the same k-jet at a point x if for some (and therefore for any) choice of local coordinates near the points x, $\varphi(x)$ the Taylor polynomials of order k of these maps coincide. The *space of k-jets* of maps $M \to N$ at a point x is the quotient of the set of all maps by this equivalence relation. This space is denoted by $J^k|_x(M, N)$.

For $N = \mathbb{R}$ the space $J^k|_x(M, N)$ also has a ring structure: it is the quotient of the ring of all germs of maps at the point x by the ideal consisting of germs with a zero of order $k+1$ or higher at x.

2.2. DEFINITION. The *space of k-jets of maps* $M \to N$ is the set of pairs (point of M, k-jet of a map $M \to N$ at this point). This space is denoted by $J^k(M, N)$.

Obviously,

$$J^0(M, N) = M \times N, \tag{1}$$

and there are sequences of obvious projections

$$\cdots \to J^k|_x(M, N) \to J^{k-1}|_x(M, N) \to \cdots \to J^0|_x(M, N) \equiv N, \tag{2}$$

$$\cdots \to J^k(M, N) \to J^{k-1}(M, N) \to \cdots \to J^0(M, N) \begin{smallmatrix} \nearrow M \\ \searrow N \end{smallmatrix}. \tag{3}$$

The space $J^k(M, N)$ has a natural structure of a smooth bundle over M (vector bundle if N is a linear space) with fiber $J^k|_x(M, N)$.

2.3. DEFINITION. The *space of jets* of maps $(M, x) \to N$ (respectively, maps $M \to N$) is the inverse limit of the spaces $J^k|_x(M, N)$ (spaces $J^k(M, N)$) with respect to projections (2) (respectively, (3)). These limits are denoted by $J|_x(M, N)$ and $J(M, N)$.

Fixing coordinates determines a correspondence between $J|_x(M^m, N^n)$ and the space of formal power series of maps $(\mathbb{R}^m, 0) \to \mathbb{R}^n$.

3. Jet extensions. To each smooth map $\varphi: M \to N$ corresponds its k-*jet extension* $j^k\varphi: M \to J^k(M, N)$, i.e. a section of the obvious bundle $J^k(M, N) \to M$; this section assigns to each point of M the k-jet of the map φ at this point.

4. Transversality.

4.1. DEFINITION. Two smooth submanifolds L_1, L_2 of a manifold N are *transversal at* $a \in N$ if one of the following two conditions is satisfied: 1) $a \notin L_1 \cap L_2$; 2) the linear hull of the tangent spaces T_aL_1, T_aL_2 coincides with the space T_aN. The submanifolds L_1, L_2 are *transversal* if they are transversal at every point of N.

For example, if $\dim L_1 + \dim L_2 < \dim N$ then transversality means that L_1 and L_2 are disjoint.

4.2. *A more general definition.* Let L_1, L_2 be two manifolds, $\varphi_1: L_1 \to N$ and $\varphi_2: L_2 \to N$ be smooth maps.

DEFINITION. The maps φ_1, φ_2 are transversal at a pair of points $a_1 \in L_1$, $a_2 \in L_2$ if either $\varphi_1(a_1) \neq \varphi_2(a_2)$ or $\varphi_{1*}T_{a_1}L_1 + \varphi_{2*}T_{a_2}L_2 = T_{\varphi_1(a_1)}N$. The maps φ_1 and φ_2 are *transversal* if they are transversal everywhere in $L_1 \times L_2$.

Definition 4.1 can be reformulated as follows: the submanifolds L_1, L_2 are transversal in L if their identical inclusions into L are transversal.

4.3. Here is an intermediate version of the transversality condition: we say that a map $\varphi: L_1 \to N$ is transversal to a submanifold $L_2 \subset N$ if φ and the identity inclusion $L_2 \hookrightarrow N$ are transversal in the sense of 4.2.

We remark that here φ does not have to be an inclusion or even an immersion.

4.4. *Residual subspaces.*

DEFINITION (see [GG]). A subspace of a topological space K is called *residual* if it is the intersection of a countable collection of open everywhere dense subsets of K. A topological space is a *Baire space* if each residual subset is dense.

Obviously, the intersection of a countable number of residual subsets is again residual.

In the space of smooth maps $M \to N$ there is a natural topology, the Whitney C^∞-topology; its definition will be given in §5 below.

