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Partial Differential Operators of Elliptic Type

Norio Shimakura

American Mathematical Society

Partial Differential Operators of Elliptic Type

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Translations of

MATHEMATICAL MONOGRAPHS

Volume 99

Partial Differential Operators of Elliptic Type

Norio Shimakura

Translated by
Norio Shimakura

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Providence, Rhode Island

楕円型偏微分作用素

DAENKEI HENBIBUN SAYŌSO (Elliptic Partial Differential Operators)
by Norio Shimakura

Originally published in Japanese by Kinokuniya Company Ltd., Publishers, Tokyo in 1978

Translated from the Japanese by Norio Shimakura

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ABSTRACT. This book gives a comprehensive study of the theory of elliptic partial differential operators. Beginning with the definitions of ellipticity for higher order operators, it discusses the Laplacian in Euclidean spaces, elementary solutions, smoothness of solutions, Vishik-Sobolev problems, general boundary value problems, the Schauder theory, and degenerate elliptic operators. The Appendix consists of preliminaries: ordinary differential equations, Sobolev spaces, etc. Taking account of the trend in mathematics that elliptic operators appear in several branches, the book collects for the users' benefit many fundamental techniques for treating elliptic operators.

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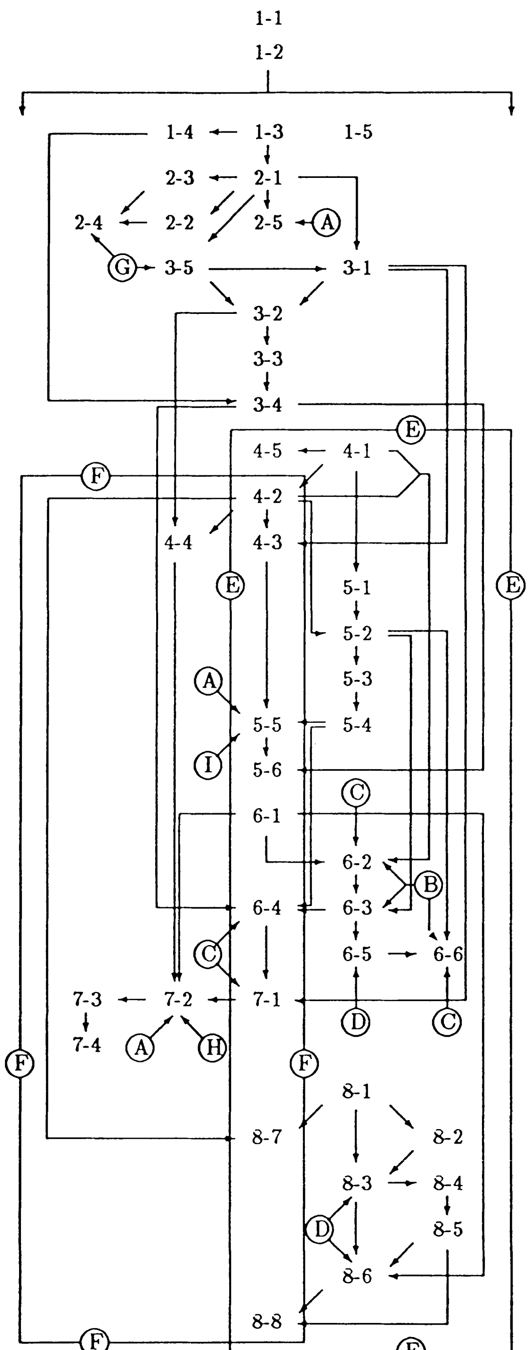
To My Parents

and Fusako

Contents

Remark. The interdependence among sections is indicated in the following diagram where *x-y* denotes chapter *x*, §*y*.



Preface to the Japanese Edition

Variational problems are discussed in several domains of mathematics and the natural sciences. It is important to investigate steady states and stationary states of quantities distributed in a region of space and their dependence on the time variable. Solutions of variational problems and descriptions of steady states and stationary states are reduced in most cases to certain partial differential equations of elliptic type. Therefore, it is not only interesting from the mathematical viewpoint but also important in applications to study under what assumptions solutions exist, whether solutions are unique and how smooth they are. Furthermore, we sometimes need more precise results: Where is the solution positive or negative, where are the values large or small, and where does the solution achieve a maximum or minimum? In the theory of elliptic equations, we discuss these questions mathematically and more or less systematically.

Since the history of the theory of elliptic equations goes back as far as that of parabolic equations and hyperbolic equations, there are many articles and textbooks treating this subject. The references listed at the end of this book refer to only a small part of the literature. Since elliptic equations appear in several branches of mathematics, there are many researchers who have investigated problems involving this type of equation. Each of them has his own idea of what types of problems are interesting and important in the theory of elliptic equations and of how to organize the contents if he writes a textbook. I wrote the present book from my own viewpoint. I have collected here some historical results established mainly up to the earlier part of the 1960s. Most of the material is concerned with linear equations of elliptic type.

However, the material is not at all complete because there are many things which I could not present here for lack of knowledge and experience. There are also many insufficiencies and inconsistencies. I sincerely look for the criticisms by the readers on such points. They will surely be fruitful to my future research.

I wrote this book for students in either an undergraduate course or a first year graduate course. This book might appear difficult since I omitted the details of the theory of distributions because of space limitations. However, in this book, I make much more use of Sobolev spaces than the space of all distributions. Thus, I devoted a section (§E) in the Appendix to Sobolev spaces. It suffices for readers to have heard of the Dirac delta function, an example of distributions. In any case, elliptic equations were investigated and solved in several ways much earlier than the appearance of the theory of distributions.

From the beginning of my research I have been indebted to Professor Sigeru Mizohata. I would like to express my hearty gratitude to him for his valuable criticism, advice, and constant encouragement throughout my career. I would also like to express my thanks to Professors Masaya Yamaguti and Takeshi Kotake for their interest in my research. I would like to express my deep gratitude to Professor Seizo Ito who gave me the opportunity to write this book.

Norio Shimakura
March 1977
Sendai

Preface to the English Translation

Almost fifteen years have passed since the publication of the first edition of this book in Japanese [164]. I am very happy and honored to have the opportunity to translate the book into English owing to the kindness of the American Mathematical Society.

In this translation, I have corrected trivial mistakes in the original book, and have modified or revised almost ten percent of the contents. I have added a new §6 to Chapter VI, §8 to Chapter VIII, §I to the Appendix, additional references, and an index.

Some of the results presented here could have been organized in a different way so that proofs of some theorems were shorter and more elegant. However, I did not change the organization in this manner largely because it is one of the roles of a textbook to present as many technical details as possible.

It is a pleasure to express my deep thanks to Mr. Setsuro Fujiié who repeatedly read the translation from beginning to end and pointed out many mistakes. Also, I am very much indebted to the Referee and Richard Porter who kindly corrected my mistakes in the English language.

Norio Shimakura
August 1991
Sendai

Appendix

§A. Maximum principles

The maximum principle is one of the most basic properties of solutions for elliptic equations. This holds for a large class of equations of second order with real coefficients. However, this cannot be generalized to either equations with complex coefficients or equations of higher order. The reason is very simple. An elliptic equation of second order with real coefficients implies something about convexity or concavity of solutions, while this is not the case for higher order equations. So, in this section, coefficients of operators, solutions, and inhomogeneous terms are assumed to be *real-valued* functions.

Let A be a partial differential operator in a bounded or unbounded domain Ω of $\mathbf{R}^n$:

$$Au(x) = \sum_{j,k=1}^{n} a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u, \tag{A.1}$$

where the coefficients $a_{jk}(x)$, $b_j(x)$, and $c(x)$ are continuous in Ω and in particular

$$a_{jk}(x) = a_{kj}(x) \quad (1 \le j, k \le n) \quad \text{and} \quad \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \ge 0 \tag{A.2}$$

hold at every point $x \in \Omega$ and every $\xi \in \mathbf{R}^n$.

A function $u(x)$ is called a *classical solution* of the equation

$$Au(x) = f(x) \quad \text{in } \Omega \tag{A.3}$$

if u is of class C^2 in Ω and satisfies (A.3) there. u is said to achieve a *local minimum* (or *local maximum*) at a point x^0 of Ω if u restricted to an appropriate neighborhood of x^0 is minimal (resp. maximal) at x^0.

THEOREM A.1 (strong maximum principle). *Suppose that either*

$$c(x) \le 0 \quad \textit{and} \quad f(x) < 0$$

or

$$c(x) < 0 \quad \textit{and} \quad f(x) \le 0 \tag{A.4}$$

at every point of Ω. *Then, a classical solution* u *of* (A.3) *achieves negative local minimum nowhere in* Ω.

PROOF. Suppose that u achieves a negative local minimum at $x^0 \in \Omega$. Then, by putting $p_{jk} = (\partial^2 u/\partial x_j \partial x_k)(x^0)$, we have

$$\sum_{j,k=1}^{n} p_{jk}\xi_j\xi_k \geq 0 \tag{A.5}$$

for any $\xi \in \mathbf{R}^n$. On the other hand, since $(a_{jk}(x^0))_{j,k=1}^n$ is nonnegative definite, there exists n^2 real numbers λ_{js} such that $a_{jk}(x^0) = \sum_{s=1}^n \lambda_{js}\lambda_{ks}$ $(1 \leq j, k \leq n)$. So by (A.3),

$$f(x^0) - c(x^0)u(x^0) = \sum_{j,k=1}^{n} a_{jk}(x^0)p_{jk} = \sum_{s=1}^{n}\sum_{j,k=1}^{n} p_{jk}\lambda_{js}\lambda_{ks}$$

because $(\partial u/\partial x_j)(x^0) = 0$ $(1 \leq j \leq n)$. The right-hand side is nonnegative by (A.5), while the left-hand side is negative by (A.4) because $u(x^0) < 0$, which is a contradiction. □

Next, we relax the condition (A.4) but we assume uniform ellipticity of A: There exists a positive number α such that

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq \alpha|\xi|^2 \quad \text{if } (x, \xi) \in \Omega \times \mathbf{R}^n. \tag{A.6}$$

Then,

THEOREM A.2 (weak maximum principle or maximum principle of E. Hopf). *Suppose that the coefficients of* A *are bounded,* (A.6) *holds, and*

$$c(x) \leq 0 \quad \textit{and} \quad f(x) \leq 0 \tag{A.7}$$

at every point of Ω. *Suppose moreover that* u *achieves a negative local minimum at* $x^0 \in \Omega$. *Then,* u *is identically equal to* $u(x^0)$ *in* Ω_0, *where* Ω_0 *is the largest connected open subset of* Ω *containing* x^0 *on which* $u(x) \geq u(x^0)$.

PROOF. Put $F = \{x \in \Omega_0 ; u(x) = u(x^0)\}$. Then F is a nonempty closed subset of Ω_0. So, if we show that $\partial F \cap \Omega_0 = \emptyset$, then $F = \Omega_0$ and we have finished the proof. Suppose on the contrary that $\partial F \cap \Omega_0 \neq \emptyset$.

First, there exist $x^1 \in \Omega_0 \backslash F$ and $x^2 \in F$ such that

$$B(x^1, \rho)\backslash\{x^2\} \subset \Omega_0\backslash F \quad \text{if } \rho = |x^1 - x^2| \tag{A.8}$$

(we denote by $B(a, r)$ the closed ball of radius r centered at $a \in \mathbf{R}^n$). In fact, let $\overline{x}$ be any point of $\partial F \cap \Omega_0$ and $d = \operatorname{dist}(\overline{x}, \partial\Omega_0)$ (> 0). If $\hat{x} \in \Omega_0\backslash F$ satisfies $|\hat{x} - \overline{x}| < d/2$, then $B(\hat{x}, \tilde{\rho}) \subset \Omega_0$ provided that $0 < \tilde{\rho} \leq d/2$. Let $\hat{r}$ be the supremum of $\tilde{\rho}$ such that $B(\hat{x}, \tilde{\rho}) \subset \Omega_0\backslash F$.

Then there exists a point, x^2 say, of $F \cap \partial B(\hat{x}, \hat{r})$. Choosing any point x^1 on the segment $\overline{\hat{x}x^2}$, these x^1 and x^2 satisfy (A.8).

Secondly, $B(x^2, \rho_1) \subset \Omega_0$ if $0 < \rho_1 < \rho$ and ρ_1 is small enough. We write $B = B(x^2, \rho_1)$ for simplicity and set

$$w(x) = u(x) + \lambda v(x),$$

where

$$v(x) = \exp(-N\rho^2) - \exp(-N|x - x^1|^2). \tag{A.9}$$

Let us prove that, for a small $\lambda > 0$ and a large $N > 0$,

$$Aw(x) < 0 \quad \text{in } B, \tag{A.10}$$

$$w(x) > w(x^2) \quad \text{on } \partial B. \tag{A.11}$$

These contradict Theorem A.1 applied to w in B because $w(x^2) = u(x^2) = u(x^0) < 0$, so the theorem will be proved.

(A.10). Setting $r = |x - x^1|$, we have

$$\begin{aligned}(Av)\exp(Nr^2) &= [\exp\{N(r^2 - \rho^2)\} - 1]c(x) \\ &+ 2N\sum_{j=1}^{n}\{a_{jj} + (x_j - x_j^1)b_j\} - 4N^2\sum_{j,k=1}^{n} a_{jk}(x)(x_j - x_j^1)(x_k - x_k^1).\end{aligned}$$

The first two terms on the right-hand side are of $O(1)$ and $O(N)$ respectively, while the last term is not larger than $-4\alpha N^2(\rho - \rho_1)^2$ in B, where we made use of (A.6). So, $Av < 0$ in B if N is sufficiently large. Since $Au \le 0$ and $\lambda > 0$, we have (A.10).

(A.11). $w(x) > w(x^2)$ on $\partial B \backslash B(x^1, \rho)$ as long as $\lambda > 0$ because $v > 0$ there. Put

$$m = \inf_{B(x^1, \rho)\cap\partial B}\{u(x) - u(x^0)\}, \qquad M = \sup_{B(x^1, \rho)\cap\partial B}\{-v(x)\}$$

(m and M are positive). Choose a λ satisfying $0 < \lambda < \min(1, m/M)$. Then, $w > w(x^2)$ also on $B(x^1, \rho) \cap \partial B$, proving (A.11). □

COROLLARY A.3. *Suppose that A satisfies the conditions in Theorem A.2 and Ω is bounded. Let u be a nonconstant classical solution of $Au = 0$ which is continuous on $\overline{\Omega}$. Then, u satisfies*

$$|u(x)| < \max_{\partial\Omega}|u| \quad \text{in } \Omega. \tag{A.12}$$

If in particular $c(x)$ is identically equal to zero, we have

$$\min_{\partial\Omega} u < u(x) < \max_{\partial\Omega} u \quad \text{in } \Omega. \tag{A.13}$$

PROOF. If (A.12) were not true, there would exist an $x \in \Omega$ at which either $u(x) \ge \max_{\partial\Omega}\max(u, 0)$ or $u(x) \le \min_{\partial\Omega}\min(u, 0)$, contradicting

Theorem A.2. Also applying Theorem A.2 to $u(x)+t$ with very large (or very small) constant t, we may prove (A.13). □

REMARK A.1. The hypothesis $c(x) \le 0$ is indispensable for the maximum principle. For example, let $\Omega = \{x \in \mathbf{R}^3 ;\, |x| < R\}$. Set

$$u(x) = \frac{\sin(k|x|)}{|x|}$$

with a positive number k. Then, u solves $\Delta u + k^2 u = 0$ and vanishes on $\partial\Omega$ if kR/π is an integer. However, $\min_\Omega u < 0 < \max_\Omega u$ if $kR/\pi \ge 2$. So, Theorem A.1 does not hold for $A = \Delta + k^2$.

REMARK A.2. As we mentioned at the beginning, the maximum principle does not hold for solutions of elliptic equations of higher order. We have infinitely many examples. Let $\Omega = \{x \in \mathbf{R}^2 ;\, |x| < 1\}$ (unit disk) and set

$$u(x) = (|x|^2 - 1)(2|x|^2 - 1).$$

Then, $\Delta^2 u = 128 > 0$ and $u = 0$ on $\partial\Omega$. However,

$$\min_\Omega u = -\frac{1}{8} < 0 < \max_\Omega u = 1.$$

So, the conclusion of Theorem A.1 does not hold for $A = \Delta^2$.

There are many good textbooks on maximum principles. For example, C. Miranda [82], M. N. Protter and H. F. Weinberger [158], D. Gilbarg and N. S. Trudinger [141] and K. Taira [167].

§B. Stokes formula and systems of boundary operators

A. Formal adjoint operators. We introduce a volume element $\mu(x)\,dx = \mu(x)\,dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ on $\mathbf{R}^n$, where $\mu(x)$ is a positive function of class C^∞ in $\mathbf{R}^n$.

Given a differential operator A of order l with coefficients of class C^∞, we can represent A in two different ways:

$$Au(x) = \sum_\alpha a_\alpha^+(x) D^\alpha u(x) = \frac{1}{\mu(x)} \sum_\alpha D^\alpha\{\mu(x) a_\alpha^-(x) u(x)\}, \tag{B.1}$$

where the summations are extended over multi-indices $\alpha \in \mathbf{N}^n$ of length at most l (see Chapter I, §1, part B). So, we have two kinds of symbols of A:

$$a^+(x,\xi) = \sum_\alpha a_\alpha^+(x)\xi^\alpha, \qquad a^-(x,\xi) = \sum_\alpha a_\alpha^-(x)\xi^\alpha. \tag{B.2}$$

Here and in what follows, ξ denotes the dual variables of x. The relationship between the two symbols is as follows:

$$\begin{aligned} a^+(x,\xi) &= \frac{1}{\mu(x)} e^{L}\{\mu(x) a^-(x,\xi)\}, \\ a^-(x,\xi) &= \frac{1}{\mu(x)} e^{-L}\{\mu(x) a^+(x,\xi)\}, \end{aligned} \tag{B.3}$$

where

$$L = \sum_{j=1}^{n} \frac{1}{i} \frac{\partial^2}{\partial x_j \partial \xi_j}, \qquad e^{\theta L} = \sum_{\alpha} \frac{\theta^{|\alpha|}}{\alpha!} D_x^{\alpha} \left(\frac{\partial}{\partial \xi}\right)^{\alpha}, \quad \theta \in \mathbf{C}. \tag{B.4}$$

Now, we define a third symbol, called the *Weyl symbol* of A:

$$\begin{aligned} a^0(x, \xi) &= \frac{1}{\mu(x)} e^{-L/2}\{\mu(x) a^+(x, \xi)\} \\ &= \frac{1}{\mu(x)} e^{L/2}\{\mu(x) a^-(x, \xi)\} = \sum_{\alpha} a_{\alpha}^0(x) \xi^{\alpha} \end{aligned} \tag{B.5}$$

(see L. Hörmander [149]). Then, $a^{\pm}(x, \xi)$ and $a^0(x, \xi)$ are polynomials of degree l with respect to ξ. The homogeneous parts of degree l of these three are the same and equal to the principal symbol $A_l(x, \xi)$. By means of the Weyl symbol, A is represented in three manners:

$$Au(x) = \frac{1}{\mu(x)} \sum_{\alpha, \beta} 2^{-|\beta|} \binom{\alpha+\beta}{\alpha} D^{\beta}\{\mu(x) a_{\alpha+\beta}^0(x)\} \cdot D^{\alpha} u(x) \tag*{(B.6)$_1$}$$

$$= \frac{1}{\mu(x)} \sum_{\alpha, \beta} (-2)^{-|\beta|} \binom{\alpha+\beta}{\alpha} D^{\alpha}[D^{\beta}\{\mu(x) a_{\alpha+\beta}^0(x)\} \cdot u(x)] \tag*{(B.6)$_2$}$$

$$= \frac{1}{\mu(x)} \sum_{\alpha, \beta} 2^{-|\alpha+\beta|} \binom{\alpha+\beta}{\alpha} D^{\beta}\{\mu(x) a_{\alpha+\beta}^0(x) D^{\alpha} u(x)\}. \tag*{(B.6)$_3$}$$

PROOF. By virtue of $e^{\theta L}\{f(x)\xi^{\alpha}\} = (\theta D_x + \xi)^{\alpha} f(x)$, we have

$$\begin{aligned} \mu(x) a^+(x, \xi) &= \sum_{\alpha, \beta} 2^{-|\beta|} \binom{\alpha+\beta}{\alpha} \xi^{\alpha} D_x^{\beta}\{\mu(x) a_{\alpha+\beta}^0(x)\} \\ &= \sum_{\alpha, \beta} (-2)^{-|\beta|} \binom{\alpha+\beta}{\alpha} (D_x + \xi)^{\alpha} D_x^{\beta}\{\mu(x) a_{\alpha+\beta}^0(x)\} \\ &= \sum_{\alpha, \beta} 2^{-|\alpha+\beta|} \binom{\alpha+\beta}{\alpha} \xi^{\alpha} (D_x + \xi)^{\beta}\{\mu(x) a_{\alpha+\beta}^0(x)\}. \end{aligned}$$

To obtain (B.6), read ξ^{γ} as $D_x^{\gamma} u$. □

The *formal adjoint* of A with respect to the volume element $\mu(x)\,dx$ is the operator A^* characterized by the following equality for all functions $u, v \in C_0^{\infty}(\mathbf{R}^n)$

$$\int (Au)(x) \overline{v(x)} \mu(x)\,dx = \int u(x) \overline{(A^* v)(x)} \mu(x)\,dx. \tag{B.7}$$

Let $a^{*+}(x, \xi)$, $a^{*-}(x, \xi)$ and $a^{*0}(x, \xi)$ be symbols of A^* defined analogously to (B.2) and (B.5) respectively. By means of symbols of A, these are represented as

$$a^{*\pm}(x, \xi) = \overline{a^{\mp}(x, \xi)}, \qquad a^{*0}(x, \xi) = \overline{a^0(x, \xi)}. \tag{B.8}$$

A is said to be *Hermitian* or *formally selfadjoint* with respect to $\mu(x)\,dx$ if $A^* = A$. This holds if and only if $a^0(x, \xi)$ is real valued for $\xi \in \mathbf{R}^n$ (see $(B.6)_3$).

