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Matching of Asymptotic Expansions of Solutions of Boundary Value Problems

A. M. Il′in

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Volume 102

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A. M. Il'in

American Mathematical Society
Providence, Rhode Island

А. М. ИЛЬИН

СОГЛАСОВАНИЕ АСИМПТОТИЧЕСКИХ РАЗЛОЖЕНИЙ РЕШЕНИЙ КРАЕВЫХ ЗАДАЧ

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Abstract. The author describes an approach to the analysis of solutions of boundary value problems for partial differential equations containing a small parameter. The asymptotic expansions of solutions are different in different regions (for example, in the boundary layer region; near the discontinuity of the limiting solution; etc.). The main problem discussed in the book is the matching problem for asymptotic solutions.

Using examples originating in various problems of fluid mechanics and continuum mechanics of solids, the author presents a rigorous construction of complete asymptotic expansions for solutions.

The book can be useful for researchers and graduate students working in various areas of analysis, partial differential equations, applied mathematics, and mechanics. It can also be used as a basis for an advanced graduate course.

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Preface

Asymptotic methods in analysis, and, especially, in the theory of equations of mathematical physics are steadily gaining in popularity among a wide range of researchers in various areas of natural sciences. This is testified by the relative increase in the number of articles appearing in the periodic publications and the considerable growth in the number of monographs published on the subject in the last 10 to 15 years. Many of these monographs touch upon the method mentioned in the title of this book. However, the expositions available are usually of a fragmentary nature, and scarcely concern the questions of justifying the asymptotics. At the same time, in the last 5 to 10 years a common approach to a class of small parameter problems frequently arising in widely different areas has been developed. We call these problems bisingular. The reader will find the precise definition in the Introduction below.

This approach is one of the versions of the method of matching different asymptotic expansions for solutions of boundary value problems. Its description can only be found in periodic publications, and, naturally, first papers do not provide the best way of presenting the subject. The purpose of the present book is, therefore, to provide a preliminary assessment of these works and to make the method available to experts in different areas. The presentation follows an inductive scheme and is based on the analysis of a series of examples. As a rule, each next example is more complicated than the preceding one.

The idea that the asymptotic analysis includes two basic steps has gained wide acceptance. The first is the actual construction of the asymptotics. One has to choose the form in which the formal asymptotic expansion of a solution (or the formal asymptotic solution, the other names are Ansatz, FAS, f.a.s., f.a.e.) is to be sought, and specify the way of constructing this f.a.s.

The second step includes the justification of the constructed asymptotics, i.e., a proof that the f.a.s. obtained is indeed an asymptotic expansion of the solution of the problem. This is achieved by providing an estimate of the difference between the true solution and partial sums of the f.a.s.

Which of the two steps is more difficult depends on the problem. Sometimes, one of them is trivial while the other requires a lot of effort. In other cases, the difficulties are distributed more or less evenly. The first part, i.e., the construction of the asymptotics, is certainly of interest to experts in many different areas, e.g., physicists, engineers or anyone who has to deal with large or small parameters in his or her problems, while the second part is mainly of interest to a much narrower community of pure mathematicians.

With that in mind, the author has endeavored to satisfy both groups of his prospective readers. The construction of asymptotics for the problems in question is given in the main text which, as far as possible, is not overloaded with unimportant details. The material necessary for the justification of the asymptotics appears in small print. If the reader's aim is just to master the methods of constructing asymptotic expansions of solutions of bisingular problems, the main text is a sufficient reading. The full text, including the small print, contains strict mathematical justifications of the asymptotics which hitherto were to be found only in periodic publications. This attempt "to trap two rabbits in one book" seems to be worth the effort. The reader has to judge whether it is a success, or the Russian proverb "If you chase two rabbits, you won't catch one" still holds true. *Spero meliora.*