THEOREM (see [GG]). *The space of C^∞-maps of smooth manifolds $M \to N$ equipped with the Whitney C^∞-topology is a Baire space.*

4.5. **THEOREM** (weak transversality theorem). *For any closed submanifold $L \subset N$, the maps transversal to L form a residual set in the space $C^\infty(M, N)$ of all smooth maps of a smooth manifold M into N. Moreover, if M is compact then this set is open in $C^\infty(M, N)$.*

4.6. **THEOREM** (Thom transversality theorem; see [GG], [AVG_1]). *Let $\mathfrak{A}$ be an arbitrary closed submanifold in the space $J^k(M, N)$. Then the set of maps whose k-jet extensions are transversal to $\mathfrak{A}$ is residual in the space of all maps $M \to N$. If M is compact then this set is also open in $C^\infty(M, N)$.*

EXAMPLE. The weak transversality theorem can be obtained from the Thom transversality theorem if $\mathfrak{A}$ is taken to be the set of k-jets that are mapped to the submanifold L by the natural projection $J^k(M, N) \to N$ (see (1), (3)).

4.7. *The Thom multijet transversality theorem.*

4.7.1. **NOTATION.** Let $M[s]$ be the space of ordered families of s distinct points in M. Denote by $J^k_{[s]}(M, N)$ the set of pairs of the form (point $(x_1, \dots, x_s) \in M[s]$, collection of k-jets of maps $M \to N$ at these points). $J^k_{[s]}(M, N)$ is the total space of a natural locally trivial bundle over $M[s]$.

To each smooth map $\varphi\colon M \to N$ we associate its s-fold k-jet extension $j^k_{[s]}\varphi\colon M[s] \to J^k_{[s]}(M, N)$ by assigning to a family of points $(x_1, \dots, x_s) \in M[s]$ the family of k-jets of the map φ at these points.

4.7.2. **THEOREM** (Thom multijet transversality theorem; see [GG]). *Let $\mathfrak{A}$ be an arbitrary regular submanifold of $J^k_{[s]}(M, N)$. Then smooth maps $M \to N$ for which the corresponding s-fold k-jet extensions are transversal to the manifold $\mathfrak{A}$ form a residual subset in the space of all smooth maps $M \to N$.*

EXAMPLE. Consider the submanifold $\mathfrak{A} \subset J^k_{[2]}(M, N)$ consisting of double jets that are mapped to the diagonal (i.e., to the set $\{(a, b) \in N \times N | a = b\}$) under the obvious projection $J^k_{[2]}(M, N) \to N \times N$. Applying Theorem 4.7.2 we get the following assertion.

THEOREM. *The maps $\varphi\colon M \to N$ such that for any pair of distinct points $a, b \in M$ either $\varphi(a) \neq \varphi(b)$ of $\varphi_* T_a M + \varphi_* T_b M = T_{\varphi(a)} N$ form a residual set in the space of smooth maps $M \to N$.*

4.8. *Transversality to stratified subsets.*

DEFINITION. Let N be a smooth manifold and L be a subset equipped with a Whitney stratification (see [Loj_2], [Wall], [Ph]) such that all open strata are C^∞-submanifolds in N. A smooth map $M \to N$ is called transversal to the set L if it is transversal to all open strata.

THEOREM. *Everywhere in Theorems* 4.5, 4.6, *and* 4.7.2 *we can replace smooth submanifolds of the manifold* N *(respectively,* $J^k(M, N)$, $J^k_{[s]}(M, N)$*) by arbitrary closed Whitney stratified subsets of these manifolds.*

5. Whitney topologies in function spaces. Let M, N be smooth manifolds. For any open subset U of the space $J^k(M, N)$ denote by $W(U)$ the set of smooth maps $f: M \to N$ such that the image of M under the action of the k-jet extension of f lies in U.

DEFINITION. The *Whitney* C^k*-topology* on the space $C^\infty(M, N)$ is the topology with base formed by subsets $W(U)$ for all open sets $U \subset J^k(M, N)$.

The *Whitney* C^∞*-topology* on $C^\infty(M, N)$ is the topology with base formed by subsets $W(U)$ for all k and all open $U \subset J^k(M, N)$.

REMARK. It follows immediately from the definitions that the statements of Theorems 4.6, 4.7.2, and 4.8 are equivalent to the same statements about the spaces $C^\infty(M, N)$ with Whitney C^{k+1}-topologies instead of the C^∞-topology.

APPENDIX 5

Homology of Local Systems

1. Definition. A *local system* on a topological space M is a covering over M with fibers endowed with the structure of a fixed abelian group depending continuously on the fiber (i.e. the group structure extends to the set of local sheets over any small domain in the base).

Examples. A) For any abelian group A the product $M \times A$ is a local system called the trivial local system with fiber A and denoted simply by A.