B. Stokes' formula. Let Ω be a domain (bounded or unbounded) of $\mathbf{R}^n$ whose boundary $S = \partial\Omega$ is a hypersurface of class C^∞. Suppose that Ω is in only one side of S at every point of S (see Chapter V, §2). Denote by $dS = dS(x')$ $(x' \in S)$ the surface element of S induced by Lebesgue measure dx of $\mathbf{R}^n$. *Stokes' formula* is the equality

$$\int_\Omega (Au \cdot \overline{v} - u \cdot \overline{A^* v})\mu\, dx = i \int_S B[u, v](x')\mu(x')\, dS(x'), \tag{B.9}$$

which is valid for all functions u, v belonging to $C_0^\infty(\overline{\Omega})$. Let us write down the sesquilinear form $B[u, v]$ explicitly by means of the Weyl symbol of A.

At first, there is a system of functions $\{b_j(x, \xi, \eta, \zeta)\}_{j=1}^n$ with which the following equality holds for any $\xi, \eta, \zeta \in \mathbf{R}^n$

$$a^0\left(x, \xi + \frac{1}{2}\zeta\right) - a^0\left(x, \eta - \frac{1}{2}\zeta\right) = \sum_{j=1}^n (\xi_j - \eta_j + \zeta_j) b_j(x, \xi, \eta, \zeta). \tag{B.10}$$

The choice of b_j's is not unique. However, it suffices to put

$$b_j(x, \xi, \eta, \zeta) = \frac{1}{2}\int_{-1}^{+1} \frac{\partial a^0}{\partial \xi_j}\left(x, \frac{1}{2}\{(1+\theta)\xi + (1-\theta)\eta + \theta\zeta\}\right) d\theta, \tag{B.10$'$}$$

for $1 \le j \le n$. So $b_j(x, \xi, \eta, \zeta)$ are polynomials with respect to (ξ, η, ζ).

Secondly, let us define $b_{j,(\gamma)}(x, \xi, \eta)$, $b_j(x, \xi, \eta)$, and $b_{j,\alpha,\beta}(x)$ to be

$$b_j(x, \xi, \eta, \zeta) = \sum_\gamma b_{j,(\gamma)}(x, \xi, \eta)\zeta^\gamma,$$

$$b_j(x, \xi, \eta) = \frac{1}{\mu(x)} \sum_\gamma D_x^\gamma\{\mu(x) b_{j,(\gamma)}(x, \xi, \eta)\} = \sum_{\alpha,\beta} b_{j,\alpha,\beta}(x)\xi^\alpha \eta^\beta. \tag{B.11}$$

Then, the integrand on the left-hand side of (B.9) is written as

$$\begin{aligned}&\{(Au)(x)\overline{v(x)} - u(x)\overline{(A^*v)(x)}\}\mu(x)\\ &\quad = \sum_{j=1}^n D_{x_j}\{\mu(x)\sum_{\alpha,\beta} b_{j,\alpha,\beta}(x) D_x^\alpha u(x) \cdot \overline{D_x^\beta v(x)}\}.\end{aligned} \tag{B.12}$$

Proof of (B.12). (B.10) and (B.5) imply

$$\begin{aligned}&\mu(x)\{a^+(x, \xi) - a^-(x, \eta)\}\\ &\quad = \sum_{j=1}^n (D_{x_j} + \xi_j - \eta_j)\{\mu(x)\sum_{\alpha,\beta} b_{j,\alpha,\beta}(x)\xi^\alpha\eta^\beta\}.\end{aligned} \tag{B.12$'$}$$

To obtain (B.12), read ξ^γ as $D^\gamma u$ and η^δ as $\overline{D^\delta v}$. □

Third, we put

$$\sum_{j=1}^{n} \nu_j(x')b_j(x', \xi, \eta) = \sum_{\alpha, \beta} b_{\alpha, \beta}(x')\xi^\alpha \eta^\beta, \tag{B.13}$$

where $\nu(x') = {}^t(\nu_1(x'), \dots, \nu_n(x'))$ is the inner unit normal to S at x'. Then, Stokes' formula (B.9) holds if we set

$$B[u, v](x') = \sum_{\alpha, \beta} b_{\alpha, \beta}(x')(D^\alpha u)(x')\overline{(D^\beta v)(x')}. \tag{B.14}$$

Note, in particular, that

$$B[u, v](x') = \overline{B[v, u](x')} \quad \text{if } A \text{ is hermitian.} \tag{B.14$'$}$$

In fact, then $A^0(x, \xi)$ has real coefficients. So $b_j(x, \xi, \eta, \zeta)$ and $b_{j(\gamma)}(x, \xi, \eta)$ are likewise, $b_j(x, \xi, \eta, \zeta) = b_j(x, \eta, \xi, -\zeta)$ by (B.10$'$) and $b_{j,(\gamma)}(x, \xi, \eta) = (-1)^{|\gamma|} b_{j,(\gamma)}(x, \eta, \xi)$ by (B.11). Therefore, $b_j(x, \xi, \eta) = \overline{b_j(x, \eta, \xi)}$ again by (B.11). From this and (B.13), we have $b_{\alpha, \beta}(x') = \overline{b_{\beta, \alpha}(x')}$, which implies (B.14$'$).

Let $\rho(x) = \operatorname{dist}(x, S)$ for $x \in \overline{\Omega}$. Then, ρ is of class C^∞ in a neighborhood of S. Since $\operatorname{grad} \rho(x) \to \nu(x')$ as x ($\in \Omega$) approaches x' ($\in S$), the directional derivative

$$D_\nu u(x') = (\operatorname{grad} \rho)(x') \cdot D_x u(x') = \sum_{j=1}^{n} \frac{\partial \rho}{\partial x_j}(x') D_{x_j} u(x') \tag{B.15}$$

is equal to the inner normal derivative of u on S. So, if we decompose D_x into D_ν and differentiations in tangential directions, the form B may be written as

$$B[u, v](x') = \sum_{j,k} P_{jk}(D_\nu^j u)(x') \cdot \overline{Q_{jk}(D_\nu^k v)(x')}, \tag{B.16}$$

where P_{jk} and Q_{jk} are differential operators on S. Since the leading part of $a^0(x, \xi)$ is equal to $A_l(x, \xi)$, we may rearrange the right-hand side of (B.16) in the following way

$$\begin{aligned} B[u, v](x') = {} & A_l(x', \nu(x')) \sum_{j=0}^{l-1} (D_\nu^j u)(x') \cdot \overline{(D_\nu^{l-1-j} v)(x')} \\ & + \sum_{j+k \le l-2} M_{jk}(D_\nu^j u)(x') \cdot \overline{N_{jk}(D_\nu^k v)(x')}, \end{aligned} \tag{B.17}$$

where M_{jk} and N_{jk} are differential operators on S with coefficients of class C^∞ satisfying $\operatorname{ord} M_{jk} + \operatorname{ord} N_{jk} \le l - j - k - 1$.

The definitive form of Stokes' formula is (B.9) with B given by (B.17). If A is elliptic of order l, then $A_l(x', \nu(x'))$ never vanishes on S.

C. Systems of boundary operators. A set $\{B_j\}_{j=1}^p$ of p operators is said to be a *system of boundary operators* if each B_j has expression

$$(B_j u)(x') = \sum_\alpha b_{j\alpha}(x')(D_x^\alpha u)(x'), \qquad x' \in S, \ 1 \le j \le p, \tag{B.18}$$

where the sum on the right-hand side is extended over a finite set of multi-indices and the $b_{j\alpha}$ are assumed to be of class C^∞ on S. So, a system of p boundary operators is a continuous linear mapping from $C^\infty(\overline{\Omega})$ into $[C^\infty(S)]^p$. If B_j is of order r_j, we can define a function

$$B_j(x', \xi) = \sum_{|\alpha|=r_j} b_{j\alpha}(x')\xi^\alpha, \qquad x' \in S, \ \xi \in \mathbf{R}^n, \ 1 \le j \le p \tag{B.19}$$

called the *principal symbol* of B_j. We can rewrite B_j as

$$(B_j u)(x') = \sum_{k=0}^{r_j} \beta_{jk}(D_\nu^k u)(x'), \qquad x' \in S, \ 1 \le j \le p, \tag{B.20}$$

where β_{jk} is a differential operator on S of order at most $r_j - k$. Let $T_{x'}^*(S)$ be the cotangent space of S at x'. Since every vector ξ of $T_{x'}^*(\mathbf{R}^n)$ can be decomposed as $\xi = \eta + \lambda\nu(x')$ with $\eta \in T_{x'}^*(S)$ and $\lambda \in \mathbf{R}$, we have an alternative expression of (B.19):

$$B_j(x', \eta + \lambda\nu(x')) = \sum_{k=0}^{r_j} \beta_{jk}(x', \eta)\lambda^k, \tag{B.21}$$

where $\beta_{jk}(x', \eta)$ is the principal symbol of β_{jk} (homogeneous polynomial of degree $r_j - k$ with respect to η).

DEFINITION B.1. *Let* $\{B_j\}_{j=1}^p$ *be a system of boundary operators.*

(i) A surface S is called *noncharacteristic* with respect to this system if none of the $B_j(x', \nu(x'))$ $(1 \le j \le p)$ vanishes at any point of S, so each B_j contains $D_\nu^{r_j}$ $(r_j = \operatorname{ord} B_j)$.

(ii) $\{B_j\}_{j=1}^p$ is called the *Dirichlet system* if S is noncharacteristic with respect to this system and if $\operatorname{ord} B_j = j - 1$ for $1 \le j \le p$.

(iii) $\{B_j\}_{j=1}^p$ is called *normal* if it is a subset of a larger Dirichlet system (so $r_1, \dots, r_p$ are distinct).

(iv) $\{B_j\}_{j=1}^p$ is called *m-normal* if it is normal, $p = m$, and if $r_j < 2m$ for $1 \le j \le m$.

For example, let $\{D_\nu^{r_j}\}_{j=1}^p$ be given. Then, S is noncharacteristic with respect to it, it is normal if $r_1, \dots, r_p$ are distinct, it is m-normal if $p = m$ and $0 \le r_1 < \cdots < r_m \le 2m - 1$ and it is a Dirichlet system if $r_j = j - 1$ for $1 \le j \le m$.

D. Equivalence of normal systems. Two normal systems $\{B_j\}_{j=1}^p$ and $\{\widetilde{B}_j\}_{j=1}^{\tilde{p}}$ are called *equivalent* if the following condition holds for functions

$u \in C^\infty(\overline{\Omega})$:

$$B_j u = 0 \textit{ identically on } S \textit{ for } 1 \le j \le p \textit{ if and only if}$$
$$\widetilde{B}_j u = 0 \textit{ identically on } S \textit{ for } 1 \le j \le \tilde{p}.$$

The proof of the following lemma is a simple exercise in linear algebra, so we omit it.

LEMMA B.1. *Two normal systems $\{B_j\}_{j=1}^p$ and $\{\widetilde{B}_j\}_{j=1}^{\tilde{p}}$ are equivalent if and only if they satisfy three conditions after renumbering the $\widetilde{B}_j$'s if necessary*:

(a) $p = \tilde{p}$;

(b) $\operatorname{ord} B_j = \operatorname{ord} \widetilde{B}_j$ (*denoted by* r_j) *for* $1 \le j \le p$;

(c) *For each* (j, k) $(1 \le j, k \le p)$, *there exists* λ_{jk}, μ_{jk}, *differential operators on* S *of order not exceeding* $r_j - r_k$ $(\lambda_{jk} = \mu_{jk} = 0$ *if* $r_j < r_k)$, *such that*

$$\widetilde{B}_j = \sum_{k=1}^{j} \lambda_{jk} B_k \textit{ and } B_j = \sum_{k=1}^{j} \mu_{jk} \widetilde{B}_k \quad \textit{for } 1 \le j, k \le p.$$

Therefore, every normal system $\{B_j\}_{j=1}^p$ is, after replacing it by an equivalent one if necessary, of the following type

$$B_j = D_\nu^{r_j} + \sum_{k \notin R,\ k<r_j} \beta_{jk} D_\nu^k, \qquad 1 \le j \le p,$$

where

$$R = \{r_j\}_{j=1}^p, \qquad 0 \le r_1 < \cdots < r_p \tag{B.22}$$

and the β_{jk} are differential operators on S of order not exceeding $r_j - k$.

E. General boundary value problem of elliptic type. Let A be properly elliptic of order $2m$ and $\{B_j\}_{j=1}^m$ be an m-normal system of boundary operators. The *general boundary value problem* $\{A, \{B_j\}_{j=1}^m\}$ is to find a function $u(x)$ on $\overline{\Omega}$ (or in Ω) satisfying

$$\begin{aligned} Au(x) &= f(x) \quad \text{in } \Omega; \\ (B_j u)(x') &= \varphi_j(x'), \quad 1 \le j \le m, \quad \text{on } S, \end{aligned} \tag{B.23}$$

for given functions $(f(x), \varphi_1(x'), \dots, \varphi_m(x'))$ (see Chapter VI, §2 for details).

DEFINITION B.4. $\{A^*, \{B_j'\}_{j=1}^n\}$ is called *formally adjoint* to $\{A, \{B_j\}_{j=1}^m\}$ with respect to $\mu(x)\,dx$ if the following condition is satisfied: Suppose that $u(x), v(x) \in C^\infty(\overline{\Omega})$. Then the equality

$$\int_\Omega Au \cdot \overline{v} \mu \, dx = \int_\Omega u \cdot \overline{A^* v} \mu \, dx \tag{B.24}$$

holds for any u satisfying

$$B_j u = 0, \quad 1 \le j \le m, \quad \text{on } S \tag{B.25}$$

if and only if v satisfies

$$B'_j v = 0, \quad 1 \le j \le m, \quad \text{on } S. \tag{B.26}$$

In this case, we write $\{A^*, \{B'_j\}_{j=1}^m\} = \{A, \{B_j\}_{j=1}^m\}^*$.

LEMMA B.2. *Let A be properly elliptic of order $2m$ and let $\{B_j\}_{j=1}^m$ be an m-normal system of boundary operators. Then, there exists an m-normal system $\{B'_j\}_{j=1}^m$ of boundary operators such that $\{A^*, \{B'_j\}_{j=1}^m\} = \{A, \{B_j\}_{j=1}^m\}^*$. The choice of $\{B'_j\}_{j=1}^m$ is unique up to equivalence.*

PROOF. Let R be the set of orders of $\{B_j\}$, and T the complement of R:

$$R = \{r_j\}_{j=1}^m, \quad r_j = \operatorname{ord} B_j \quad (1 \le j \le m); \qquad T = \{0, 1, \dots 2m-1\} \backslash R.$$

Since $\{B_j\}$ is m-normal, R and T are sets of m elements. We denote temporarily B_j by $B_{(r)}$ $(r = r_j)$ so that the index of operator is equal to its order. Then, by virtue of the notion of equivalence, we may assume that each $\{B_{(r)}\}$ is of type (B.22), that is,

$$B_{(r)} u = D_\nu^r u + \sum_{t \in T,\ t<r} \beta_{rt} D_\nu^t u, \qquad r \in R. \tag{B.27}$$

We use (B.17) and denote $A_{2m}(x', \nu(x'))$ by $a_{2m}(x')$. We can identify $M_{jk} f \cdot \overline{N_{jk} g}$ with $f \cdot \overline{M_{jk}^* N_{jk} g}$ (integration by parts on S with respect to $\mu\, dS$). We can eliminate $\{D_\nu^r u\}_{r \in R}$ from the right-hand side of (B.17) by solving (B.27) with respect to $D_\nu^r u$ for each $r \in R$. We denote furthermore

$$\begin{aligned} r' &= 2m - 1 - r \quad \text{for } r \in R, & R' &= \{r'\}_{r \in R}, \\ t' &= 2m - 1 - t \quad \text{for } t \in T, & T' &= \{t'\}_{t \in T}. \end{aligned}$$

Then, (B.17) is rewritten as

$$B[u, v] = \sum_{r \in R} B_{(r)} u \cdot \overline{C'_{(r')} v} + \sum_{t \in T} D_\nu^t u \cdot \overline{B'_{(t')} v}, \tag{B.28}$$

where $B'_{(t')}$ and $C'_{(r')}$ are of type

$$\begin{aligned} B'_{(t')} v &= \overline{a_{2m}} D_\nu^{t'} v + \sum_{k=0}^{t'-1} \beta'_{t',k} D_\nu^k v, & t' &\in T', \\ C'_{(r')} v &= \overline{a_{2m}} D_\nu^{r'} v + \sum_{k=0}^{r'-1} \gamma'_{r',k} D_\nu^k v, & r' &\in R', \end{aligned} \tag{B.29}$$

$\beta'_{t',k}$ and $\gamma'_{r',k}$ are differential operators on S (ord $\beta'_{t',k} \le t' - k$, ord $\gamma'_{r',k} \le r' - k$). Note that $\{B'_{(t')}\}_{t' \in T'}$ is also m-normal because $\{B'_{(t')}\}_{t' \in T'} \cup \{C'_{(r')}\}_{r' \in R'}$ is a Dirichlet system.

We show that $\{A^*, \{B_{(t')}\}_{t' \in T'}\} = \{A, \{B_{(r)}\}_{r \in R}\}^*$. Obviously by (B.28), (B.25) and (B.26) imply (B.24). So, we have only to show that (B.24) and (B.25) imply (B.26). Since $\{B_{(r)}\}_{r \in R} \cup \{D^t_\nu\}_{t \in T}$ is a Dirichlet system, there exists a $u \in C^\infty(\overline{\Omega})$ satisfying

$$B_{(r)} u = 0 \text{ for } r \in R \quad \text{and } D^t_\nu u = B'_{(t')} v \quad \text{for } t \in T$$

on S for a given $v \in C^\infty(\overline{\Omega})$. For these v and u, we have

$$B[u, v] = \sum_{t \in T} |B'_{(t')} v|^2 .$$

And hence (B.24) and (B.25) imply (B.26). Any other system $\{\widetilde{B}'_j\}_{j=1}^m$ is equivalent to this $\{B'_{(t')}\}_{t' \in T'}$ if $\{A^*, \{\widetilde{B}'_j\}_{j=1}^m\} = \{A, \{B_{(r)}\}_{r \in R}\}^*$. □

DEFINITION B.3. The general boundary value problem $\{A, \{B_j\}_{j=1}^m\}$ is called *Hermitian* or *formally selfadjoint* with respect to $\mu\, dx$ if and only if

$$\{A, \{B_j\}_{j=1}^m\}^* = \{A, \{B_j\}_{j=1}^m\}. \tag{B.30}$$

If $\{A, \{B_{(r)}\}_{r \in R}\}$ is Hermitian, then at first $A^* = A$, $T' = R$ (or equivalently $R' \cap R = \varnothing$) and further algebraic conditions on the symbols of A and $\{B_{(r)}\}$ should be satisfied (see Chapter VI, §6). The simplest case is the Dirichlet problem. In this case, we have

$$\{A, \{D^{j-1}_\nu\}_{j=1}^m\}^* = \{A^*, \{D^{j-1}_\nu\}_{j=1}^m\}. \tag{B.31}$$

So, it is Hermitian if $A^* = A$.

The results of this section are correct not only in the case where the coefficients of A, $\{B_j\}$, and S are of class C^∞ but also in the case where these are sufficiently smooth so that the computations make sense.

Readers are invited to refer to N. S. Aronszajn and A. N. Milgram [7] and M. Schechter [108] for the contents of this section.

§C. Preliminaries from ordinary differential equations

One of the best approaches to the general boundary value problems is to study an ordinary differential operator $a(D_t)$ on the half-line $\mathbf{R}_+ = (0, +\infty)$ under a boundary condition $\{B_\alpha\}$ at $t = 0$. We can then understand what kind of algebraic condition for $\{B_\alpha\}$ with respect to $a(D_t)$ is necessary for the existence and uniqueness of solutions to hold. And from this, the context of Chapter VI will be made clear.

In this book, we study mainly single elliptic equations. Therefore, it suffices to treat single ordinary differential equations of order l. However, in this section, we discuss at the same time first order systems of ordinary differential equations with l unknowns, because the discussions are quite parallel.