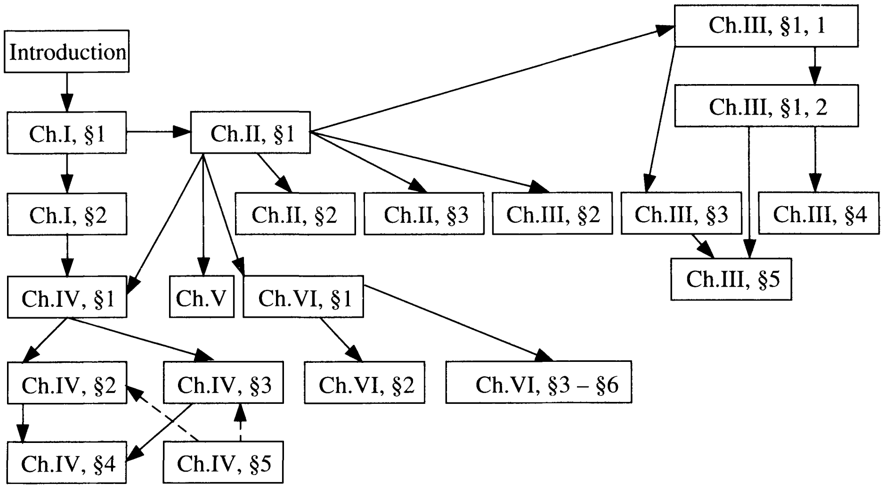
Much of the material appearing in the book has been included in the lecture course given at the Bashkir State University. Most of it is based on the three-year university mathematics course, and for the first two chapters even two years of mathematics at a university or a technical school is sufficient.

The author did not make it his goal to compile a comprehensive list of all significant publications on the subject considered in this book. The references in the text are reduced to a minimum and refer mainly to the justification of asymptotics. All mention of sources and articles relevant to the subject is relegated to the end of the book.

All the results presented in the book, with the exception of Chapter 1 which is of an auxiliary, tutorial nature, have been obtained by a group of Russian mathematicians working in the cities of Ufa and Sverdlovsk, in the Ural region. I take this opportunity to thank my colleagues and students whose research and discussions of results made an important contribution to the publication of this book. In writing the book I received direct assistance from E. F. Lelikova and Yu. Z. Shaygardanov who helped me in my work on Chapter IV, and from L. A. Kalyakin, who helped with Chapter V. T. N. Nesterova and O. B. Sokolova performed extensive work preparing the manuscript. To all of them I express my deep gratitude.

Interdependence of Chapters

Provision is made for a selective study of the book. For the reader's convenience, the following diagram indicates how the different chapters and sections depend on each other.



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Notes and Comments on Bibliography

Chapter I

What we now call singular perturbation problems for differential equations have been considered for a long time going back at least to the last century. References to such works can be found, for example, in [85]. Nevertheless, a sufficiently general theory appeared relatively recently. The Cauchy problem for systems of differential equations was considered in [116], [117] (more details on the development of this line can be found in [121]–[123], [127]).

Rigorous mathematical analysis of singular perturbation boundary value problems for partial differential equations appeared only in the 1950s (see [69], [94], [95], [55]). A detailed investigation of boundary value problems of the type considered in Examples 1–3 has been made in [124]–[126]. Problems of this type are often called boundary layer type problems. The same name is sometimes also applied to the problems considered in this book and called bisingular. In [124] the situation described in Chapter I is called that of regular degeneracy. There is no accepted terminology. It is apparently convenient to call the problems described in Chapter I problems with exponential boundary layer (or a similar term taking into account the fact that coefficients of the outer expansion are smooth functions, while the boundary layer functions decay exponentially at infinity). The method suggested in [124]–[126] is often called the Vishik-Lyusternik method. Its subsequent development is described in [119]. Problems with corner boundary layers of exponential type similar to Example 4 are considered in [11]–[14]. Taken together, problems with exponential boundary layer and bisingular problems can naturally be called problems of boundary layer type thus singling them out among a large number of other questions in the theory of differential equations with a small parameter which are not treated in this book.

Among other branches of the asymptotic approach to differential equations with small parameter we note the investigation of equations with rapidly oscillating coefficients and boundary value problems in perforated domains (see, for example, [7], [129]). The Krylov-Bogolyubov-Mitropolskiĭ method of averaging [9] is not considered at all.

There are closer problems, also not considered here, for differential equations whose coefficients are smooth and of slow variation, while the solutions are nevertheless rapidly oscillating. The method used for the investigation of these problems is usually called the WKB method, and, roughly speaking, reduces to the study of the "fast" phase and "slow" amplitude of oscillations. The method is described in the books [33], [28], [71]–[73]. The resulting equation for the amplitude also often turns out to be bisingular, and can be analyzed using the method of matched asymptotic expansions given above.