B) Let $\varphi: M' \to M$ be a τ-fold covering and A an abelian group. Then there is a local system $\varphi_! A$ on M with the fiber over any point $x \in M$ isomorphic to A^τ and consisting of all possible A-valued functions on the fiber $\varphi^{-1}(x)$. This local system is called the *direct image* of the trivial local system with fiber A on M'.

In general, any local system L on M' defines a system $\varphi_! L$ on M called the direct image of the system L with fiber isomorphic to the direct sum of τ copies of the fiber of the system L.

C) On the set of local systems on M there are obviously defined operations of direct sums, tensor product, Hom, factorization of a local system by its subsystem.

D) Let M be a manifold. An *orientation local system* (or *orientation sheaf*) on M is a local system whose fiber is isomorphic to $\mathbb{Z}$, and moving over a closed path in M sends every sheet to itself or its opposite depending on whether the orientation of M is preserved or reversed along this path. In terms of the previous examples this local system is the quotient system of the direct image of the trivial $\mathbb{Z}$-system on the orientation two-fold covering of M by the trivial $\mathbb{Z}$-system on M. In general, to any vector bundle over a topological space M corresponds its orientation sheaf with fiber $\mathbb{Z}$: it is isomorphic to the trivial local system if and only if the bundle is orientable.

An isomorphism of local systems is an isomorphism of the coverings preserving the group structure in fibers.

Any local system over M with fiber A defines a representation $\pi_1(M) \to \mathrm{Aut}(A)$: to any loop we associate the permutation of the sheets of the covering over the base point corresponding to this loop. The classification of nonisomorphic local systems on a path-connected space M coincides with the classification of such representations up to conjugacy.

EXAMPLE. Any representation of an arbitrary group π in $\mathrm{Aut}(A)$ defines a unique (up to isomorphism) local system on the space $K(\pi, 1)$.

2. For each local system $L \to M$ there is a related chain complex $C_*(M, L)$; its elements are formal sums of singular simplices of the space of the covering L with the following relations: the sum of a simplex Δ in a sheet U of the covering L and the analogous simplex lying precisely over Δ in a sheet U' is identified with similar simplex in the sheet $U + U'$; any simplex in the zero sheet is equal to 0.

The boundary operator of this complex is defined in the standard way; its homology is called the *homology of M with coefficients in the local system L* (or, shorter, the *homology of the system L*).

The dual construction defines cohomology groups of the system L.

As usual, it is possible to define homology of L by using finite and locally finite chains (a locally finite chain with coefficients in L is a sum of simplices in the space of the corresponding covering whose projection into M is a locally finite chain in M). These homology groups are denoted by $H_*(M, L)$ and $\overline{H}_*(M, L)$ or $H_*^{\mathrm{lf}}(M, L)$, respectively.

EXAMPLES. A) The homology and cohomology of M with coefficients in the trivial local A-system are just the usual homology and cohomology of M with coefficients in the group A: a singular simplex in M with coefficient a can be viewed as a simplex in the sheet $M \times \{a\}$.

B) For any group π and any representation of π in $\mathrm{Aut}(A)$ the (co)homology groups of the space $K(\pi, 1)$ with coefficients in the corresponding local system with fiber A are called the (co)homology groups of π with coefficients in this representation. This definition is equivalent to the abstract algebraic definition from [Brown], [FF].

C) For any k-dimensional vector bundle $E \to B$ we have the *Thom isomorphism*

$$\overline{H}^i(E, \mathbb{Z}) \cong \overline{H}^{i-k}(B, \mathrm{Or}(E)),$$

where $\mathrm{Or}(E)$ is the orientation sheaf of the bundle.

3. Now let G be one of the groups $\mathbb{Z}$, $\mathbb{Z}_q$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$; let A be a free G-module, and L be a local system on M with fiber A.

Define the local system $L^\vee \to M$ dual to L as a local system with fiber $A^* \equiv \mathrm{Hom}(A, G)$ whose fibers are dual to the corresponding fibers of L.

The representation $\pi_1(M) \to \mathrm{Aut}(A^*)$ given by the system $L^\vee$ is conjugate to the representation in $\mathrm{Aut}(A)$ defined by the initial system L.

POINCARÉ DUALITY THEOREM. *For any oriented n-dimensional manifold M and any local system L on M with fiber a free G-module there is a canonical isomorphism*

$$H^i(M, L) \cong \overline{H}_{n-i}(M, L^\vee).$$

In particular, there is a pairing

$$H_i(M, L) \otimes \overline{H}_{n-i}(M, L^\vee) \to G;$$

similarly to the case of homology of trivial systems, it is given by intersection indices and defines a nonsingular pairing on G-free parts of the groups $H_i(M, L)$ and $\overline{H}_{n-i}(M, L)$.

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