Throughout the section, t denotes a real independent variable. We discuss a system of l equations of first order with l unknown functions:

$$D_t\mathbf{u}(t) - M\mathbf{u}(t) = \mathbf{f}(t), \qquad D_t = -i\frac{d}{dt}. \tag{C.1}$$

where $\mathbf{u}(t) = {}^t(u_1(t), \dots, u_l(t))$ is a column vector of complex-valued unknowns, $\mathbf{f}(t)$ is a column vector of given functions and M is a given $l \times l$ constant complex matrix. At the same time we discuss a single equation of order l:

$$a(D_t)u(t) = f(t), \tag{C.2}$$

where $u(t)$ is a complex-valued unknown function, $f(t)$ is a complex-valued given function and $a(\lambda)$ is a given monic of degree l:

$$a(\lambda) = \sum_{p=0}^{l} a_p\lambda^p, \qquad a_p \in \mathbf{C} \quad (0 \le p \le l), \quad a_l = 1, \tag{C.3}$$

and the a_p's are independent of t.

(C.2) is reduced to a system of type (C.1) if we put

$$\mathbf{u}(t) = {}^t(u(t), D_tu(t), \dots, D_t^{l-1}u(t)), \qquad \mathbf{f}(t) = {}^t(0, \dots, 0, f(t)), \tag{C.4}$$

and $M = M_a$, where

$$M_a = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & \cdots & & \\ 0 & 0 & \cdots & \cdots & 1 \\ -a_0 & -a_1 & \cdots & & -a_{l-1} \end{pmatrix}. \tag{C.5}$$

Denoting the unit matrix of order l by I, we have

$$\det(\lambda I - M_a) = a(\lambda). \tag{C.6}$$

For (C.1) and (C.2), we set

$$\sigma_M = \{\lambda \in \mathbf{C};\, \det(\lambda I - M) = 0\}, \qquad \sigma_a = \{\lambda \in \mathbf{C};\, a(\lambda) = 0\}. \tag{C.7}$$

A. Hypotheses. Since we are discussing (C.1) and (C.2) as models of elliptic equations, we assume that

$$\sigma_M \cap \mathbf{R} = \varnothing, \qquad \sigma_a \cap \mathbf{R} = \varnothing. \tag{C.8}$$

So, σ_M and σ_a are divided by the real-axis into two parts

$$\sigma_M^{\pm} = \{\lambda \in \sigma_M;\, \pm\Im\lambda > 0\}, \qquad \sigma_a^{\pm} = \{\lambda \in \sigma_a;\, \pm\Im\lambda > 0\}. \tag{C.9}$$

Let γ be a simple closed rectifiable curve in a complex λ-plane enclosing σ_M or σ_a. Similarly, let $\gamma^{\pm}$ be simple closed rectifiable curves in the upper (or lower) half plane enclosing $\sigma_M^{\pm}$ or $\sigma_a^{\pm}$. These are oriented in the counter-clockwise sense. We define

$$P_{\pm} = \frac{1}{2\pi i}\int_{\gamma^{\pm}} (\lambda I - M)^{-1}\, d\lambda. \tag{C.10}$$

The matrices $P_\pm$ satisfy the following:

$$P_+ + P_- = I, \qquad P_\pm^2 = P_\pm, \qquad P_+P_- = P_-P_+ = 0. \tag{C.11}$$

Similarly, $a(\lambda)$ is factored as

$$a(\lambda) = a^+(\lambda)a^-(\lambda), \tag{C.12}$$

where $a^+(\lambda)$ has zeros only in the upper half plane, while $a^-(\lambda)$ has zeros only in the lower half plane. The $a^\pm(\lambda)$ are assumed to be monics. We set

$$r^\pm = \operatorname{rank} P_\pm \text{ for (C.1)} \quad \text{and} \quad r^\pm = \deg a^\pm(\lambda) \text{ for (C.2)} \tag{C.13}$$

(note that $\operatorname{rank} P_\pm = \deg a^\pm(\lambda)$ if $M = M_a$). Then $r^\pm$ is equal to the sum of multiplicities of eigenvalues of M or of zeros of $a(\lambda)$ in the upper or lower half plane. (Naturally, $r^\pm \geq 0$ and $r^+ + r^- = l$.)

B. Particular solutions of (C.1) and (C.2). We define a matrix-valued kernel $K(t)$ and a complex-valued kernel $k(t)$ by

$$K(t) = \lim_{N\uparrow+\infty} \frac{1}{2\pi} \int_{-N}^{+N} e^{i\lambda t}(\lambda I - M)^{-1}\, d\lambda,$$
$$k(t) = \lim_{N\uparrow+\infty} \frac{1}{2\pi} \int_{-N}^{+N} \frac{e^{i\lambda t}}{a(\lambda)}\, d\lambda. \tag{C.14}$$

We do not define them at $t = 0$ but

$$K(t) = ie^{itM}P_+, \qquad k(t) = \frac{1}{2\pi}\int_{\gamma^+} \frac{e^{i\lambda t}}{a(\lambda)}\, d\lambda \quad \text{if } t > 0;$$
$$K(t) = -ie^{itM}P_-, \qquad k(t) = \frac{-1}{2\pi}\int_{\gamma^-} \frac{e^{i\lambda t}}{a(\lambda)}\, d\lambda \quad \text{if } t < 0.$$

Then $K(t)$ and $k(t)$ are elementary solutions of $D_t - M$ and $a(D_t)$ respectively.

PROPOSITION C.1. *If $\mathbf{f}(t)$ (or $f(t)$) is bounded and measurable on $\mathbf{R}$, there exists one and only one bounded solution $\mathbf{u}(t)$ of (C.1) (or $u(t)$ of (C.2)) which is given by*

$$\mathbf{u}(t) = \int_{-\infty}^{+\infty} K(t-s)\mathbf{f}(s)\, ds \quad (\textit{resp. } u(t) = \int_{-\infty}^{+\infty} k(t-s)f(s)\, ds). \tag{C.15}$$

PROOF. We have $(D_t - M)K(t) = O$, $a(D_t)k(t) = 0$ except at $t = 0$ and

$$K(0+) - K(0-) = iI,$$
$$D_t^{j-1}k(0+) - D_t^{j-1}k(0-) = \begin{cases} 0 & \text{if } 1 \leq j \leq l-1, \\ i & \text{if } j = l. \end{cases} \tag{C.16}$$

Moreover, $K(t)$, $k(t)$, $D_tk(t)$, …, $D_t^{l-1}k(t)$ are absolutely integrable on $\mathbf{R}$. Therefore, the solutions given in (C.15) are bounded. Uniqueness will be proved in part D. □

C. General solutions of homogeneous equations. Next we solve the initial value problems

$$D_t\mathbf{u}(t) - M\mathbf{u}(t) = \mathbf{0}, \tag{C.1$_0$}$$
$$\mathbf{u}(0) = \mathbf{u}_0; \tag{C.17}$$

and

$$a(D_t)u(t) = 0, \tag{C.2$_0$}$$
$$D_t^{j-1}u(0) = u_j, \qquad 1 \le j \le l. \tag{C.18}$$

For the latter, we need a system of polynomials

$$a_k(\lambda) = \sum_{p=k}^{l} a_p\lambda^{p-k}, \qquad 1 \le k \le l. \tag{C.19}$$

PROPOSITION C.2. *The solution of* $(\mathrm{C.1})_0$–(C.17) (*or* $(\mathrm{C.2})_0$–(C.18)) *is given by*

$$\mathbf{u}(t) = e^{itM}\mathbf{u}_0 \quad (\textit{resp. } u(t) = \frac{1}{2\pi i}\int_\gamma \frac{e^{i\lambda t}}{a(\lambda)}\sum_{k=1}^{l} u_k a_k(\lambda)\, d\lambda). \tag{C.20}$$

PROOF. The first formula is well known. To obtain the second, we remark that

$$\lambda^{j-1}a_k(\lambda) = \begin{cases} \lambda^{j-k-1}a(\lambda) + p_{j,k}(\lambda) & \text{if } 1 \le k < j \le l, \\ \delta_{jk}\lambda^{l-1} + p_{j,k}(\lambda) & \text{if } 1 \le j \le k \le l, \end{cases}$$

where $p_{j,k}(\lambda)$ are polynomials of degree at most $l-2$. From this, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{\lambda^{j-1}a_k(\lambda)}{a(\lambda)}\, d\lambda = \delta_{jk}, \qquad 1 \le j, k \le l. \tag{C.21}$$

Therefore, $u(t)$ defined as above satisfies (C.18). □

D. Bounded solutions on $\mathbf{R}_+ = (0, +\infty)$.

PROPOSITION C.3. (i) *The solution of* $(\mathrm{C.1})_0$–(C.17) *is bounded on* $\mathbf{R}_+$ *if and only if* $P_-\mathbf{u}_0 = \mathbf{0}$.

(ii) *The solution of* $(\mathrm{C.2})_0$–(C.18) *is bounded on* $\mathbf{R}_+$ *if and only if* $a^-(\lambda)$ *divides* $\sum_{k=1}^{l} u_k a_k(\lambda)$.

(iii) *No solution of* $(\mathrm{C.1})_0$ (*or* $(\mathrm{C.2})_0$) *other than* $\mathbf{0}$ (*or* 0) *is bounded on* $\mathbf{R}$.

PROOF. If $M = M_a$, we can verify that

$$(j, k)\text{-entry of } (\lambda I - M_a)^{-1} = \frac{\lambda^{j-1}a_k(\lambda)}{a(\lambda)} + \text{polynomial}, \tag{C.22}$$

and hence

$$(j, k)\text{-entry of } P_\pm = \frac{1}{2\pi i}\int_{\gamma^\pm} \frac{\lambda^{j-1}a_k(\lambda)}{a(\lambda)}\, d\lambda, \qquad 1 \le j, k \le l. \tag{C.23}$$

So, (i) implies (ii). Also, (i) and (ii) imply (iii) (by replacing t by $-t$ for $t < 0$). Therefore, it suffices to show (i).

For each $\zeta \in \sigma_M$, we set

$$\begin{aligned} P_\zeta &= \frac{1}{2\pi i}\int_{|\lambda-\zeta|=\varepsilon} (\lambda I - M)^{-1}\, d\lambda, \\ N_\zeta &= \frac{1}{2\pi i}\int_{|\lambda-\zeta|=\varepsilon} (\lambda-\zeta)(\lambda I - M)^{-1}\, d\lambda, \end{aligned} \tag{C.24}$$

where ε is a small positive number. The following properties of P_ζ and N_ζ are known in linear algebra: P_ζ and N_ζ commute with M; P_ζ is idempotent and N_ζ is nilpotent;

$$\begin{aligned} &N_\zeta = N_\zeta P_\zeta = P_\zeta N_\zeta, \qquad P_\zeta P_{\zeta'} = O \quad \text{if } \zeta \neq \zeta'; \\ &P_\pm = \sum_{\zeta\in\sigma_M^\pm} P_\zeta, \qquad e^{itM}P_\zeta = e^{it\zeta}e^{itN_\zeta}P_\zeta. \end{aligned} \tag{C.24$'$}$$

Assertion (i) follows from the last two of these. □

In what follows, we are interested only in bounded solutions on $\mathbf{R}_+$. So, we say a solution is *bounded* if it is bounded on $\mathbf{R}_+$. Boundedness is equivalent to the exponential decay as $t \to +\infty$ because no eigenvalue of M (no zero of $a(\lambda)$) is real.

Let E^+ be the set of $\mathbf{u}_0 \in \mathbf{C}^l$ for which $\mathbf{u}(t)$ is bounded. By Proposition C.3 and (C.11), E^+ is a linear subspace of $\mathbf{C}^l$ and it is characterized as follows:

$$\begin{aligned} E^+ &= P_+\mathbf{C}^l = \operatorname{Ker} P_- \\ &= \{\mathbf{v} \in \mathbf{C}^l\, ;\, (\lambda I - M)^{-1}\mathbf{v} \text{ is holomorphic in } \Im\lambda \le 0\}. \end{aligned} \tag{C.25}$$

Changing the sign, we put $E^- = P_-\mathbf{C}^l$. Then we have

$$\mathbf{C}^l = E^+ + E^- \quad \text{and} \quad E^+ \cap E^- = \{\mathbf{0}\}. \tag{C.26}$$

We can also define $E^\pm$ for $(\mathrm{C}.2)_0$ by setting

$$E^\pm = \{{}^t(v_1, \dots, v_l) \in \mathbf{C}^l\, ;\, a^\mp(\lambda) \text{ divides } \sum_{k=1}^{l} v_k a_k(\lambda)\}. \tag{C.25$'$}$$

In any case, (C.13) implies

$$\dim E^\pm = r^\pm. \tag{C.27}$$

So, the dimension of the vector space of bounded solutions of $(\mathrm{C}.1)_0$ or $(\mathrm{C}.2)_0$ is equal to r^+.

E. Dirichlet problem for $(\mathrm{C}.2)_0$ as $r^+ > 0$. Next we solve $(\mathrm{C}.2)_0$ under the Dirichlet boundary condition

$$D_t^{\alpha-1}u(0) = c_\alpha, \qquad 1 \le \alpha \le r^+. \tag{C.28}$$

The bounded solution of $(C.2)_0$–(C.28) is unique. (If $r^- > 0$, there are infinitely many unbounded solutions.) Analogously to (C.19), we define $\{a_\alpha^+(\lambda)\}$ by means of $a^+(\lambda)$:

$$a_\alpha^+(\lambda) = \sum_{k=\alpha}^{r^+} a_k^+ \lambda^{k-\alpha} \quad \text{for } 1 \le \alpha \le r^+ \quad \text{if } a^+(\lambda) = \sum_{k=0}^{r^+} a_k^+ \lambda^k. \tag{C.29}$$

PROPOSITION C.4. *For every* ${}^t(c_1, \ldots, c_{r^+}) \in \mathbf{C}^{r^+}$, *there exists one and only one bounded solution of* $(C.2)_0$–(C.28) *and it is given by*

$$u(t) = \frac{1}{2\pi i} \int_{\gamma^+} \frac{e^{i\lambda t}}{a^+(\lambda)} \sum_{\alpha=1}^{r^+} c_\alpha a_\alpha^+(\lambda)\, d\lambda. \tag{C.30}$$

PROOF. Similarly to the proof of Proposition C.2, we make use of

$$\frac{1}{2\pi i} \int_{\gamma^+} \frac{\lambda^{\alpha-1} a_\beta^+(\lambda)}{a^+(\lambda)}\, d\lambda = \delta_{\alpha\beta} \quad \text{for } 1 \le \alpha, \beta \le r^+. \quad \square \tag{C.31}$$

If we set

$$s_{p\alpha} = \frac{1}{2\pi i} \int_{\gamma^+} \frac{\lambda^{p-1} a_\alpha^+(\lambda)}{a^+(\lambda)}\, d\lambda \quad \text{for } 1 \le \alpha \le r^+ < p \le l, \tag{C.32}$$

we have

$$D_t^{p-1} u(0) = \sum_{\alpha=1}^{r^+} s_{p\alpha} D_t^{\alpha-1} u(0), \qquad r^+ + 1 \le p \le l \tag{C.33}$$

for every bounded solution $u(t)$ of $(C.2)_0$. On other words,

$$v_p = \sum_{\alpha=1}^{r^+} s_{p\alpha} v_\alpha \qquad (r^+ + 1 \le p \le l) \quad \text{if } {}^t(v_1, \ldots, v_l) \in E^+. \tag{C.34}$$

In this way, the Dirichlet data of bounded solutions of $(C.2)_0$ determine the Cauchy data.

F. General boundary conditions as $r^+ > 0$. We now formulate a model of the general boundary value problems studied in Chapter VI. For equation (C.1), a general boundary condition is of type

$$B\mathbf{u}(0) = \vec{c}, \tag{C.35}$$

where B is a given $r^+ \times l$ complex matrix. For equation (C.2), the boundary condition is of type

$$B_\alpha u(0) = \sum_{p=1}^{l} b_{\alpha p} D_t^{p-1} u(0) = c_\alpha, \qquad 1 \le \alpha \le r^+, \tag{C.36}$$

where $b_{\alpha p}$ are given complex numbers (if we reduce (C.2) to (C.1) as in part A, $(b_{\alpha p})$ plays the role of B in (C.35)). Vectors $\vec{c}$ and ${}^t(c_1, \ldots, c_{r^+})$ are given elements of $\mathbf{C}^{r^+}$. The Dirichlet condition is the simplest among all general boundary conditions for (C.2). On the contrary for (C.1), there is no particular one which can be considered as *standard*.

The matrix B or $\{B_\alpha\}_{\alpha=1}^{r^+}$ determining a general boundary condition may not be arbitrary but should satisfy a condition for the sake that the setting be natural. There are a number of versions, equivalent to one another, to represent this: (a)–(e) for (C.35) and (a′)–(f′) for (C.36). We need further notation for the latter:

$$b_\alpha(\lambda) = \sum_{p=1}^{l} b_{\alpha p}\lambda^{p-1}, \quad \tilde{b}_{\alpha\beta} = \frac{1}{2\pi i}\int_{\gamma^+} \frac{b_\alpha(\lambda)a_\beta^+(\lambda)}{a^+(\lambda)}\,d\lambda, \qquad 1 \le \alpha, \beta \le r^+. \tag{C.37}$$

Then $(\tilde{b}_{\alpha\beta})$ is a square matrix of order r^+.

DEFINITION. The general boundary condition B (or $\{B_\alpha\}_{\alpha=1}^{r^+}$) is said to satisfy the *Condition of Shapiro-Lopatinski* with respect to $D_t - M$ (resp. $a(D_t)$) or *covers* $D_t - M$ (resp. $a(D_t)$) if and only if one of (a)–(e) (resp. (a′)–(f′)) below holds:

(a)(a′) For the bounded solution $\mathbf{u}(t)$ of $(\text{C.1})_0$ (resp. $u(t)$ of $(\text{C.2})_0$), $B\mathbf{u}(0) = \vec{0}$ implies $\mathbf{u}(t) = \mathbf{0}$ (resp. $B_\alpha u(0) = 0$, $1 \le \alpha \le r^+$, imply $u(t) = 0$).

(b)(b′) The inequality

$$\sum_{j=0}^{l} \sup_{\mathbf{R}_+} |D_t^j \mathbf{u}(t)| \le K|B\mathbf{u}(0)|$$

$$\left(\text{resp. } \sum_{j=0}^{l} \sup_{\mathbf{R}_+} |D_t^j u(t)| \le K \sum_{\alpha=1}^{r^+} |B_\alpha u(0)|\right)$$

holds for bounded solutions, where K is a positive number independent of $\mathbf{u}$ (resp. u).

(c)(c′) For every $\vec{c}$ (resp. ${}^t(c_1, \dots, c_{r^+})) \in \mathbf{C}^{r^+}$, there exists one and only one bounded solution of $(\text{C.1})_0$–(C.35) (resp. $(\text{C.2})_0$–(C.36)).

(d)(d′) $\operatorname{rank}(BP_+) = r^+$ (resp. $\det(\tilde{b}_{\alpha\beta})_{\alpha,\beta=1}^{r^+} \ne 0$).

(e)(e′) There exists an $l \times r^+$ matrix Q satisfying

$$Q = P_+Q, \quad QBP_+ = P_+, \quad \text{and} \quad BQ = I$$

$$(\text{resp. } (\tilde{b}_{\alpha\beta})_{\alpha,\beta=1}^{r^+} \text{ has inverse } (\rho_{\alpha\beta})_{\alpha,\beta=1}^{r^+}).$$

(f′) $\{b_\alpha(\lambda)\}_{\alpha=1}^{r^+}$ are linearly independent modulo $a^+(\lambda)$.

PROPOSITION C.5. *For* $(\text{C.1})_0$–(C.35) (*or* $(\text{C.2})_0$–(C.36)), *conditions* (a)–(e) (*resp.* (a′)–(f′)) *above are equivalent one another. If one of them is satisfied,* *there exists one and only one bounded solution and it is given by*

$$\mathbf{u}(t) = e^{itM}Q\overrightarrow{c}$$

$$\left(resp.\ u(t) = \frac{1}{2\pi i}\int_{\gamma^+}\frac{e^{i\lambda t}}{a^+(\lambda)}\sum_{\alpha,\beta=1}^{r^+} c_\beta \rho_{\alpha\beta} a_\alpha^+(\lambda)\,d\lambda \right). \tag{C.38}$$

PROOF. Clearly, (a) $\Leftrightarrow$ (b) and (d) $\Leftrightarrow$ (e). Proposition C.3 implies

$$\sum_{j=0}^{l}\sup_{\mathbf{R}_+}|D_t^j\mathbf{u}(t)| \le K|P_+\mathbf{u}(0)|$$

for any bounded solution of $(C.1)_0$ with a positive number K independent of $\mathbf{u}$. We can replace $|P_+\mathbf{u}(0)|$ by $|B\mathbf{u}(0)|$ of and only if (a) or (b) holds. And hence (a), (b), (d), and (e) are equivalent. Any one of these is equivalent to (c) by virtue of the formula (C.38). Similarly, we may prove equivalence of (a′)–(f′). □

G. General boundary value problems as $r^+ > 0$. We are going to write down the solution of the inhomogeneous problem (C.1)–(C.35) (or (C.2)–(C.36)). For this, we set

$$G(t, s) = K(t - s) - e^{itM}Q\{BK(\tau - s)\}|_{\tau\downarrow 0},$$

$$g(t, s) = k(t-s) - \sum_{\alpha=1}^{r^+}\sum_{p=1}^{l}\rho_\alpha(t)b_{\alpha p}\{D_\tau^{p-1}k(\tau - s)\}|_{\tau\downarrow 0} \tag{C.39}$$

for $t, s > 0$ and $t \neq s$, where

$$\rho_\alpha(t) = \frac{1}{2\pi i}\int_{\gamma^+}\frac{e^{i\lambda t}}{a^+(\lambda)}\sum_{\beta=1}^{r^+}\rho_{\beta\alpha}a_\beta^+(\lambda)\,d\lambda, \qquad 1 \le \alpha \le r^+. \tag{C.40}$$

$K(t, s)$ (or $k(t, s)$) is called the *Green function* for (C.1)–(C.35) (resp. (C.2)–(C.36)) and $e^{itM}Q$ (resp. $\{\rho_\alpha(t)\}_{\alpha=1}^{r^+}$) is called the *Poisson kernel(s)*.
The following is a consequence of Proposition C.1 and C.5.

PROPOSITION C.6. *Suppose that a general boundary condition B (or $\{B_\alpha\}$) satisfies the condition of Shapiro-Lopatinski for $D_t - M$ (resp. $a(D_t)$). Then, every bounded function $\mathbf{f}(t)$ (resp. $f(t)$) on $\mathbf{R}_+$ and every $\overrightarrow{c}$ (resp. ${}^t(c_1, \dots, c_{r^+})) \in \mathbf{C}^{r^+}$, there exists one and only one solution $\mathbf{u}(t)$ of* (C.1)–(C.35) (*resp.* $u(t)$ *of* (C.2)–(C.36)) *and it is given by*

$$\mathbf{u}(t) = \int_0^{+\infty} G(t, s)\mathbf{f}(s)\,ds + e^{itM}Q\overrightarrow{c}$$

$$\left(resp.\ u(t) = \int_0^{+\infty} g(t, s)f(s)\,ds + \sum_{\alpha=1}^{r^+}c_\alpha\rho_\alpha(t) \right). \tag{C.41}$$

H. Case where $r^+ = 0$. In this case, the bounded solution of (C.1) (or (C.2)) is unique without imposing any boundary condition at $t = 0$.

PROPOSITION C.7. *For every bounded function* $\mathbf{f}(t)$ (*or* $f(t)$) *on* $\mathbf{R}_+$, *there exists one and only one bounded solution* $\mathbf{u}(t)$ *of* (C.1) (*resp.* $u(t)$ *of* (C.2)) *and it is given by* (C.15).

I. Summary. The formulas for solutions obtained above are available also in the L^p-framework without any modification. Therefore, we summarize the results of this section in the following way by making use of Sobolev spaces which are defined in §E.

PROPOSITION C.8. *Suppose that* $1 \le p \le +\infty$.

(i) *If* $r^+ > 0$ *and* B (*or* $\{B_\alpha\}_{\alpha=1}^{r^+}$) *satisfies the condition of Shapiro-Lopatinski with respect to* $D_t - M$ (*resp.* $a(D_t)$), *we have an isomorphism*

$$(D_t - M, B) \text{ from } \{W^{1,p}(\mathbf{R}_+)\}^l \text{ onto } \{L^p(\mathbf{R}_+)\}^l \times \mathbf{C}^{r^+}$$

$$(\text{resp. } (a(D_t), B_1, \dots, B_{r^+}) \text{ from } W^{l,p}(\mathbf{R}_+) \text{ onto } L^p(\mathbf{R}_+) \times \mathbf{C}^{r^+});$$

(ii) *if* $r^+ = 0$, *we have an isomorphism*

$$D_t - M \text{ from } \{W^{1,p}(\mathbf{R}_+)\}^l \text{ onto } \{L^p(\mathbf{R}_+)\}^l$$

$$(\text{resp. } a(D_t) \text{ from } W^{l,p}(\mathbf{R}_+) \text{ onto } L^p(\mathbf{R}_+)).$$

§D. Fredholm operators

In this section, $X, Y, Z, \dots$ are assumed to be Banach spaces (in general of infinite dimension). We denote by $\mathscr{L}(X, Y)$ the set of continuous linear mappings from X to Y. Then $\mathscr{L}(X, Y)$ is a Banach space equipped with the operator norm. Let $\mathscr{K}(X, Y)$ be the set of compact (or completely continuous) linear mappings from X to Y. $\mathscr{K}(X, Y)$ is a closed linear subspace of $\mathscr{L}(X, Y)$. Furthermore, let $\mathscr{L}_f(X, Y)$ be the set of elements of $\mathscr{L}(X, Y)$ which are of finite rank. $\mathscr{L}_f(X, Y)$ is a dense linear subspace of $\mathscr{K}(X, Y)$. We denote the identity mapping in X by I_X.

DEFINITION D.1. A mapping u is called a *Fredholm operator* if it satisfies four conditions:

(i) $u \in \mathscr{L}(X, Y)$;

(ii) $\dim \operatorname{Ker} u < +\infty$;

(iii) $\operatorname{Im} u$ (image of X by u) is a closed linear subspace of Y;

(iv) $\dim \operatorname{Coker} u < +\infty$, where $\operatorname{Coker} u = Y / \operatorname{Im} u$.

If these are satisfied, the integer

$$\chi(u) = \dim \operatorname{Ker} u - \dim \operatorname{Coker} u \tag{D.1}$$

is called the *index* of u. The set of Fredholm operators X to Y is denoted by $\Phi(X, Y)$.

In this way, a Fredholm operator (called also Fredholm mapping, Φ-operator on Φ-mapping) is a linear mapping which is invertible *modulo* a subspace of finite dimension. Denoting by $\mathbf{Z}$ the set of integers, χ is a $\mathbf{Z}$-valued function defined in $\Phi(X, Y)$. As we shall see in Theorem D.3, χ is continuous. What is interesting in the definition is that elements of $\Phi(X, Y)$ are classified by their indices. That is, if we set

$$\Phi_n(X, Y) = \{u \in \Phi(X, Y); \chi(u) = n\} \quad \text{for } n \in \mathbf{Z}, \tag{D.2}$$

then $\Phi(X, Y)$ is equal to the disjoint union of $\Phi_n(X, Y)$'s:

$$\Phi(X, Y) = \bigcup_{n=-\infty}^{+\infty} \Phi_n(X, Y), \qquad \Phi_m(X, Y) \cap \Phi_n(X, Y) = \emptyset \quad \text{if } m \neq n. \tag{D.3}$$

The structure of $\Phi_n(X, Y)$ will be studied in (a)–(f) below. First, we summarize some of the basic properties of indices.

THEOREM D.1. *An element u of $\mathscr{L}(X, Y)$ belongs to $\Phi(X, Y)$ if and only if there exists an element u' of $\mathscr{L}(Y, X)$ such that $u' \circ u - I_X \in \mathscr{K}(X, X)$ and that $u \circ u' - I_Y \in \mathscr{K}(Y, Y)$. Then, u' belongs to $\Phi(Y, X)$ and we have*

$$\chi(u) + \chi(u') = 0. \tag{D.4}$$

Analogously,

THEOREM D.2. *An element u of $\mathscr{L}(X, Y)$ belongs to $\Phi(X, Y)$ if and only if there exists an element $u' \in \mathscr{L}(Y, X)$ such that $u' \circ u - I_X \in \mathscr{L}_f(X, X)$ and that $u \circ u' - I_Y \in \mathscr{L}_f(Y, Y)$. Then, $u' \in \Phi(Y, X)$ and* (D.4) *holds.*

The index remains invariant under a large class of perturbations. To see this, we need

DEFINITION D.2. We define

$$|u|_c = \inf\{\|u + k\|; k \in \mathscr{K}(X, Y)\}. \tag{D.5}$$

($|\ |_c$ is not a norm but a seminorm because $|u|_c = 0$ if $u \in \mathscr{K}(X, Y)$. Note also that $|u|_c \le \|u\|$.)

THEOREM D.3. *Given an element $u \in \Phi(X, Y)$, there exists a positive number r (depending on u) such that, if $w \in \mathscr{L}(X, Y)$ and if $|w|_c < r$, then $u + w \in \Phi(X, Y)$ and*

$$\chi(u + w) = \chi(u). \tag{D.6}$$

From these, we can deduce some results on the question of classification.

(a) (D.3) *is a classification of $\Phi(X, Y)$ modulo $\mathscr{K}(X, Y)$.*

(b) *Each of $\{\Phi_n(X, Y)\}_{n=-\infty}^{+\infty}$ is an open subset of $\mathscr{L}(X, Y)$, and hence $\Phi(X, Y)$ is also an open subset of $\mathscr{L}(X, Y)$.*

The following propositions and corollaries may be helpful to compute indices of concrete operators.

Proposition D.4. *If* $u \in \Phi(X, Y)$, *the transposed operator* ${}^t u$ *belongs to* $\Phi(Y', X')$ (X' *is the dual space of* X) *and we have*

$$\chi(u) + \chi({}^t u) = 0. \tag{D.7}$$

Proposition D.5. *Given* $u_1 \in \Phi(X_1, Y_1)$ *and* $u_2 \in \Phi(X_2, Y_2)$, *define* $u = (u_1, u_2)$ *by setting* $u(x_1, x_2) = (u_1(x_1), u_2(x_2))$. *Then,* $u \in \Phi(X_1 \times X_2, Y_1 \times Y_2)$ *and we have*

$$\chi(u) = \chi(u_1) + \chi(u_2). \tag{D.8}$$

Proposition D.6. *If* $u \in \Phi(X, Y)$ *and* $v \in \Phi(Y, Z)$, *then* $v \circ u \in \Phi(X, Z)$ *and we have*

$$\chi(v \circ u) = \chi(u) + \chi(v). \tag{D.9}$$

Proposition D.7 (Theorem of F. Riesz and J. Schauder).

$$I_X + k \in \Phi_0(X, X) \quad \text{if } k \in \mathscr{K}(X, X).$$

As is known, Proposition D.7 generalizes the alternative theorem of I. Fredholm. Theorems D.1–D.3 are consequences of Proposition D.4–D.7. Theorem D.3 and Proposition D.7 imply Corollary D.8 below and Theorem D.1 (or D.2) and Proposition D.6 imply Corollary D.9.

Corollary D.8. *If* $u \in \mathscr{L}(X, X)$ *and* $|u|_c < 1$, *then* $I_X + u \in \Phi_0(X, X)$.

Corollary D.9. *Suppose that* $u \in \mathscr{L}(X, Y)$ *and* $v \in \mathscr{L}(Y, Z)$. *If two of* u, v, *and* $v \circ u$ *are Fredholm operators, then the other is also a Fredholm operator and* (D.9) *holds.*

We consider again the question of classification. By Proposition D.6, D.8, and Theorem D.1, we see that

(c) $\Phi_0(X, X)$ *is nonempty because it contains* I_X;

(d) *if* $\Phi_1(X, X)$ *is nonempty, then for every* $n \in \mathbf{Z}$, $\Phi_n(X, X)$ *is continuously in one-to-one correspondence with* $\Phi_0(X, X)$ *modulo* $\mathscr{K}(X, X)$, *that is, any two of* $\Phi_n(X, X)$*'s are homeomorphic;*

(e) *if one of* $\Phi_1(X, X)$ *and* $\Phi_1(Y, Y)$ *is nonempty, then* $\{\Phi_n(X, Y)\}_{n=-\infty}^{+\infty}$ *has properties analogous to those in* (d);

(f) $\Phi(X, X)$ *is a topological group modulo* $\mathscr{K}(X, X)$ *with respect to the operator product, and the index* χ *is a representation of this group.*

Here is an important fact to notice. The index $\chi(u)$ is fairly stable under perturbations as is shown above, while each of $\dim \operatorname{Ker} u$ and $\dim \operatorname{Coker} u$ is not stable. For example, we have

Proposition D.10. *Suppose that* $u \in \Phi_n(X, Y)$, $n \geq 0$ *and that* $\dim \operatorname{Coker} u > 0$. *Then, for any positive number* ε, *there exists a* $w \in \mathscr{L}_f(X, Y)$ *such that* $\|w\| < \varepsilon$ *and* $\operatorname{Coker}(u+w) = \{0\}$. *Analogously, if* $n \leq 0$

and $\dim \operatorname{Ker} u > 0$, *we can find arbitrarily small perturbation* $w \in \mathscr{L}_f(X, Y)$ *for which* $\operatorname{Ker}(u + w) = \{0\}$.

The contents of this section are collected from I. C. Gohberg and M. G. Krein [34], where there are proofs of the results stated above. The same authors investigated in [35] the integral operators (*Wiener-Hopf operators*) on the half line $\mathbf{R}_+$ defined by

$$f(t) \to Kf(t) = \int_0^{+\infty} K(t-s)f(s)\,ds, \qquad t > 0, \tag{D.10}$$

and they represented $\chi(K)$ by means of the Fourier transform of $K(t)$. This is one of the backgrounds of index theory for elliptic operators due to M. F. Atiyah and I. M. Singer [130]. In §C, we treated ordinary differential operators on the half line. We may restate the results obtained there as examples of Wiener-Hopf operators (the elementary solution of $a(D_t)$ there plays the role of $K(t)$).

§E. Sobolev spaces

As spaces of functions with the aid of which we set up the problems in the theory of partial differential equations, the largest possible spaces are those of all distributions or hyperfunctions. However, these are sometimes so large that, we are not satisfied with solutions found in these spaces. Sobolev spaces which we are going to study are subspaces of L^p-spaces equipped with structures of Banach spaces or Hilbert spaces. Therefore, it is not so difficult to discuss the solutions belonging to these spaces. Hence, Sobolev spaces are very frequently used in the theory of partial differential equations. Sobolev spaces sometimes play essential roles not only for elliptic equations but also for parabolic or hyperbolic equations.

A. Definition of Sobolev spaces. Let Ω be a bounded or unbounded domain of $\mathbf{R}^n$ and p be a real number satisfying $1 \le p < +\infty$. For the moment, we make no assumption on smoothness of the boundary $S = \partial\Omega$. Denote by $L^p(\Omega)$ the vector space of complex-valued measurable functions whose pth powers are summable in Ω. $L^p(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p\,dx\right)^{1/p} \tag{E.1}$$

(more precisely, an element of $L^p(\Omega)$ represents not an individual function but an equivalence class of functions obtained by identifying all functions which coincide except for values on null sets of Ω). Every element of $L^p(\Omega)$ is locally summable in Ω, so it defines a distribution in Ω. Therefore, $L^p(\Omega)$ is a linear subspace of $\mathscr{D}'(\Omega)$ (see Chapter I, §1, part F).

Given a nonnegative integer k, let $W^{k,p}(\Omega)$ be the vector space of elements of $L^p(\Omega)$ whose partial derivatives (in the sense of distributions) up to order k belong to $L^p(\Omega)$. Then, $W^{k,p}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p} . \tag{E.2}$$

In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ and $W^{k,p}(\Omega)$ is a proper subspace of $W^{k-1,p}(\Omega)$ if $k > 0$.

Let γ be a real number satisfying $0 < \gamma < 1$. We denote by $W^{\gamma,p}(\Omega)$ the vector space of elements u of $L^p(\Omega)$ for which the seminorm

$$|u|_{W^{\gamma,p}(\Omega)} = \left\{ \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+\gamma p}} \, dx \, dy \right\}^{1/p} \tag{E.3}$$

is finite. Furthermore, given a nonnegative integer k, let $W^{k+\gamma,p}(\Omega)$ be the set of elements of $W^{k,p}(\Omega)$ whose partial derivatives of order k belong to $W^{\gamma,p}(\Omega)$. Then, $W^{k+\gamma,p}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{W^{k+\gamma,p}(\Omega)} = \left\{ \|u\|^p_{W^{k,p}(\Omega)} + \sum_{|\alpha| = k} |D^\alpha u|^p_{W^{\gamma,p}(\Omega)} \right\}^{1/p} . \tag{E.4}$$

Thus we defined $W^{a,p}(\Omega)$ for any nonnegative real number a. Note that $C_0^\infty(\Omega)$ is contained in any one of the $W^{a,p}(\Omega)$'s. Let $W_0^{a,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in the space $W^{a,p}(\Omega)$. Then, $W_0^{a,p}(\Omega)$ is a closed linear subspace of $W^{a,p}(\Omega)$. (We may verify that $W_0^{a,p}(\Omega) = W^{a,p}(\Omega)$ if $0 \le a \le 1/p$. See Proposition E.10 below.) Note that $W_0^{a,p}(\mathbf{R}^n) = W^{a,p}(\mathbf{R}^n)$ for any a and p.

Assuming that $1 < p < +\infty$, define p' to be $p' = p/(p-1)$ or $1/p + 1/p' = 1$ (so $1 < p' < +\infty$). For $a > 0$, let $W^{-a,p}(\Omega)$ be the dual space of $W_0^{a,p'}(\Omega)$ or the space of continuous linear forms on $W_0^{a,p'}(\Omega)$. So, $W^{-a,p}(\Omega)$ is a Banach space.

In this way, we defined $W^{a,p}(\Omega)$ for $(a, p) \in \mathbf{R} \times (1, +\infty)$ or $(a, p) \in [0, +\infty) \times [1, +\infty)$. Any one of these spaces is called a *Sobolev space* on Ω in the L^p-framework. There are continuous inclusions among these:

$$W^{b,p}(\Omega) \subset W^{a,p}(\Omega) \quad \text{if} \begin{cases} \text{either } 1 < p < +\infty \text{ and} \\ -\infty < a \le b < +\infty, \\ \text{or } p = 1 \text{ and } 0 \le a \le b < +\infty; \end{cases}$$

$$W_0^{b,p}(\Omega) \subset W_0^{a,p}(\Omega) \quad \text{if } 1 \le p < +\infty \text{ and } 0 \le a \le b < +\infty . \tag{E.5}$$

Especially for $p = 2$, the notation H is more popular than W:

$$\begin{aligned} W^{a,2}(\Omega) &= H^a(\Omega) \quad \text{for } -\infty < a < +\infty, \\ W_0^{a,2}(\Omega) &= H_0^a(\Omega) \quad \text{for } 0 \le a < +\infty . \end{aligned} \tag{E.6}$$

$H^a(\Omega)$ and $H_0^a(\Omega)$ are Hilbert spaces.

The index a of the Sobolev space $W^{a,p}(\Omega)$ (or $W_0^{a,p}(\Omega)$) is called the *order*. If $a \geq 0$, this means that any element u of $W^{a,p}(\Omega)$ (or $W_0^{a,p}(\Omega)$) is "a times differentiable" in $L^p(\Omega)$. The notion of differentiability of fractional order is not elementary in general. For $p = 2$, $\Omega = \mathbf{R}^n$, and $0 < \gamma < 1$, we have

$$|u|^2_{H^\gamma(\mathbf{R}^n)} = C(\gamma, n) \int_{\mathbf{R}^n} |\xi|^{2\gamma} |\mathscr{F} u(\xi)|^2 \, d\xi. \tag{E.7}$$

So, the notion is interpreted by means of Fourier transform. For every $a \in \mathbf{R}$, there exist positive numbers $C_j = C_{j,a}$ $(j = 1, 2)$ such that

$$C_1 \|u\|_{H^a(\mathbf{R}^n)} \leq \|\langle\xi\rangle^a \mathscr{F} u(\xi)\|_{L^2(\mathbf{R}^n)} \leq C_2 \|u\|_{H^a(\mathbf{R}^n)} \tag{E.8}$$

holds for every $u \in H^a(\mathbf{R}^n)$, where $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$.

Spaces $W^{a,p}(\Omega)$ with $p \neq 2$ are not used much in this book. So, we study mainly the properties of $H^a(\Omega)$ and $H_0^a(\Omega)$. Some results or methods of proof which we shall present here may no longer be available for the case where $p \neq 2$. Also, for simplicity of notation, we denote sometimes by $\| \ \|$ the norm in $L^2(\Omega)$.

B. Inequality of H. Poincaré.

LEMMA E.1. *Suppose that Ω is contained in the cube $Q = \{x \in \mathbf{R}^n ; |x_j| < a \ (1 \leq j \leq n)\}$. Then the inequality*

$$\|D^\alpha u\|^2 \leq C a^{2(k-|\alpha|)} \sum_{|\beta|=k} \|D^\beta u\|^2 \tag{E.9}$$

holds for every $u \in H_0^k(\Omega)$ and $|\alpha| \leq k$ with a positive number C independent of (a, u).

PROOF. It suffices to prove (E.9) for $k = 1$ and $\alpha = 0$. The smallest eigenvalue of the Dirichlet problem for Laplacian in Q is equal to $4a^2/n\pi^2$ (see Chapter II, §5). Therefore, eigenfunction expansion yields

$$\|u\|^2 \leq \frac{4a^2}{n\pi^2} \sum_{j=1}^n \left\| \frac{\partial u}{\partial x_j} \right\|^2 \tag{E.10}$$

for every $u \in H_0^1(Q)$. Since $H_0^1(\Omega)$ is imbedded in $H_0^1(Q)$ (extension by 0 to outside Ω), (E.10) holds also for $u \in H_0^1(\Omega)$. Consequently, (E.9) is proved for $k = 1$. □

Analogously by making use of the Neumann problem, we may prove that

$$\|u\|^2 \leq C \sum_{j=1}^n \left\| \frac{\partial u}{\partial x_j} \right\|^2 + \frac{1}{|\Omega|} \left| \int_\Omega u(x) \, dx \right|^2 \tag{E.10$'$}$$

holds for every $u \in H^1(\Omega)$ with a positive number C independent of u ($C = 4a^2/\pi^2$ if $\Omega = Q$).

C. Extension mapping from $H^a(\Omega)$ to $H^a(\mathbf{R}^n)$ for $a \geq 0$. The restriction to Ω of an element of $H^a(\mathbf{R}^n)$ is naturally an element of $H^a(\Omega)$. The following is the converse of this. Suppose that Ω is a bounded domain of $\mathbf{R}^n$ whose boundary S is a hypersurface of class C^∞ and that Ω is in only one side of S at every point of S.

LEMMA E.2. *If $a \geq 0$, then for every positive number ε, there exists a linear mapping E from $H^a(\Omega)$ to $H^a(\mathbf{R}^n)$ satisfying three conditions*:

(i) *E is continuous,*

(ii) *$(Eu)|_\Omega = u$ for every $u \in H^a(\Omega)$,*

(iii) *the support of Eu is contained in an ε-neighborhood of Ω.*

PROOF. Suppose that Ω is the half space $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n \,;\, x_n > 0\}$. Let k be a positive integer not smaller than a. For arbitrary k distinct positive numbers $t_1, \ldots, t_k$, let $c_1, \ldots, c_k$ be real numbers satisfying the system of equations

$$\sum_{j=1}^{k} (-t_j)^l c_j = 1 \quad \text{for } 0 \leq l \leq k-1. \tag{E.11}$$

For $u \in H^k(\mathbf{R}^n_+)$, we define $E_0 u$ by

$$(E_0 u)(x) = \begin{cases} u(x) & \text{for } x_n > 0, \\ \displaystyle\sum_{j=1}^{k} c_j u(x_1, \ldots, x_{n-1}, -t_j x_n) & \text{for } x_n < 0. \end{cases} \tag{E.12}$$

Moreover, let $\zeta(x_n)$ be a function of class C^∞ which is equal to 1 for $x_n \geq -\varepsilon/2$ and to 0 for $x_n \leq -\varepsilon$. Finally we put $Eu(x) = \zeta(x_n) E_0 u(x)$. Then, the mapping E has the required properties. Also for the general domain Ω, we may construct an extension mapping E by making use of partition of unity, local change of coordinates which makes S flat, and of E constructed for the half space. □

COROLLARY E.3. *Let E^e (and E^o) be even (resp. odd) extension of functions in $\mathbf{R}^n_+$ to $\mathbf{R}^n$ with respect to x_n. If $u \in H^1(\mathbf{R}^n_+)$, then $E^e u \in H^1(\mathbf{R}^n)$. And $E^o u \in H^1(\mathbf{R}^n)$ if and only if $u \in H^1_0(\mathbf{R}^n_+)$.*

The proof is omitted. See part G below.

D. Theorem of R. Rellich. The following is an L^2-version of the theorem of Ascoli-Arzelà (see R. Rellich [159]).

THEOREM E.4. *Suppose that Ω is bounded and $0 \leq b < a < +\infty$. Then, the imbedding from $H^a_0(\Omega)$ into $H^b_0(\Omega)$ is compact.*

PROOF. We prove the assertion in the case where $a = 1$ and $b = 0$. (The idea of proof is the same for the general case.) For this, it suffices to show that the unit ball B of $H^1_0(\Omega)$ is relatively compact in $L^2(\Omega)$. Let u be an element of B. Extending it by 0 to outside Ω, it may be identified with an element of $H^1(\mathbf{R}^n)$.

Taking a positive number R, we put $u = v_R + w_R$, where

$$v_R(x) = (2\pi)^{-n} \int_{|\xi|<R} e^{ix\cdot\xi} \mathscr{F} u(\xi)\, d\xi,$$
$$w_R(x) = (2\pi)^{-n} \int_{|\xi|>R} e^{ix\cdot\xi} \mathscr{F} u(\xi)\, d\xi.$$

Then, at first

$$\begin{aligned} \|w_R\|^2 \le \|w_R\|^2_{L^2(\mathbf{R}^n)} &= (2\pi)^{-n} \int_{|\xi|>R} |\mathscr{F} u(\xi)|^2\, d\xi \\ &\le \frac{(2\pi)^{-n}}{1+R^2} \int_{|\xi|>R} \langle\xi\rangle^2 |\mathscr{F} u(\xi)|^2\, d\xi \le \frac{1}{1+R^2}. \end{aligned}$$

So, $\|w_R\|$ is uniformly small for any $u \in B$ if R is large. We fix a large R. As for v_R, it is the inverse Fourier image of functions with support contained in $|\xi| < R$. Not only v_R itself but also any derivative of v_R is uniformly bounded in $\mathbf{R}^n$

$$\begin{aligned} |D^\alpha v_R(x)|^2 &\le (2\pi)^{-2n} \left\{ \int_{|\xi|<R} |\xi^\alpha \mathscr{F} u(\xi)|\, d\xi \right\}^2 \\ &\le (2\pi)^{-2n} \int_{|\xi|<R} |\xi^\alpha|^2\, d\xi \int |\mathscr{F} u(\xi)|^2\, d\xi \le C R^{n+2|\alpha|}. \end{aligned}$$

And hence the family $\{v_R(x)|_\Omega\,;\, u \in B\}$ is uniformly bounded and equicontinuous. Therefore by the Ascoli-Arzelà theorem, any sequence of B contains a subsequence which is uniformly convergent in Ω. Consequently, B is relatively compact in $L^2(\Omega)$. □

An analogous fact holds for the imbedding $H^a(\Omega) \subset H^b(\Omega)$. We omit the proof of the following corollary (see Lemma E.2).

COROLLARY E.5. *Let Ω be a bounded domain satisfying the smoothness assumption in part* C. *The imbedding from $H^a(\Omega)$ into $H^b(\Omega)$ is compact if $-\infty < b < a < +\infty$.*

Neither Theorem E.4 nor Corollary E.5 is correct in general if Ω is unbounded. For example, let $\Omega = \mathbf{R}^n$, $\varphi(x) \in C_0^\infty(\mathbf{R}^n)$ and ξ be a fixed unit vector. Then the set $\{\varphi(x + j\xi)\}_{j=0}^\infty$ is bounded in $H_0^1(\mathbf{R}^n)$, but no subsequence of it is convergent in $L^2(\mathbf{R}^n)$.

E. Inequalities of interpolation. We define a seminorm in $H^k(\Omega)$ to be

$$|u|_{H^k(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|^2 \right)^{1/2} \tag{E.13}$$

for positive integer k. If $0 \le k < l < m < +\infty$, there exists a positive number $C = C(k, l, m)$ such that

$$\sum_{|\alpha|=l} \xi^{2\alpha} \le \varepsilon \sum_{|\alpha|=m} \xi^{2\alpha} + C\varepsilon^{-(l-k)/(m-l)} \sum_{|\alpha|=k} \xi^{2\alpha}$$

holds for any positive number ε and $\xi \in \mathbf{R}^n$. From this and the Fourier transform, we may prove the following

LEMMA E.6. *Let k, l, m be integers satisfying $0 \le k < l < m < +\infty$. Then*

(i) *There exists a positive number $C = C(k, l, m, \Omega)$ such that*

$$|u|_{H^l(\Omega)} \le \varepsilon |u|_{H^m(\Omega)} + C\varepsilon^{-(l-k)/(m-l)} |u|_{H^k(\Omega)} \tag{E.14}$$

holds for any positive number ε and $u \in H_0^m(\Omega)$.

(ii) *If Ω satisfies the condition in part* C, *then inequality* (E.14) *holds also for $u \in H^m(\Omega)$ possibly with a larger constant C.*

The second assertion is also verified by making use of Theorem E.4 and

LEMMA E.7. *Let X, Y and Z be Banach spaces satisfying $X \subset Y \subset Z$ Suppose that the imbedding from X to Y is compact and the imbedding from Y to Z is continuous. Then, for every positive number ε, there exists a positive number $C(\varepsilon)$ such that*

$$\|u\|_Y \le \varepsilon \|u\|_X + C(\varepsilon)\|u\|_Z \quad \textit{for any } u \in X. \tag{E.15}$$

F. Representation of elements of $H^a(\Omega)$ for $a < 0$. If $\Omega = \mathbf{R}^n$, then

$$H^a(\mathbf{R}^n) = \{u \in \mathscr{S}'(\mathbf{R}^n)\,;\, \langle\xi\rangle^a \mathscr{F}u(\xi) \in L^2(\mathbf{R}^n)\} \tag{E.16}$$

whatever a may be. In particular, assume that $a = -k$ and k is a positive integer. Then, every $u \in H^{-k}(\mathbf{R}^n)$ is represented as

$$u = (1 - \Delta)^k v, \qquad v \in H^k(\mathbf{R}^n), \tag{E.17}$$

by setting $v = \mathscr{F}^{-1}\{(1 + |\xi|^2)^{-k} \mathscr{F}u(\xi)\}$. Or again

$$u = \sum_{|\alpha| \le k} D^\alpha v_\alpha, \quad v_\alpha \in L^2(\mathbf{R}^n), \tag{E.18}$$

by setting $v_\alpha = \mathscr{F}^{-1}\{(\xi^\alpha / \sum_{|\beta| \le k} \xi^{2\beta}) \mathscr{F}u(\xi)\}$ Analogous representations are true for a general domain Ω (the proof is omitted):

LEMMA E.8. *Let u be an element of $H^{-k}(\Omega)$ for a positive integer k. Then,*

(i) *There exists one and only one element v of $H_0^k(\Omega)$ such that*

$$u = (1 - \Delta)^k v.$$

(ii) *There exists a family $\{v_\alpha\,;\, |\alpha| \le k\}$ of $L^2(\Omega)$ such that*

$$u = \sum_{|\alpha| \le k} D^\alpha v_\alpha.$$

G. Traces on hyperplanes. If $u(x)$ belongs to $\mathscr{S}(\mathbf{R})$, we have

$$2\pi u(x) = \int_{-\infty}^{+\infty} e^{ix\cdot\xi}(\xi^2+s^2)^{-a/2}\{(\xi^2+s^2)^{a/2}\mathscr{F}u(\xi)\}\,d\xi \tag{E.19}$$

for any positive number s. Suppose that $a > 1/2$. Then we have a bound

$$|u(x)| \leq CK_{u,s}, \tag{E.20}$$

where

$$4\pi^2 K_{u,s}^2 = \int_{-\infty}^{+\infty}(\xi^2+s^2)^a|\mathscr{F}u(\xi)|^2\,d\xi, \qquad C^2 = \int_{-\infty}^{+\infty}(\xi^2+s^2)^{-a}\,d\xi.$$

If $1/2 < a < 3/2$, then

$$\int_{-\infty}^{+\infty}(\xi^2+s^2)^{-a}|e^{it\xi}-1|^2\,d\xi \leq A^2\min(|t|^{2a-1}, s^{1-2a}) \tag{E.21}$$

for $s > 0$ and $t \in \mathbf{R}$, where $A = A_a$ is a positive number independent of (t, s). We represent $u(x) - u(y)$ as in (E.19) and make use of (E.21). Then

$$|u(x) - u(y)| \leq AK_{u,s}|x-y|^{a-1/2} \quad \text{for } x, y \in \mathbf{R}. \tag{E.22}$$

Inequalities (E.20) and (E.22) hold for every $u \in H^a(\mathbf{R})$. That is, if $a > 1/2$, each element of $H^a(\mathbf{R})$ is a bounded continuous function (after a correction of values on a null set if necessary). Moreover, if $1/2 < a < 3/2$, it is Hölder continuous of exponent $a - 1/2$. In particular if $a = 1$, (E.20) and (E.22) can be rewritten as

$$|u(x)|^2 \leq \frac{1}{2}\left(\frac{1}{s}\|u'\|^2 + s\|u\|^2\right) \quad \text{for } s > 0,$$
$$|u(x) - u(y)| \leq \sqrt{|x-y|}\,\|u'\| \quad \text{for } x, y \in \mathbf{R} \tag{E.23}$$

($|u(x) - u(y)|$ is in fact of $o(\sqrt{|x-y|})$).

Let us generalize this to n variables. In $\mathbf{R}^n$, we identify the generic point on the hyperplane $x_n = 0$ with $x' = (x_1, \dots, x_{n-1})$ of $\mathbf{R}^{n-1}$. Given a $u(x) \in C_0^\infty(\mathbf{R}^n)$, we define

$$\gamma_j u(x') = \lim_{x_n \to 0} D_{x_n}^j u(x) \quad \text{for } j = 0, 1, 2, \dots. \tag{E.24}$$

Also for $u(x) \in C_0^\infty(\overline{\mathbf{R}_+^n})$, we set

$$\gamma_j u(x') = \lim_{x_n \downarrow 0} D_{x_n}^j u(x) \quad \text{for } j = 0, 1, 2\dots. \tag{E.24$'$}$$

γ_j is said to be the *trace operator* of order j.

PROPOSITION E.9. *Let j be a nonnegative integer and let a be a real number satisfying $a > j + 1/2$. Then, the trace operator γ_j: $C_0^\infty(\mathbf{R}^n) \to$*

$C_0^\infty(\mathbf{R}^{n-1})$ can be extended to a continuous linear mapping from $H^a(\mathbf{R}^n)$ to $H^{a-j-1/2}(\mathbf{R}^{n-1})$.

PROOF. It suffices to prove the case where $j = 0$. For $u \in H^a(\mathbf{R}^n)$, let $\mathscr{F}'u(\xi', x_n)$ be the partial Fourier transform with respect to x'. Since $\langle\xi\rangle^a \mathscr{F}u(\xi)$ is square summable with respect to ξ_n for almost every ξ', there exists a null set N in $\mathbf{R}^{n-1}_{\xi'}$ such that $\mathscr{F}'u(\xi', x_n)$ is finite as long as $\xi' \notin N$ and

$$2\pi\mathscr{F}'u(\xi', x_n) = \int_{-\infty}^{+\infty} e^{ix_n\xi_n}\mathscr{F}u(\xi', \xi_n)\,d\xi_n.$$

Applying (E.20) with $s = \langle\xi'\rangle$, we have

$$\langle\xi'\rangle^{sa-1}|\mathscr{F}'u(\xi', x_n)|^2 \le C\int_{-\infty}^{+\infty}\langle\xi\rangle^{2a}|\mathscr{F}u(\xi)|^2\,d\xi_n$$

if $\xi' \notin N$ (C is a positive number independent of (u, ξ', x_n)). Since the right-hand side is integrable with respect to ξ', the left-hand side is also integrable. This shows by (E.8) that $u(\cdot, x_n)$ is an element of $H^{a-1/2}(\mathbf{R}^{n-1})$ and that

$$\|u(\cdot, x_n)\|_{H^{a-1/2}(\mathbf{R}^{n-1})} \le C'\|u\|_{H^a(\mathbf{R}^n)}$$

with a positive number C' independent of (u, x_n). By putting $x_n = 0$, we see that γ_j can be extended to a continuous linear mapping from $H^a(\mathbf{R}^n)$ to $H^{a-1/2}(\mathbf{R}^{n-1})$. Since $C_0^\infty(\mathbf{R}^n)$ is dense in $H^a(\mathbf{R}^n)$, this continuous extension is unique. □

REMARK E.1. γ_j is no longer defined on $H^{j+1/2}(\mathbf{R}^n)$. For example,

$$u(x) = \mathscr{F}^{-1}\left\{\frac{1}{\langle\xi\rangle\log(1+\langle\xi\rangle)}\right\} \in H^{1/2}(\mathbf{R}).$$

However $u(0) = +\infty$.

PROPOSITION E.10.

(i) $H_0^a(\mathbf{R}^n_+) = H^a(\mathbf{R}^n_+)$ *if* $0 \le a \le 1/2$.

(ii) *If $a > 1/2$, an element u of $H^a(\mathbf{R}^n_+)$ belongs to $H_0^a(\mathbf{R}^n_+)$ if and only if*

$$\gamma_j u = 0 \quad \textit{for every integer } j \textit{ satisfying } 0 \le j < a - 1/2. \tag{E.25}$$

PROOF. For simplicity of description, we restrict ourselves to the one-dimensional case. We set $V = \{\varphi(x) \in C_0^\infty(\mathbf{R});\ 0 \notin \operatorname{supp}\varphi\}$.

For the case $0 \le a \le 1/2$, it suffices to show that an element f of $H^{-a}(\mathbf{R})$ is equal to zero if f annihilates V (see part C above). If $f|V = 0$, then f is a linear combination of a finite number of derivatives of the Dirac δ function, so $\mathscr{F}f(\xi)$ is a polynomial. Since $\langle\xi\rangle^{-a}\mathscr{F}f(\xi) \in L^2(\mathbf{R})$, $\mathscr{F}f(\xi)$ should be identically equal to zero. Therefore, $f = 0$.

Suppose that $a > 1/2$. If $v \in H^a(\mathbf{R})$ and if $v(0) \ne 0$, then v cannot be approximated, even in $L^2(\mathbf{R})$, by a sequence of V because v is continuous (see (E.22)). Therefore, $u(0) = 0$ if u is an extension to $\mathbf{R}$ of an element of $H_0^a(\mathbf{R}_+)$ (also $u^{(j)}(0+) = 0$ for $0 \le j < a - 1/2$).

Conversely, suppose that $u \in H^a(\mathbf{R}_+)$ and $u^{(j)}(0+) = 0$ for all j such that $0 \le j < a - 1/2$. Begin with the case where $a \le 3/2$. Then, $u' \in H^{a-1}(\mathbf{R}_+)$ and $H^{a-1}(\mathbf{R}_+) = H_0^{a-1}(\mathbf{R}_+)$ by (i). So, there exists a sequence $\{\varphi_k\}_{k=1}^{\infty}$ of $C_0^{\infty}(\mathbf{R}_+)$ which tends to u' in $H^{a-1}(\mathbf{R}_+)$ as $k \to \infty$. We set

$$\psi_k(x) = \int_0^x \varphi_k(t)\, dt.$$

Then $\{\psi_k\}_{k=1}^{\infty}$ is also a sequence in $C_0^{\infty}(\mathbf{R}_+)$ and tends to u in $H^a(\mathbf{R}_+)$. (Note that $\int_{-\infty}^{+\infty} \varphi_k(x)\, dx = 0$.) Therefore, u belongs to $H_0^a(\mathbf{R}_+)$. For the case where $a > 3/2$, we can proceed by induction with respect to the integral part of $a - 1/2$. □

Next, we prove that γ_j is in fact surjective.

PROPOSITION E.11. *Suppose that $a > 1/2$ and let k be an integer satisfying $k < a - 1/2 \le k + 1$. Then, for every positive number λ, there exists a linear mapping*

$$R_\lambda \colon \prod_{j=0}^{k} H^{a-j-1/2}(\mathbf{R}^{n-1}) \to H^a(\mathbf{R}^n)$$

satisfying three conditions:

(i) *For every $\vec{\varphi}(x') = (\varphi_0(x'), \dots, \varphi_k(x')) \in \prod_{j=0}^{k} H^{a-j-1/2}(\mathbf{R}^{n-1})$,*

$$\gamma_j R_\lambda \vec{\varphi} = \varphi_j \quad \textit{for } 0 \le j \le k.$$

(ii) *For any b satisfying $0 \le b \le a$, the inequality*

$$\|R_\lambda \vec{\varphi}\|_{H^b(\mathbf{R}^n)} \le C(\lambda + 1)^b \sum_{j=0}^{k} \lambda^{-j-1/2} \|\varphi_j\|_{H^{b-j-1/2}(\mathbf{R}^{n-1})} \tag{E.26}$$

holds with a positive number $C = C(a, b)$ independent of $(\vec{\varphi}, \lambda)$.

(iii) *The support of $R_\lambda \vec{\varphi}$ is contained in $\{|x_n| \le 1/\lambda\}$.*

PROOF. Choose a function $\zeta(t)$ of class C^∞ of a single variable satisfying $\zeta = 1$ for $|t| \le 1/2$ and $\zeta = 0$ for $|t| \ge 1$. We define

$$R_\lambda \vec{\varphi}(x) = \sum_{j=0}^{k} \frac{1}{j!} (ix_n)^j \mathscr{F}'^{-1}\{\zeta(\lambda \langle \xi' \rangle x_n) \mathscr{F}' \varphi_j(\xi')\},$$

where $\mathscr{F}'$ is the partial Fourier transform with respect to x'. Then, R_λ has the required properties. □

Let Ω be the interior or exterior of a bounded hypersurface S satisfying the smoothness condition in part C. Then we define the trace operator γ_j of order j to be

$$\gamma_j u(x') = \lim_{x \in \Omega,\, x \to x'} D_\nu^j u(x) \quad \text{for } x' \in S, \tag{E.27}$$

where $D_\nu = -i\partial/\partial\nu$ and ν is the interior unit normal vector to S at x'. Similarly to Propositions E.9–E.11, we may prove that, if $a > j + 1/2$, then $\gamma_j\colon C^\infty(\overline{\Omega}) \to C^\infty(S)$ is extended to a continuous linear mapping from $H^a(\Omega)$ onto $H^{a-j-1/2}(S)$. An element u of $H^a(\Omega)$ belongs to $H_0^a(\Omega)$ if and only if $\gamma_j u = 0$ for all j satisfying $0 \le j < a - 1/2$. To construct a mapping R_λ as above, we take λ sufficiently large.

H. Sobolev spaces on compact manifolds. Let M be an n-dimensional, compact oriented manifold of class C^∞ without boundary. Let $\{U_k\}_{k=1}^N$ be an open covering of M consisting of local coordinate neighborhoods and $\{\zeta_k\}_{k=1}^N$ be a partition of unity subordinate to this covering:

$$\zeta_k \in C^\infty(M), \quad \operatorname{supp}\zeta_k \subset U_k \quad \text{and} \quad \sum_{k=1}^N \zeta_k = 1 \quad \text{on } M.$$

A distribution u on M is said to belong to $H^a(M)$ if and only if $\zeta_k u$, regarded as distribution on $\mathbf{R}^n$, belongs to $H^a(\mathbf{R}^n)$ for every k. The set $H^a(M)$ depends on the choice of neither the covering nor the partition of unity. We have $H_0^a(M) = H^a(M)$ for $a \ge 0$ because the boundary of M is assumed to be empty. The imbedding from $H^a(M)$ into $H^b(M)$ is compact if $a > b$.

If M is endowed with a Riemannian metric g, we can define Sobolev spaces in another way without making use of partition of unity. First, the metric g determines a volume element dv, so the space $L^2(M)$ is defined. The Laplace-Beltrami operator Δ is defined, too (see §G). $I - \Delta$ is revealed to be essentially selfadjoint and strictly positive definite. Denote also by $I-\Delta$ the selfadjoint extension. Then, by means of spectral decomposition, we may define complex powers $(I - \Delta)^c$ for $c \in \mathbf{C}$. The space $H^a(M)$ is equal to the domain of definition of $(I - \Delta)^{a/2}$ if $a \ge 0$ and to the dual space of $H^{-a}(M)$ (or image of $L^2(M)$ by $(I - \Delta)^{a/2}$) if $a < 0$.

I. Imbedding theorem of Sobolev. Inclusions of type $W^{a,p}(\Omega) \subset W^{b,q}(\Omega)$ are collectively called the *imbedding theorem of Sobolev.* This is one of the most important properties of Sobolev spaces in applications. We have proved in part G that elements of $H^1(\mathbf{R}^1)$ are bounded and Hölder continuous functions of exponent $1/2$. Corresponding results for 2- and 3-dimensional cases are as follows:

$$H^1(\mathbf{R}^2) \subset L^q(\mathbf{R}^2) \quad \text{for any } q \text{ satisfying } 2 \le q < +\infty; \tag{E.28}$$

$$H^1(\mathbf{R}^3) \subset L^6(\mathbf{R}^3). \tag{E.29}$$

(Elements of $H^1(\mathbf{R}^2)$ are not bounded in general. A counter-example is

$$u(x) = \mathscr{F}^{-1}\left\{\frac{1}{\langle\xi\rangle^2 \log(1 + \langle\xi\rangle)}\right\},$$

for which $u(0) = +\infty$. See Theorem E.15 below.)

Let us verify (E.28) and (E.29). If $u(t) \in C_0^\infty(\mathbf{R}^1)$, both of

$$|f(t)|^k \le k \int_{-\infty}^t |f|^{k-1}|f'|\,dt, \qquad |f(t)|^k \le k \int_t^{+\infty} |f|^{k-1}|f'|\,dt$$

hold. So,

$$\max |f(t)|^k \le \frac{k}{2}\|f'\|_{L^2}\|f\|_{L^{2k-2}}^{k-1} \quad \text{for } k \ge 2. \tag{E.30}$$

Proof of (E.28). Suppose that $u(x) \in C_0^\infty(\mathbf{R}^2)$. Then by (E.30),

$$\|u\|_{L^{2p}}^{2p} \le \left(\int \max_{x_2} |u|^p\,dx_1\right)\left(\int \max_{x_1} |u|^p\,dx_2\right) \le \frac{p^2}{4}\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2}\|u\|_{L^{2p-2}}^{2p-2}$$

for $p = 2, 3, \ldots$. Repeated use of this yields

$$\|u\|_{L^{2p}}^{2p} \le 4^{1-p}p!^2(\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2})^{p-1}\|u\|_{L^2}^2$$

for $p = 1, 2, \ldots$. This holds for $u \in H^1(\mathbf{R}^2)$, proving (E.28). □

Proof of (E.29). Put $J = \|u\|_{L^6(\mathbf{R}^3)}^6$ for $u \in C_0^\infty(\mathbf{R}^3)$. Again by (E.30), we have

$$\begin{aligned} J &\le \int \left(\int \max_{x_2} |u|^3\,dx_3\right)\left(\int \max_{x_3} |u|^3\,dx_2\right) dx_1 \\ &\le \frac{9}{4}\int \left\{\left(\iint |u|^4\,dx_2\,dx_3\right)\left(\iint |\partial_2 u|^2\,dx_2\,dx_3\right)^{1/2} \right. \\ &\qquad\qquad \left. \cdot \left(\iint |\partial_3 u|^2\,dx_2\,dx_3\right)^{1/2}\right\} dx_1 \\ &\le \frac{9}{4}\left(\iint \max_{x_1} |u|^4\,dx_2\,dx_3\right)\|\partial_2 u\|_{L^2}\|\partial_3 u\|_{L^2} \le \frac{9}{2}\sqrt{J}\prod_{j=1}^3 \|\partial_j u\|_{L^2}, \end{aligned}$$

or

$$\|u\|_{L^6}^3 \le \frac{9}{2}\prod_{j=1}^3 \|\partial_j u\|_{L^2}.$$

This is correct also for $u \in H^1(\mathbf{R}^3)$. (E.29) is proved. See (E.39) below applied to $n = 3$, $p = 2$ for inequality with the best possible constant factor. □

To state a comprehensive version of the theorem, we need some notation. In addition to $W^{a,p}(\Omega)$, we introduce the spaces $\mathscr{B}^a$ to treat the case $p = +\infty$. In what follows, let k, l be nonnegative integers and γ, δ be fractions: $0 < \gamma, \delta < 1$. We define

$\mathscr{B}^k(\Omega)$: *the set of functions in* Ω *whose derivatives of order up to* k *are bounded and continuous in* Ω;

$\mathscr{B}^{0-}(\Omega) = L^\infty(\Omega)$: *the set of bounded measurable functions in* Ω;

$\mathscr{B}^{k-}(\Omega)$ $(k \geq 1)$: *the set of elements of* $\mathscr{B}^{k-1}(\Omega)$ *whose derivatives of order* $k-1$ *are uniformly Lipschitz continuous in* Ω;

$\mathscr{B}^{k+\gamma} = C^{k+\gamma}(\overline{\Omega})$: *the set of elements of* $\mathscr{B}^{k}(\Omega)$ *whose derivatives of order* k *are uniformly Hölder continuous of exponent* γ *on* $\overline{\Omega}$ (*see* §H, *part* A *below*).

Definition. Let Ω be a (bounded or unbounded) domain of $\mathbf{R}^n$. Ω is said to have the *cone property* $C(T, N)$ (T is a positive number or $+\infty$ and N is a positive number) if and only if there exists an $\mathbf{R}^n$-valued function Ψ defined in Ω satisfying two conditions:

(a) $\Psi \in \mathscr{B}^{N-}(\Omega)$;

(b) $x + tz + t\Psi(x) \in \Omega$ if $x \in \Omega$, z is in the open unit ball of $\mathbf{R}^n$, and if $t \in [0, T]$.

This is a smoothness assumption on the boundary of Ω. It is more relaxed than any differentiability condition. Now, the imbedding theorem of S. L. Sobolev ([165] and [116]) is

Theorem E.12. *Suppose that the domain* Ω *has the cone property* $C(T, N)$ *for a sufficiently large* N. *Let* k, l, γ, δ *be as above. Supposing that* $1 \leq p \leq q < +\infty$, *we set*

$$\lambda = \frac{n}{p} - \frac{n}{q}.$$

Then, the imbedding

$$W^{a,p}(\Omega) \subset W^{b,q}(\Omega) \tag{E.31}$$

is continuous if

$$\begin{array}{ll} \textit{either} & a-b>\lambda, \\ & a-b=\lambda,\ a=k,\ \textit{and}\ 1<p<q<+\infty, \\ & a-b=\lambda,\ a=k+\gamma,\ b=l,\ \textit{and}\ 1\leq p<q<+\infty, \\ \textit{or} & a-b=\lambda,\ a=k+\gamma,\ b=l+\delta,\ \textit{and}\ 1\leq p\leq q<+\infty. \end{array}$$

The imbedding

$$W^{a,p}(\Omega) \subset \mathscr{B}^{b}(\Omega) \tag{E.32}$$

is continuous if

$$\begin{array}{ll} \textit{either} & a-b>n/p, \\ & a-b=n/p,\ b=l+\delta,\ \textit{and}\ 1<p<+\infty, \\ \textit{or} & a-b=n/p,\ a=k+\gamma,\ b=l+\delta,\ \textit{and}\ 1\leq p<+\infty. \end{array}$$

The imbedding

$$W^{k+\gamma,p}(\Omega) \subset \mathscr{B}^{l-}(\Omega) \tag{E.33}$$

is continuous if $k+\gamma-l > n/p$.

Remark E.2. If we replace $W^{a,p}(\Omega)$ by $W_0^{a,p}(\Omega)$, the imbeddings hold without assuming cone property on the boundary because then the proof is reduced to that in the case where $\Omega = \mathbf{R}^n$.

REMARK E.3. If $a - b > \lambda$, the imbeddings in (E.31) and (E.32) are compact. On the contrary, if $a - b = \lambda$, none of the imbeddings in (E.31)–(E.33) are compact.

SKETCH OF PROOF OF THEOREM E.12. We restrict ourselves to the simplest case where $a = k$, $b = l$, and $a - b > \lambda$. Let $\zeta(x)$ be a function of class C^∞ with support contained in the open unit ball B of $\mathbf{R}^n$ and satisfying $\int_B \zeta\, dx = 1$. Given a function $u(x) \in C^\infty(\Omega)$ and a multi-index β with $|\beta| = l$, we may prove the following equality (see Chapter I, §1, (1.12))

$$\left(\frac{\partial}{\partial x}\right)^\beta u(x) = T^{-l}\int_B \zeta_\beta(z, y) u(x + Tz + Ty)\, dz + \int_0^T t^{k-l-1}\, dt \int_B \sum_{|\alpha|=k-1} \omega_\alpha(z, y)\left(\frac{\partial}{\partial x}\right)^{\alpha+\beta} \cdot u(x + tz + ty)\, dz, \quad \text{(E.34)}$$

where y is a fixed point of $\mathbf{R}^n$ and

$$\zeta_\beta(z, y) = (-1)^l \sum_{|\mu| \le k-l-1} \frac{1}{\mu!}\left(\frac{\partial}{\partial z}\right)^{\mu+\beta} \{(z+y)^\mu \zeta(z)\},$$

$$\omega_\alpha(z, y) = (-1)^{k-l}\frac{|\alpha|}{\alpha!}(z+y)^\alpha \zeta(z).$$

(E.34) represents the derivatives of order l by means of u itself and the derivatives of order k. Let us substitute here $y = \Psi(x)$. Then, the right-hand side is the sum of a finite number of $V(t, x)$'s (and the integrals of it with respect to t) defined to be

$$V(t, x) = \int_B \varphi(z) v(x + tz + t\Psi(x))\, dz, \quad \text{(E.35)}$$

where $t > 0$ and $\varphi(x) \in C_0^\infty(B)$. A key estimate is

$$\|V(t, \cdot)\|_{L^q(\Omega)} \le Ct^{-\lambda}\|v\|_{L^p(\Omega)} \quad \text{for } 1 \le p \le q \le +\infty, \quad \text{(E.36)}$$

where $\lambda = n/p - n/q$ and C is a positive number independent of (t, v). (If this is proved for two cases $q = p$ and $q = +\infty$, then the intermediate cases follow by interpolation.) The first term on the right-hand side of (E.34) is of type $V(T, x)$ and each one of the other terms may be written as

$$W(x) = \int_0^T t^{k-l-1} V(t, x)\, dt. \quad \text{(E.37)}$$

Since we are assuming that $k - l > \lambda$, it is easy to see from (E.36) that

$$\|W\|_{L^q(\Omega)} \le C\|v\|_{L^p(\Omega)}.$$

These two inequalities give the proof of (E.31) or (E.32) for the case where $a = k$, $b = l$, and $k - l > \lambda$.

The proofs for other cases are more complicated. Especially if $a-b=\lambda$, we need delicate considerations based on the theory of interpolations. In any case, the proof starts from (E.34). For details see T. Muramatu [92], [93] and R. A. Adams [127]. □

The most popular version of the theorem is the assertion applied to $H^a(\Omega)$.

COROLLARY E.13. *Let k be an integer greater than $n/2$. Then,*

$$H^k(\Omega) \subset C^b(\overline{\Omega}) \quad \text{if } n \text{ is odd}, \qquad H^k(\Omega) \subset \mathscr{B}^{b-}(\Omega) \quad \text{if } n \text{ is even}, \tag{E.38}$$

where $b = k - n/2$.

The original proof of Theorem E.12 by S. L. Sobolev [165] was based on an estimate of an integral of type

$$\iint f(x)g(y)h(x-y)\,dx\,dy$$

with the aid of Steiner's symmetrization (see also G. H. Hardy, J. E. Littlewood, and G. Pólya [40], Chapter 10). Many authors have investigated alternative proofs and generalizations of the theorem. Some of these contain the finest results in real analysis (see Sobolev [116], J. Nečas [96], D. Gilbarg and N. S. Trudinger [141], and H. Triebel [174]). The theorem was one of the motivations of the theory of interpolation of spaces (see A. P. Caldéron [17], J. L. Lions and J. Peetre [75], and Lions and E. Magenes [73]). Recently, T. Horiuchi [147] proved the imbedding theorems for weighted Sobolev spaces in very general domains.

G. Talenti [168] established the best possible constant factor in the typical inequality of imbedding. His proof is based on Schwarz symmetrization of functions, geometric integration theory, and calculus of variations (see also §I below). The constant factor in inequality (E.39) below is the best when applied to $u \in W_0^{1,p}(\Omega)$ whatever Ω may be. See also T. Aubin [131], Chapter 2, §8 for the corresponding results for Sobolev spaces on manifolds.

THEOREM E.14. *Suppose that $n \geq 2$ and $1 < p < n$. Then, the inequality*

$$\|u\|_{L^q(\mathbf{R}^n)} \leq C \left[\int_{\mathbf{R}^n} \left\{ \sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j} \right|^2 \right\}^{p/2} dx \right]^{1/p} \tag{E.39}$$

holds for every $u(x) \in W^{1,p}(\mathbf{R}^n)$, where

$$q = \frac{np}{n-p}, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

$$C = \frac{1}{\sqrt{\pi}} n^{-1/p} \left(\frac{p-1}{n-p} \right)^{1/p'} \left\{ \frac{\Gamma(n)\Gamma(1+\frac{n}{2})}{\Gamma(\frac{n}{p})\Gamma(1+\frac{n}{p'})} \right\}^{1/n}.$$

The equality sign holds in (E.39) *if and only if* $u(x) = (a + b|x|^{p'})^{-p/q}$ *with positive numbers* a, b.

On the other hand, by virtue of the remark after (E.28), elements of $H_0^1(\Omega)$ (Ω is a bounded domain of $\mathbf{R}^2$) are not necessarily bounded. The following theorem of N. S. Trudinger [176] and J. Moser [154] shows that $\exp\{4\pi|u(x)|^2/\|\operatorname{grad} u\|^2\}$ is integrable in Ω (see also T. Aubin [131], Chapter 2, §17 and R. A. Adams [127], §8.25).

THEOREM E.15. *Suppose that* Ω *is a bounded domain of* $\mathbf{R}^n$. *Then, the inequality*

$$\int_\Omega \exp\{\alpha_n |u(x)|^r / K_u^r\}\, dx \le c_n \operatorname{vol}(\Omega) \tag{E.40}$$

holds for every $u \in W_0^{1,n}(\Omega)$, *where*

$$K_u = \left\{ \int_\Omega \left(\sum_{j=1}^n \left| \frac{\partial u}{\partial x_j} \right|^2 \right)^{n/2} dx \right\}^{1/n}, \quad r = \frac{n}{n-1}, \quad \alpha_n = n \left(\frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \right)^{1/(n-1)},$$

and c_n *is a positive number depending only on* n.

§F. Hölder spaces and Schauder spaces

Let Ω be a bounded or unbounded domain of $\mathbf{R}^n$. In this section, we introduce two kinds of spaces of functions which are sufficiently smooth in Ω in the ordinary sense. Throughout the section, $k, l, m, p, \ldots$ are nonnegative integers and $\gamma, \delta, \ldots$ are fractions ($0 < \gamma, \delta < 1$).

A. Hölder spaces. Let $C^k(\overline{\Omega})$ be the vector space of complex-valued functions whose derivatives of order up to k are bounded and continuous on $\overline{\Omega}$. Let $C^{k+\gamma}(\overline{\Omega})$ be the vector space of elements of $C^k(\overline{\Omega})$ whose kth order derivatives are Hölder continuous of exponent γ uniformly on $\overline{\Omega}$. Let us define seminorms and norms in these spaces as follows:

$$|u|_l = \sum_{|\alpha|=l} \frac{1}{\alpha!} \sup\{|D^\alpha u(x)|\,;\, x \in \overline{\Omega}\},$$

$$|u|_{l+\gamma} = \sum_{|\alpha|=l} \frac{1}{\alpha!} \sup\left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma}\,;\ (x, y) \in \Omega \times \Omega \backslash \Delta \right\}; \tag{F.1}$$

$$\|u\|_k = \sum_{l=0}^k |u|_l, \qquad \|u\|_{k+\gamma} = \|u\|_k + |u|_{k+\gamma}. \tag{F.2}$$

$C^k(\overline{\Omega})$ endowed with the norm $\|\ \|_k$ and $C^{k+\gamma}(\overline{\Omega})$ endowed with the norm $\|\ \|_{k+\gamma}$ are Banach algebras too because

$$\|uv\|_k \le \|u\|_k \|v\|_k, \qquad \|uv\|_{k+\gamma} \le \|u\|_{k+\gamma} \|v\|_{k+\gamma}. \tag{F.3}$$

The most useful inequalities of interpolation are (see [23]).

LEMMA F.1. *Suppose that* Ω *has the cone property* $C(T, 0)$ (*see* §E, *part* I). *Then, for a sufficiently small* $\varepsilon_0 > 0$, *the inequalities*

(i) $|u|_1 \leq \varepsilon |u|_2 + C|u|_0/\varepsilon$,

(ii) $|u|_1 \leq \varepsilon |u|_{1+\gamma} + C\varepsilon^{-1/\gamma}|u|_0$,

(iii) $|u|_\gamma \leq \varepsilon |u|_1 + C\varepsilon^{-\gamma/(1-\gamma)}|u|_0$,

(iv) $|u|_\gamma \leq \varepsilon |u|_\delta + C\varepsilon^{-\gamma/(\delta-\gamma)}|u|_0$ $(0 < \gamma < \delta < 1)$,

hold for every u *and* $0 < \varepsilon < \varepsilon_0$, *where the* C *'s are positive numbers independent of* (u, ε).

PROOF. (i) and (ii). We may assume that u is real valued. For an arbitrary point $x^0 \in \Omega$, there exists an $a \in \mathbf{R}^n$ such that $x^0 + t(a+B) \subset \Omega$ as long as $0 \leq t \leq T$, where a may depend on x^0 but T (> 0) is independent of x^0 (B is the unit ball of $\mathbf{R}^n$ centered at the origin). Put $C = \bigcup_{0\leq t\leq T} t(a+B)$. Choosing an $s_0 > 0$ and a unit vector ω such that $s_0\omega \in C$, we define $v(s) = u(x^0 + s\omega)$ for $0 \leq s \leq s_0$.

By the mean value theorem, there exists a σ satisfying $0 < \sigma < s_0$ and $v(s_0) - v(0) = s_0 v'(\sigma)$, so $|v'(\sigma)| \leq 2|u|_0/s_0$ at this point. Next, $|v'(s) - v'(\sigma)| \leq c|s-\sigma|^\gamma |u|_{1+\gamma} \leq cs_0^\gamma |u|_{1+\gamma}$ if $0 \leq s \leq s_0$, where c depends only on (n, γ). Therefore,

$$|v'(s)| \leq cs_0^\gamma |u|_{1+\gamma} + \frac{2}{s_0}|u|_0 \qquad (0 \leq s \leq s_0)$$

holds for all s_0 and ω as long as $s_0\omega \in C$, moreover $x^0 \in \Omega$ is arbitrary, too. This applied to $s = 0$ implies (ii) by putting $\varepsilon = cs_0^\gamma$. The reasoning remains available as $\gamma \uparrow 1$, so (i) holds. In (i) and (ii), ε_0 may depend on T.

(iii) and (iv). For arbitrary $x^0 \in \Omega$, we set

$$A_\gamma = \sup\left\{ \frac{|u(y) - u(x^0)|}{|y - x^0|^\gamma} ; y \in \Omega \backslash \{x^0\} \right\}.$$

Then, there exists a $y^0 \in \Omega$ such that $|u(y^0) - u(x^0)|/\rho^\gamma \geq A_\gamma/2$ where $\rho = |y^0 - x^0|$. So $A_\gamma \leq 4|u|_0/\rho^\gamma$ or $\rho \leq (4|u|_0/A_\gamma)^{1/\gamma}$. On the other hand, $A_\delta \geq |u(y^0) - u(x^0)|/\rho^\delta \geq (A_\gamma/2)\rho^{\gamma-\delta}$, so $A_\gamma \leq 2\rho^{\delta-\gamma}A_\delta$. Eliminating ρ from the two inequalities, $A_\gamma^\delta \leq 2^{2\delta-\gamma}A_\delta^\gamma |u|_0^{\delta-\gamma}$. Since $A_\delta \leq |u|_\delta$, we have $A_\gamma^\delta \leq c|u|_\delta^\gamma |u|_0^{\delta-\gamma}$, where c depends only on (γ, δ). Again x^0 being arbitrary, we obtain $|u|_\gamma^\delta \leq c|u|_\delta^\gamma |u|_0^{\delta-\gamma}$, proving (iv). The reasoning is available also as $\delta \uparrow 1$, so (iii) holds. (iii) and (iv) hold for any positive number ε. □

Repeated use of Lemma F.1 yields the following:

(v) $|u|_l \leq \varepsilon |u|_m + C(\varepsilon)|u|_k$ if $0 \leq k < l < m$,

(vi) $|u|_m \leq \varepsilon |u|_{m+\gamma} + C(\varepsilon)|u|_0$,

(vii) $|u|_{k+\gamma} \leq \varepsilon |u|_m + C(\varepsilon)|u|_0$ if $0 \leq k < m$.

B. Schauder spaces. In this part, we introduce norms and seminorms involving weights defined by means of powers of the distance function $\rho(x) = \operatorname{dist}(x, \partial\Omega)$:

$$|u|_{p,l} = \sum_{|\alpha|=l} \frac{1}{\alpha!} \sup_{x\in\Omega}\{\rho(x)^{p+l}|D^\alpha u(x)|\},$$

$$|u|_{p,l+\gamma} = \sum_{|\alpha|=l} \frac{1}{\alpha!} \sup_{(x,y)\in\Omega\times\Omega\setminus\Delta} \left\{\min(\rho(x), \rho(y))^{p+l+\gamma} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma}\right\}; \tag{F.4}$$

$$\|u\|_{p,k} = \sum_{l=0}^{k} |u|_{p,l}, \qquad \|u\|_{p,k+\gamma} = \|u\|_{p,k} + |u|_{p,k+\gamma}. \tag{F.5}$$

Let $CS_p^k(\Omega)$ be the set of complex-valued functions $u(x)$ of class C^k in Ω for which $\|u\|_{p,k} < +\infty$. Also, let $CS_p^{k+\gamma}(\Omega)$ be the set of $u(x) \in CS_p^k(\Omega)$ for which $\|u\|_{p,k+\gamma} < +\infty$. The boundedness assumption on u is more relaxed here than in part A. In fact, $CS_p^{k+\gamma}(\Omega)$ contains unbounded functions if $p > 0$. If $k > 0$ and $u \in CS_0^k(\Omega)$, then u itself is bounded. But $D^\alpha u$ ($|\alpha| \le k$) may diverge by $O(\rho^{-|\alpha|})$ near the boundary.

$CS_p^k(\Omega)$ and $CS_p^{k+\gamma}(\Omega)$ are Banach spaces. Moreover we have

$$\|uv\|_{p+q,k} \le \|u\|_{p,k}\|v\|_{q,k}, \qquad \|uv\|_{p+q,k+\gamma} \le \|u\|_{p,k+\gamma}\|v\|_{q,k+\gamma}. \tag{F.6}$$

So, $CS_0^k(\Omega)$ and $CS_0^{k+\gamma}(\Omega)$ are Banach algebras. We again have inequalities of interpolation. However, we assume no smoothness on the boundary of Ω:

LEMMA F.2.
(i) $|u|_{p,1} \le \varepsilon|u|_{p,2} + C|u|_{p,0}/\varepsilon$;
(ii) $|u|_{p,1} \le \varepsilon|u|_{p,1+\gamma} + C\varepsilon^{-1/\gamma}|u|_{p,0}$;
(iii) $|u|_{p,\gamma} \le \varepsilon|u|_{p,1} + C\varepsilon^{-\gamma/(1-\gamma)}|u|_{p,0}$;
(iv) $|u|_{p,\gamma} \le \varepsilon|u|_{p,\delta} + C\varepsilon^{-\gamma/(\delta-\gamma)}|u|_{p,0}$ *if* $0 < \gamma < \delta < 1$.

PROOF. First we prove (ii). For an arbitrary $x^0 \in \Omega$ and unit vector ω, we put $v(s) = u(x^0 + s\omega)$ $(0 \le s < \rho)$, where $\rho = \rho(x^0)$. If $0 < s_0 < \rho$, we have $v(s_0) - v(0) = s_0 v'(\sigma)$ for a σ satisfying $0 < \sigma < s_0$. So, $|v'(\sigma)| \le (|v(s_0)| + |v(0)|)/s_0$. Multiplying ρ^{p+1} to both sides, we see that

$$\rho^{p+1}|v'(\sigma)| \le \left(\frac{\rho}{\rho - s_0}\right)^p \frac{\rho}{s_0}(\rho - s_0)^p(|v(s_0)| + |v(0)|) \le \left(\frac{\rho}{\rho - s_0}\right)^p \frac{2\rho}{s_0}|u|_{p,0}.$$

Therefore, by setting $\lambda = \rho/(\rho - s_0)$ and $\mu = s_0/\rho$, we have

$$\rho^{p+1}|v'(\sigma)| \le \frac{2\lambda^p}{\mu}|u|_{p,0}.$$

Next, if $0 \le s \le s_0$, we can rewrite $\rho^{p+1}|v'(s) - v'(\sigma)|$ as

$$\left(\frac{\rho}{\rho - s_0}\right)^{p+1+\gamma} \left(\frac{|s-\sigma|}{\rho}\right)^{\gamma} (\rho - s_0)^{p+1+\gamma} \frac{|v'(s) - v'(\sigma)|}{|s-\sigma|^{\gamma}}.$$

So

$$\rho^{p+1}|v'(s) - v'(\sigma)| \le \lambda^{p+1+\gamma}\mu^{\gamma}|u|_{p,1+\gamma}.$$

From these two, we have

$$\rho^{p+1}|v'(s)| \le \lambda^{p+1+\gamma}\mu^{\gamma}|u|_{p,1+\gamma} + \frac{2\lambda^p}{\mu}|u|_{p,0}$$

for $0 \le s \le s_0$. If $0 \le s_0 \le \rho/2$, then $1 < \lambda \le 2$ and $0 < \mu \le 1/2$. So by setting $\varepsilon = 2^{p+1+\gamma}\mu^{\gamma}$, we have in particular that

$$\rho^{p+1}|v'(0)| \le \varepsilon|u|_{p,1+\gamma} + c\varepsilon^{-1/\gamma}|u|_{p,0},$$

where c depends only on (p, γ). This proves (ii). (i) follows from this by the limit procedure $\gamma \uparrow 1$. We can prove (iii) and (iv) analogously. □

Readers may refer to A. Douglis and L. Nirenberg [23] and D. Gilbarg and L. Hörmander [140]. Our spaces $CS_p^a(\omega)$ are denoted as $H_a^{(p)}$ in [140] and defined also in the case where $p < 0$ and $p + a \ge 0$.

§G. Geodesic distance

Let M be an oriented Riemannian manifold of class C^∞. Our discussion being local, we do not assume compactness of M. At any point p of M, the tangent space $T_p(M)$ and cotangent space $T_p^*(M)$ are endowed with scalar products induced by Riemannian metric g on M. Both of the scalar products are denoted by $\langle\ ,\ \rangle_p$. Let $x = (x^1, \dots, x^n)$ be a local coordinate system in an open subset U of M. Identifying $p \in U$ with its coordinates x, we have a basis $\{\partial/\partial x^j\}_{j=1}^n$ of $T_p(M)$ and a basis $\{dx^j\}_{j=1}^n$ of $T_p^*(M)$. If we denote

$$g^{jk}(x) = \langle dx^j, dx^k\rangle_p, \quad g_{jk}(x) = \left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right\rangle_p, \qquad 1 \le j, k \le n, \tag{G.1}$$

the covariant tensor $(g_{jk}(x))$ represents the metric g, that is, the line element is given by

$$ds^2 = g_{jk}(x)\,dx^j\,dx^k \tag{G.2}$$

(only in this section do we make use of the summation convention of Einstein). Matrices $(g_{jk}(x))$ and $(g^{jk}(x))$ are symmetric, positive definite and inverse to each other (the entries are functions of class C^∞ in U).

A *Geodesic* on M is an integral curve of a system of ordinary differential equations which is represented as follows by means of a local coordinate system:

$$\frac{d^2x^j}{ds^2} = -\Gamma^j_{kl}(x)\frac{dx^k}{ds}\frac{dx^l}{ds}, \qquad 1 \le j \le n, \tag{G.3}$$

where Γ^j_{kl} stands for the Christoffel symbol

$$\Gamma^j_{kl}(x) = \frac{1}{2} g^{jm} \left(\frac{\partial g_{lm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^l} - \frac{\partial g_{kl}}{\partial x^m} \right) .$$

If we introduce a vector $\xi = (\xi_1, \dots, \xi_n) \in T_x(M)$ and a quadratic form

$$\mathscr{H}(x, \xi) = g^{jk}(x)\xi_j \xi_k , \tag{G.4}$$

the system (G.3) is reduced to the first order one

$$\frac{dx^j}{ds} = \frac{1}{2}\frac{\partial \mathscr{H}}{\partial \xi_j}(x, \xi), \quad \frac{d\xi_j}{ds} = -\frac{1}{2}\frac{\partial \mathscr{H}}{\partial x^j}(x, \xi), \qquad 1 \le j \le n. \tag{G.5}$$

We denote by $(x(s; y, \eta), \xi(s; y, \eta))$ the solution subject to the initial condition

$$(x, \xi) = (y, \eta) \quad \text{at } s = 0 . \tag{G.6}$$

Then for any positive number λ, the equality

$$\left(x\left(\lambda s; y, \frac{1}{\lambda}\eta\right), \xi\left(\lambda s; y, \frac{1}{\lambda}\eta\right) \right) = \left(x(s; y, \eta), \frac{1}{\lambda}\xi(s; y, \eta) \right) \tag{G.7}$$

holds as long as both sides make sense. Equality $\mathscr{H}(x, \xi) = \langle \xi, \xi \rangle_x$ holds if ξ is identified with $g^{jk}(x)\xi_k(\partial/\partial x^j)_p \in T_p(M)$. On the integral curve of (G.5)–(G.6), we have

$$\mathscr{H}(x(s; y, \eta), \xi(s; y, \eta)) = \mathscr{H}(y, \eta) \tag{G.8}$$

because the left-hand side is constant along the curve. We sometimes denote the x-component of the solution as

$$x(s; y, \eta) = \operatorname{Exp}(s\eta)\, y . \tag{G.9}$$

Local existence, uniqueness and smoothness of geodesics may be summarized as follows:

LEMMA G.1. *Let p_0 be an arbitrary point of M. If we choose a small closed neighborhood V of p_0, any two points x and y of V can be joined in V by one and only one geodesic $\{\operatorname{Exp}(s\eta)y ; 0 \le s \le 1\}$ such that $x = \operatorname{Exp}(\eta)y$. The mapping $(y, x) \in V \times V \to \eta \in T_y(M)$ is of class C^∞ and $x \in V \to \eta \in T_y(M)$ (for fixed y) is one-to-one.*

An alternative version is

LEMMA G.1′. *Let p_0 be an arbitrary point of M. If we choose a small closed neighborhood W of p_0 and a neighborhood $\widetilde{W}$ of 0 in $T_{p_0}(M)$, then the equation $x = \operatorname{Exp}(\eta)p_0$ defines a diffeomorphism: $x \to \eta$ of class C^∞ from W onto $\widetilde{W}$.*

Let $ST_x(M)$ be the unit ball of $T_y(M)$:

$$ST_y(M) = \{\eta \in T_y(M) ; \langle \eta, \eta \rangle_y = \mathscr{H}(y, \eta) = 1\} . \tag{G.10}$$

Then, for every $(x, y) \in V \times V$, the equations

$$x = \operatorname{Exp}(r\eta)y, \quad r \geq 0, \quad \text{and} \quad \eta \in ST_y(M), \tag{G.11}$$

determine a nonnegative real number r which is called the *geodesic distance* from y to x and is denoted by $r(x, y)$. By virtue of uniqueness of solution of (G.5)–(G.6), we have $y = \operatorname{Exp}(-r\xi)x$, where $\xi = \xi(r; y, \eta) \in ST_x(M)$, so

$$r(x, y) = r(y, x). \tag{G.12}$$

We are now going to deduce a number of equalities involving geodesic distance. First, we introduce a new covector

$$\dot{x}^j = g^{jk}\xi_k = \frac{1}{2}\frac{\partial \mathscr{H}}{\partial \xi_j}(x, \xi), \qquad 1 \leq j \leq n, \tag{G.13}$$

and a quadratic form

$$\mathscr{L}(x, \dot{x}) = g_{jk}(x)\dot{x}^j\dot{x}^k. \tag{G.14}$$

Then $\mathscr{L}(x, \dot{x}) = \mathscr{H}(x, \xi)$. So $\mathscr{L}(x, \dot{x})$ is constant on an integral curve of (G.5). And (G.5) is equivalent to

$$\dot{x}^j = \frac{dx^j}{ds}, \quad \frac{d}{ds}\left\{\frac{\partial \mathscr{L}}{\partial \dot{x}^j}(x, \dot{x})\right\} = \frac{\partial \mathscr{L}}{\partial x^j}(x, \dot{x}), \qquad 1 \leq j \leq n. \tag{G.15}$$

If we make use of a new parameter $\sigma = s/r$ instead of s, we have by (G.11)

$$x|_{\sigma=0} = y, \qquad x|_{\sigma=1} = x. \tag{G.16}$$

Setting $x' = dx/d\sigma$ $(= r\dot{x})$, (G.15) is rewritten as

$$\frac{d}{d\sigma}\left\{\frac{\partial \mathscr{L}}{\partial x'^j}(x, x')\right\} = \frac{\partial \mathscr{L}}{\partial x^j}(x, x'), \qquad 1 \leq j \leq n, \tag{G.15$'$}$$

and

$$r(x, y)^2 = \int_0^1 \mathscr{L}(x(\sigma), x'(\sigma))\, d\sigma. \tag{G.17}$$

Given two points x and y sufficiently close to each other, consider all curves $\{x(\sigma); 0 \leq \sigma \leq 1\}$ satisfying (G.16). Among them, the curve minimizing the integral on the right-hand side of (G.17) should be a solution of (G.15$'$). Conversely, if $x(\sigma)$ is a solution of (G.15$'$), then $\mathscr{L}$ is constant on this curve, and it also minimizes

$$\int_0^1 \sqrt{\mathscr{L}(x(\sigma), x'(\sigma))}\, d\sigma,$$

and hence it is the geodesic joining y and x.

LEMMA G.2. *Let* $r = r(x, y)$. *Then*

$$g^{jk}(x)\frac{\partial r}{\partial x^j}\frac{\partial r}{\partial x^k} = 1. \tag{G.18}$$

And for a scalar function f, *we have*

$$g^{jk}(x)\frac{\partial r}{\partial x^j}\frac{\partial f}{\partial x^k} = \frac{\partial f}{\partial r}. \tag{G.19}$$

PROOF. Perturbing the end point x by δx, we compute the first variation of the functional on the right-hand side of (G.17). Then we can verify that

$$\frac{\partial r}{\partial x^j} = \xi_j. \tag{G.20}$$

So, the left-hand side of (G.18) is equal to $\mathscr{H}(x, \xi)$ or 1. And (G.19) follows from (G.5) and (G.20). □

Now, let us represent $r(x, y)$ by means of x and y. For an arbitrary point y, let x be given by (G.11). Then (G.3) and (G.6) imply

$$x^j = y^j + r\eta^j - \frac{1}{2}r^2\Gamma^j_{pq}(y)\eta^p\eta^q + O(r^3), \quad \text{where } \eta^j = g^{jk}(y)\eta_k. \tag{G.21}$$

Solving these with respect to $r\eta^j$ and substituting them into $r^2 = \mathscr{H}(y, r\eta)$, we obtain an approximation up to the third order

$$\begin{aligned} r(x, y)^2 &= g_{jk}(y)(x^j - y^j)(x^k - y^k) \\ &\quad + g_{jm}(y)\Gamma^m_{kl}(y)(x^j - y^j)(x^k - y^k)(x^l - y^l) + O(|x - y|^4). \end{aligned} \tag{G.22}$$

Therefore, $r(x, y)^2$ is a function of class C^∞ with respect to (x, y).

Taking account of $\eta^j = dx^j/ds|_{s=0}$ (see (G.21)), we set

$$z^j = r\eta^j = r\left.\frac{dx^j}{ds}\right|_{s=0}, \qquad 1 \le j \le n. \tag{G.23}$$

$z = (z^1, \dots, z^n)$ are called the *normal coordinates* of x with origin at y. Representing x^j by means of z, we have $x^j = y^j + z^j + O(|z|^2)$ by (G.21) so that

$$\left.\frac{\partial x^j}{\partial z^k}\right|_{x=y} = \delta^j_k. \tag{G.24}$$

Let $(\tilde{g}_{jk}(z))$ be the metric tensor with respect to z. Then

$$\tilde{g}_{jk}(0) = g_{jk}(y). \tag{G.25}$$

If x' is another local coordinate system near y and z' is a normal coordinate system with origin at y defined above, the mapping $z \to z'$ is linear. The first of the remarkable properties of a normal coordinate system is that any geodesic issued from y becomes a straight line passing through the origin $z = 0$, and vice versa. Second, the Christoffel symbol vanishes at $z = 0$. More precisely, if a given coordinate system x is such that $g_{jk}(y) = \delta_{jk}$, then

$$\tilde{g}_{jk}(z) = \delta_{jk} + \frac{1}{3}\widetilde{R}_{jlkm}(0)z^l z^m + \frac{1}{6}\widetilde{R}_{jlkm/p}(0)z^l z^m z^p + O(|z|^4), \tag{G.26}$$

where $\widetilde{R}_{jlkm}$ is the curvature tensor represented by means of z:

$$\widetilde{R}_{jlkm} = \tilde{g}_{ja} \left\{ \frac{\partial}{\partial z^m} \widetilde{\Gamma}^{\,a}_{lk} - \frac{\partial}{\partial z^k} \widetilde{\Gamma}^{\,a}_{lm} + \widetilde{\Gamma}^{\,a}_{bm} \widetilde{\Gamma}^{\,b}_{lk} - \widetilde{\Gamma}^{\,a}_{bk} \widetilde{\Gamma}^{\,b}_{lm} \right\}$$

and $\widetilde{R}_{jlkm/p}(0) = (\partial \widetilde{R}_{jlkm}/\partial z^p)(0)$ (see E. Cartan [135], Chapter X, §226).

A metric g on M defines a particular operator $\Delta = \Delta_{(M,g)}$ of second order which is called the *Laplace-Beltrami operator.* It has local expression

$$\Delta u = \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x^j} \left\{ \sqrt{g(x)} g^{jk}(x) \frac{\partial u}{\partial x^k} \right\}, \quad \text{where } g(x) = \deg(g_{jk}(x)). \tag{G.27}$$

For $M = S^{n-1}$, the unit sphere endowed with the standard metric, we represented it by means of the polar coordinate system in Chapter II, §3, (3.11) (we denoted Δ by Λ there). If M is the Euclidean space $\mathbf{R}^n$, then $G_{jk}(x) = \delta_{jk}$ and hence Δ is nothing but the ordinary Laplacian. We have constructed an elementary solution of Δ in Chapter III, §5. There we have made use of Lemmas G.2, G.3, and G.4 below:

LEMMA G.3. *If* $r = r(x, y)$,

$$\Delta(r^2) = 2n + r\frac{d}{dr}\{\log \tilde{g}(z)\} = 2n + O(r^2). \tag{G.28}$$

PROOF. In a general coordinate system x,

$$\Delta(r^2) = \frac{2}{\sqrt{g(x)}} \frac{\partial}{\partial x^j} \left(\sqrt{g}\, r \,\frac{dx^j}{dr} \right).$$

However in a normal coordinate system, we have $z^j = r\,dx^j/dr$ so

$$\Delta(r^2) = \frac{2}{\sqrt{\tilde{g}}} \frac{\partial}{\partial z^j} (\sqrt{\tilde{g}} z^j) = 2n + z^j \frac{\partial}{\partial z^j} \log \tilde{g}.$$

Since $z^j \partial/\partial z^j = r\partial/\partial r$, we have first equality. And (G.26) implies the second. □

LEMMA G.4. *If* $r = r(x, y)$ *and* y *is a fixed point, then for every function* $f(t)$ *of a single variable and another function* $u(x)$, *we have*

$$\Delta\{f(r^2)u(x)\} = f(r^2)\Delta u + 4rf'(r^2)\frac{\partial u}{\partial r} + \{f'(r^2)\Delta(r^2) + 4r^2 f''(r^2)\}u. \tag{G.29}$$

PROOF. This follows from Lemma G.2 and

$$\Delta(uv) = v\Delta u + 2g^{jk} \frac{\partial u}{\partial x^j} \frac{\partial v}{\partial x^k} + u\Delta v. \quad \square$$

See M. Riesz [104] and other textbooks on differential geometry for the contents of this section.

§H. Lemma for approximation of domains

We state a lemma for approximation of domains which was used in Chapter VII, §2. Let Ω be a bounded domain of $\mathbf{R}^n$ whose boundary is a hypersurface of class $C^{k+\gamma}$ where k is an integer not smaller than 2 and $0 < \gamma < 1$. Suppose that Ω is in only one side of S at every point of S. Then,

LEMMA H.1. *If V is an appropriate neighborhood of S and ε is a small positive number, there exists a mapping χ satisfying four conditions*:

(i) *χ is a one-to-one mapping from $\mathbf{R}^n$ onto itself, $\chi = I$ (identity) outside V and the Jacobian of χ is equal to $1 + O(\varepsilon)$ in $\mathbf{R}^n$*;

(ii) *χ is of class $C^{k+\gamma}$ in $\mathbf{R}^n$*;

(iii) $\|\chi - I\|_{C^k} = O(\varepsilon)$;

(iv) *The image Σ of S by χ is of class C^∞*.

PROOF. Let $\rho(x)$ be the signed distance from x to S:

$$\rho(x) = \begin{cases} \operatorname{dist}(x, S) & \text{if } x \in \overline{\Omega}, \\ -\operatorname{dist}(x, S) & \text{if } x \notin \Omega. \end{cases} \tag{H.1}$$

In what follows, we denote $F_\alpha = \{x \in \mathbf{R}^n;\, |\rho(x)| \le \alpha\}$ for $\alpha > 0$. Since S is of class $C^{k+\gamma}$, there exist a $\delta > 0$ and a function $f(x)$ such that

$$\begin{gathered} f(x) \in C^{k+\gamma}(\mathbf{R}^n); \qquad f(x) > 0 \quad \text{if and only if} \quad x \in \Omega; \\ f(x) = -1 \quad \text{if } x \text{ is very far from } \Omega; \qquad \frac{1}{2} \le |\operatorname{grad} f(x)| \le 2 \quad \text{in } F_{8\delta}. \end{gathered} \tag{H.2}$$

Given $\varepsilon > 0$, we can find a function $\tilde{f} \in C^\infty(\mathbf{R}^n)$ such that $\|\tilde{f}\|_{C^{k+\gamma}(\mathbf{R}^n)} \le 2\|f\|_{C^{k+\gamma}(\mathbf{R}^n)}$ and $\|f - \tilde{f}\|_{C^k(\mathbf{R}^n)} \le \varepsilon$. Then,

$$|f(x) - \tilde{f}(x)| + |\operatorname{grad} f(x) - \operatorname{grad} \tilde{f}(x)| \le \varepsilon \quad \text{in } \mathbf{R}^n. \tag{H.3}$$

We will define a χ which maps Ω onto the set

$$\mathscr{D} = \{x \in \mathbf{R}^n;\, \tilde{f}(x) > 0\}. \tag{H.4}$$

For this, we again modify $\tilde{f}$. Let $\varphi(x)$ be of class $C^\infty(\mathbf{R}^n)$ with values in $[0, 1]$ such that

$$\begin{gathered} \varphi(x) = 1 \quad \text{in } F_{4\delta}, \qquad \varphi(x) = 0 \quad \text{outside } F_{5\delta}, \\ |\operatorname{grad} \varphi(x)| \le \frac{2}{\delta} \quad \text{in } \mathbf{R}^n. \end{gathered} \tag{H.5}$$

We set

$$F(x) = \varphi(x)\tilde{f}(x) + (1 - \varphi(x))f(x). \tag{H.6}$$

We are going to construct a χ which maps level surfaces $f(x) = c$ onto level surfaces $F(y) = c$ for every c with small absolute value. First note that $F \in C^{k+\gamma}(\mathbf{R}^n) \cap C^\infty(F_{4\delta})$. Second, $\|F - f\|_{C^k(\mathbf{R}^n)} = O(\varepsilon)$,

$$|F(x) - f(x)| \leq \varepsilon, \qquad |\operatorname{grad} F(x) - \operatorname{grad} f(x)| \leq \left(1 + \frac{2}{\delta}\right)\varepsilon \tag{H.7}$$

in $\mathbf{R}^n$. So, if ε is so small that $0 < \varepsilon < \delta/(8 + 4\delta)$, we have

$$|\operatorname{grad} F(x)| \geq \frac{1}{4} \quad \text{in } F_{8\delta}. \tag{H.8}$$

Now, we define

$$W(x) = \frac{1}{|\operatorname{grad} F(x)|} \operatorname{grad} F(x) \quad \text{in } F_{8\delta}. \tag{H.9}$$

Then $W(x) \in C^{k-1+\gamma}(F_{8\delta}) \cap C^\infty(F_{4\delta})$. Let $Y(\theta, x)$ be the integral curve of W:

$$\frac{d}{d\theta} Y(\theta, x) = W(Y(\theta, x)), \qquad Y(0, x) = x. \tag{H.10}$$

Then, $Y(\theta, x) \in C^{k-1+\gamma}([-\delta, \delta] \times F_{7\delta}) \cap C^\infty([-\delta, \delta] \times F_{3\delta})$.

Next, there exists a value $\Theta(x)$ of θ for which

$$F(Y(\Theta(x), x)) = f(x) \quad \text{in } F_{7\delta}. \tag{H.11}$$

In fact by (H.8), we have $dF(Y(\theta, x))/d\theta \geq 1/4$ for $x \in F_{8\delta}$. So we may find one and only one $\Theta(x)$ satisfying (H.11), $\Theta(x) \in C^{k-1+\gamma}(F_{7\delta}) \cap C^{k+\gamma}(F_{2\delta})$ and moreover

$$|\Theta(x)| \leq 4\varepsilon \quad \text{in } F_{7\delta}, \qquad \Theta(x) = 0 \quad \text{in } F_{7\delta} \backslash F_{6\delta}. \tag{H.12}$$

A mapping χ satisfying the conditions of the lemma can be defined in the following way:

$$\chi : x \to y(x) = \zeta(x) Y(\Theta(x), x) + (1 - \zeta(x))x, \tag{H.13}$$

where $\zeta(x)$ is of class $C^\infty(\mathbf{R}^n)$ with values in $[0, 1]$ and satisfying $\zeta = 1$ in F_δ, $\zeta = 0$ outside $F_{2\delta}$ and $|\operatorname{grad} \zeta| \leq 2/\delta$ in $F_{2\delta} \backslash F_\delta$. Note that χ is equal to I outside

$$V = F_{2\delta} = \{x \in \mathbf{R}^n ; |\rho(x)| \leq 2\delta\}.$$

We prove that, if ε is small, χ satisfies the requirements of the lemma.

(ii) Obvious.

(iv) Let ε_0 be so small that $0 < \varepsilon_0 < \delta/(8 + 4\delta)$ and $|f(x)| \geq |\rho(x)|/4$ in $F_{8\delta_0}$ (see (H.1)). If $\rho_\Sigma(x)$ is the signed distance from x to $\Sigma = \chi(S)$,

$$|\rho_\Sigma(x) - \rho(x)| \leq 4\varepsilon \tag{H.14}$$

in $\mathbf{R}^n$ if $0 < \varepsilon \leq \varepsilon_0$. Therefore, $y(x) \in \mathscr{D} = \chi(\Omega)$ if $\rho(x) \geq \delta$. (In fact, if $\rho(x) \geq \delta$, then $\rho(y(x)) \geq \delta - 4\varepsilon$ because (H.13) holds and

$|y(x)-x| \leq 4\varepsilon$. So $\rho_\Sigma(y(x)) \geq \delta - 8\varepsilon > 0$ by (H.14).) Similarly, $y(x) \notin \overline{\mathscr{D}}$ if $\rho(x) \leq -\delta$. Therefore $y(x) \in \Sigma$ only if $x \in F_\delta$. However on F_δ, we have $y(x) = Y(\Theta(x), x)$, so (H.11) is equivalent to $\tilde{f}(y(x)) = f(x)$ there. When x is fixed, the curve $Y(\theta, x)$ passes through Σ once and only once, Consequently, $x \in \Omega$ if and only if $y(x) \in \mathscr{D}$ and $x \in S$ if and only if $y(x) \in \Sigma$.

(i) (H.13) implies

$$\frac{\partial y_j}{\partial x_k} - \delta_{jk} = (Y_j - x_j)\frac{\partial \zeta}{\partial x_k} + \zeta(\eta_{jk} - \delta_{jk}) + \zeta w_j \frac{\partial \Theta}{\partial x_k}, \qquad 1 \leq j, k \leq n,$$

where

$$\eta_{jk} = \left.\frac{\partial Y_j}{\partial x_k}(\theta, x)\right|_{\theta=\Theta(x)}, \quad w_j = W_j(Y(\Theta(x), x)), \quad Y_j = Y_j(\Theta(x), x).$$

This implies $\partial y_j/\partial x_k = \delta_{jk} + O(\varepsilon)$. So the Jacobian $|\partial y/\partial x|$ never vanishes in $\mathbf{R}^n$ if ε is small.

(iii) Similarly, we can estimate higher order derivatives of $y(x)$.

Lemma H.1 is proved. □

§I. A priori estimates of Talenti

Consider the Dirichlet problem in a domain Ω of $\mathbf{R}^n$ (with finite volume if $n = 2$):

$$-\sum_{j,k=1}^{n} \frac{\partial}{\partial x_k}\left\{a_{jk}(x)\frac{\partial u}{\partial x_j}\right\} + c(x)u = f(x) \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega \quad \text{(I.1)}$$

We denote the left-hand side by $-\nabla \cdot (a\nabla u) + cu$ for brevity. All functions appearing in this section are assumed to be *real valued.* In particular, we suppose that the coefficients $a_{jk}(x)$, $c(x)$ are measurable functions on Ω satisfying

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq \sum_{j=1}^{n} \xi_j^2 \quad \text{for all } \xi \in \mathbf{R}^n \quad \text{and} \quad c(x) \geq 0 \qquad \text{(I.2)}$$

almost everywhere in Ω. Our basic assumption on f and u is the following:

$$f \in L^p(\Omega) \quad \text{and} \quad u \in H_0^1(\Omega),$$

where

$$1 < p < +\infty \quad \text{if } n = 2, \qquad p = \frac{2n}{n+2} \quad \text{if } n \geq 3. \qquad \text{(I.3)}$$

So, the existence and uniqueness of solution are guaranteed by maximum principle.

Let $\Omega^\star$ be the n-dimensional ball centered at the origin with the same volume as Ω. Given a measurable function f in Ω, a function $f^\star$ in $\Omega^\star$

is said to be the *Schwarz symmetrization* (or spherically symmetric rearrangement) of f if $f^\star(y)$ is nonnegative, depends only on $|y|$, nonincreasing with respect to $|y|$ and moreover

$$\begin{aligned} \operatorname{meas}\{y \in \Omega^\star;\ f^\star(y) > t\} &= \operatorname{meas}\{x \in \Omega;\ |f(x)| > t\} \\ &\text{for almost every } t > 0. \end{aligned} \tag{I.4}$$

(see (I.11) below).

G. Talenti was the first who made a systematic use both of rearrangement and isoperimetric inequality in the study of equations with variable coefficients. One of his ideas was to evaluate the Schwarz symmetrization $u^\star$ of the solution u of (I.1) by means of the solution $v(y)$ of the Dirichlet problem in $\Omega^\star$:

$$-\Delta v(y) = f^\star(y) \quad \text{in } \Omega^\star; \qquad v = 0 \quad \text{on } \partial\Omega^\star. \tag{I.5}$$

Note that $v(y)$ is nonnegative, nonincreasing, and radial.

THEOREM I.1 (G. Talenti [169]). *Under the assumptions above, we have two inequalities*:

$$u^\star(y) \le v(y) \quad \textit{almost everywhere in } \Omega^\star; \tag{I.6}$$

$$\int_\Omega \left\{ \sum_{j,k=1}^n a_{jk} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} \right\}^{q/2} dx \le \int_{\Omega^\star} \left\{ \sum_{j=1}^n \left(\frac{\partial v}{\partial y_j} \right)^2 \right\}^{q/2} dy, \quad \textit{if } 0 < q \le 2. \tag{I.7}$$

Both (I.6) *and* (I.7) *hold with the equality sign if and only if* Ω *is a ball,* $a_{jk}(x) = \delta_{jk}$, $c(x) = 0$, *and if* f *(or* $-f$*) is nonnegative, nonincreasing, and radial.*

A rigorous proof of the theorem is based on geometric integration theory. We cannot present it here in detail. However, we shall sketch the proof supposing that the boundary $\partial\Omega$, the coefficients a_{jk}, c, the inhomogeneous term f, and the solution u are sufficiently smooth.

We begin by introducing some notation. We denote

$$\begin{gathered} E_t = \{x \in \Omega;\ |u(x)| < t\}, \\ \mu(t) = \operatorname{meas}(E_t) \quad \text{for } t > 0, \qquad \mu(0) = \operatorname{meas}[\operatorname{supp} u]. \end{gathered} \tag{I.8}$$

μ is nonincreasing and right continuous. Let $u^*(s)$ be the inverse function of μ:

$$u^*(s) = \inf\{t \ge 0;\ \mu(t) < s\}, \qquad 0 \le s < +\infty. \tag{I.9}$$

u^* is said to be the *decreasing rearrangement* of u. Furthermore set

$$\overline{u}(s) = \frac{1}{s} \int_0^s u^*(\sigma)\, d\sigma, \qquad 0 < s < +\infty. \tag{I.10}$$

$\overline{u}$ is called the *Hardy-Littlewood maximal function* of u. Now, the Schwarz symmetrization of u is given by

$$u^\star(y) = u^*(c_n|y|^n), \quad \text{where } c_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \text{ (volume of the unit ball).} \tag{I.11}$$

The basic properties of $u^\star$ and u^* which we need are as follows:

$$\int_\Omega |u|^p\,dx = \int_0^{+\infty} u^*(s)^p\,ds = \int_{\Omega^\star} u^\star(y)^p\,dy \quad \text{for } 0 < p < +\infty; \tag{a}$$

$$\text{ess.}\sup|u| = u^*(0) = \text{ess.}\sup u^* = \text{ess.}\sup u^\star; \tag{b}$$

$$\int_A |u|\,dx \le \int_0^{|A|} u^*(s)\,ds \quad \text{for every } A \subset \Omega. \tag{c}$$

The solution v of the symmetrized problem (I.5) is revealed to be

$$v(y) = n^{-2}c_n^{-2/n}\int_{c_n|y|^n}^{|\Omega|} r^{2/n-1}\overline{f}(r)\,dr. \tag{I.12}$$

Therefore,

$$|\nabla v(y)| = \frac{|y|}{n}\overline{f}(c_n|y|^n). \tag{I.13}$$

Let e be the set of noncritical values of $|u|$, that is, $t \in e$ if and only if $\operatorname{grad} u = 0$ nowhere on ∂E_t. Then, $\mathbf{R}\backslash e$ is of measure zero by a theorem of Sard. ∂E_t is the union of a finite number of smooth closed surfaces of dimension $n-1$ if $t \in e$ and we have the isoperimetric inequality

$$nc_n^{1/n}\mu(t)^{1-1/n} \le |\partial E_t|, \tag{I.14}$$

where $|\partial E_t|$ is the area (or $(n-1)$-dimensional Hausdorff measure) of ∂E_t. Moreover,

$$-\mu'(t) = \int_{\partial E_t} \frac{dS}{|\nabla u|} \quad \text{if } t \in e. \tag{d}$$

We introduce auxiliary functions $\Phi_q(t)$ and $\Psi_q(y)$ by setting

$$\Phi_0(t) = \Psi_0(t) = \mu(t) \quad \text{for } q = 0,$$

$$\Phi_q(t) = \int_{E_t} (a\nabla u, \nabla u)^{q/2}\,dx, \qquad \Psi_q(t) = \int_{E_t} |\nabla u|^q\,dx \quad \text{for } 0 < q \le 2.$$

By virtue of the hypothesis of ellipticity,

$$\Psi_q(t) \le \Phi_q(t), \qquad \Psi_q(t) - \Psi_q(t+h) \le \Phi_q(t) - \Phi_q(t+h)$$

hold for all $t > 0$, $h > 0$. By Hölder's inequality, we have

$$\Phi_q(t) - \Phi_q(t+h) \le \{\mu(t) - \mu(t+h)\}^{1-q/2}\{\Phi_2(t) - \Phi_2(t+h)\}^{q/2} \tag{e}$$

if $t > 0$, $h > 0$ and the analogous inequality holds for Ψ_q.

Now, let us evaluate $-\Phi_2'$ by means of $\overline{f}$. Since u solves (I.1) and $u \in H_0^1(\Omega)$,

$$\int_\Omega \{(a\nabla u, \nabla\alpha) + (cu - f)\alpha\}\, dx = 0$$

holds for every $\alpha \in H_0^1(\Omega)$. A particular choice $\alpha = (\operatorname{sgn} u)\max(|u| - t, 0)$ gives

$$\Phi_2(t) = \int_{E_t} (|u| - t)(f \operatorname{sgn} u - c|u|)\, dx .$$

This implies

$$\Phi_2(t) - \Phi_2(t+h) = h \int_{E_t} (f \operatorname{sgn} u - c|u|)\, dx + R_h ,$$

with

$$R_h = \int_{E_t \setminus E_{t+h}} (|u| - t - h)(f \operatorname{sgn} u - c|u|)\, dx .$$

The error term R_h is of $O(h^2)$. Therefore, Φ_2 is Lipschitz continuous and

$$-\Phi_2'(t) \le \int_{E_t} (|f| - c|u|)\, dx \le \int_{E_t} |f|\, dx$$

(because $c \ge 0$), so by (c)

$$-\Phi_2'(t) \le \mu(t)\overline{f}(\mu(t)) . \tag{f}$$

Next, we evaluate u^* and $-\Phi_q'$ by means of $-\Phi_2'$. By (d), (e), and (f), Φ_q is absolutely continuous and

$$-\Phi_q'(t) \le -\mu'(t)\left\{\frac{\Phi_2'(t)}{\mu'(t)}\right\}^{q/2} , \qquad -\Psi_q'(t) \le -\mu'(t)\left\{\frac{\Psi_2'(t)}{\mu'(t)}\right\}^{q/2} \tag{g}$$

almost everywhere. Since $-\Psi_1'(t) = |\partial E_t|$, (g) with $q = 1$ and (I.14) yield

$$-n^2 c_n^{2/n}/\mu'(t) \le -\mu(t)^{2/n-2}\Psi_2'(t) \le -\mu(t)^{2/n-2}\Phi_2'(t) \tag{h}$$

almost everywhere. Substituting this into (g), we have

$$-\Phi_q'(t) \le -\mu'(t)\left\{\frac{-\mu(t)^{1/n-1}\Phi_2'(t)}{nc_n^{1/n}}\right\}^q . \tag{i}$$

Proof of (I.6). (f) and (h) imply

$$n^2 c_n^{2/n} \le -\mu'(t)\mu(t)^{2/n-1}\overline{f}(\mu(t))$$

almost everywhere. Integrating both sides with respect to t, we have

$$u^*(s) \le n^{-2} c_n^{-2/n} \int_s^{|\Omega|} r^{2/n-1}\overline{f}(r)\, dr$$

(see (I.9)). We obtain (I.6) by putting $s = c_n|y|^n$ (see (I.12)).

PROOF OF (I.7). Substituting (f) into (i) and integrating both sides, we have

$$\int_\Omega (a\nabla u, \nabla u)^{q/2}\, dx \le \int_0^{|\Omega|} \left\{ \frac{r^{1/n}}{nc_n^{1/n}} \overline{f}(r) \right\}^q dr.$$

This is nothing but (I.7) (see (I.13)).

EQUALITY SIGNS. Suppose that Ω is a ball, $a_{jk}(x) = \delta_{jk}$, $c(x) = 0$, and f is a nonnegative, nonincreasing radial function. Then, the solution $u(x)$ is also radial, in fact equal to $v(x-a)$ if Ω is centered at a. So, (I.6) and (I.7) hold with equality signs.

Conversely, suppose that (I.6) holds with equality sign. Then at first, (f) should be equality. So, $c(x) = 0$, f should be a monotone radial function of constant sign. Next, (h) should also be equality. And for this in turn, the isoperimetric inequality (I.14) should be equality, that is, E_t should be ball for all t. This means that u (or $-u$) is radial and nonincreasing. Consequently, Ω should be ball and $a_{jk}(x) = \delta_{jk}$. □

REMARK I.1. (I.6) combined with (b) and (I.12) yield in particular

$$\text{ess.}\sup |u| \le n^{-2} c_n^{-2/n} \int_0^{|\Omega|} r^{2/n-1} \overline{f}(r)\, dr. \tag{I.15}$$

Equality takes place under the same conditions as in the theorem. Since $\overline{f}(r) \le \text{ess.}\sup |f|$, we have

$$\text{ess.}\sup |u| \le \frac{1}{2n} \left(\frac{|\Omega|}{c_n} \right)^{2/n} \text{ess.}\sup |f|. \tag{I.15$'$}$$

Equality holds if and only if f is a constant function in addition to the conditions in the theorem.

Talenti [169] proved also a number of a priori estimates for solutions of (I.1) in the L^p-framework with precision of the best possible constant factors. He generalized the results to equations with first order terms (see [171]) and some quasilinear equations of divergence form (see [170]).

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