Chapter II

The method of matched asymptotic expansions arose, under different names given in the Introduction, in mechanics, and made it possible to construct the first terms of the asymptotics, to solve the arising paradoxes, etc. We only note the articles [56], [57], [63], [103]. The history of the question can be found in the monographs [121], [15], [85], [62]. For ordinary differential equations, the method has been used in different situations in [19], [101], [83]. For partial differential equations a rigorous justification of the asymptotics has been obtained relatively recently (see [1], [40], [42], [65], [77], [113], [27] et al.) For the problems of short-wave diffraction mentioned above a rigorous mathematical investigation was carried out in [2]–[5] et al.

§**1.** Example 5 is of an educational nature and is provided for the explanation of the technique developed in the last 10 to 12 years (see [35], [36], [40], [42], [47], [49], [65], [66], [90]–[92], [109]–[111]). This method is quite close to the one described in [121], [29], but in our view is much more convenient and consistent. Another, but essentially close approach is developed in [77]–[81].

For elliptic partial differential equations the problem mentioned in the Remark to §1 is considered in [45].

§**2.** Problems similar to Example 6 are considered in [22].

§**3.** This section describes the contents of the research thesis of F. M. Sattarova made at the Bashkir State University in 1979.

Chapter III

§**1.** The problem is of an auxiliary, illustrative nature. In [36] the construction of the asymptotics of solutions having singularities at a point is set forth in the form convenient for the purposes of the problem under consideration. Much earlier, the asymptotics of Green's function has been constructed in a somewhat different form in [32]. In the case where the singularity of the solution is located near the boundary, the asymptotics of Green's function for an elliptic second order equation was studied in [47].

§**2.** The section presents, in a somewhat simplified form, the paper [35]. This paper considers the problem in the case of a general elliptic second order equation. The two-dimensional problem of the flow past a thin solid body was treated in detail in [121]. Exterior boundary value problems for thin solid bodies have been considered in [78], [80], [81], [106], [25]–[27].

§**3.** The exposition follows that of [36] where the same problem is considered for the equation with variable coefficients. The method can also be applied to elliptic problems of higher order, as well as to other problems with singular perturbations in boundary conditions. Problems of these kind have been studied in [31], [77], [79]. In the so called critical cases, as in the problem of §3, rational functions of $\ln \varepsilon$ appear as the gauge functions. Note that such gauge functions are typical for a rather general situation (see [120, Chapter 9, §3]).

§**4.** The section presents, in a simplified form, the results of [38].

Chapter IV

§**1.** The results of this section have been obtained in [42]. However, the approach used in [42] is far from perfect. The treatment in this book is based on the methods developed in [66] which is more natural. Such an approach makes it possible to obtain the asymptotics for a wide range of problems. The analysis of boundary value problems, for which the characteristic of the limit first order equation coincides with a part of the boundary, was conducted in [66] for the three-dimensional case, and in [48], [49] for a system of elliptic equations. Note that the asymptotics of the boundary value problems (1.1), (1.2) for constant coefficients was earlier investigated in [16] by directly analyzing the explicit formulas for a solution.

§**2.** The exposition mainly follows the article [67]. In the examination of the inner expansion the methods of [43] are used. For domains with the nonsmooth boundary, the asymptotics of the solution of an elliptic equation with a small parameter was studied in [115] (where there is no "inner" boundary layer), in [68], [77], [86], [87].

§**3.** The problem considered in this section was discussed back in [124], some estimates were given in [20], but a complete solution was obtained only in [65]. A similar analysis for an elliptic equation of higher order was conducted in [110].

§**4.** The asymptotics of the solution of the problem considered in this section was first obtained in [40], where both the exposition and techniques are far from perfect. In this section we use the technique developed in the subsequent papers ([66], [109], etc.). For an elliptic equation of higher order the asymptotics was studied, in a similar situation, in [111]. The explicit formulas mentioned in the remark to §4 were obtained in [37].

§**5.** The estimates of subsection 2 of this section were obtained in [110]. We also note that the method of matched asymptotic expansions is used not only in the case of elliptic equations but for a wide range of other boundary value problems. The asymptotics of solutions of pseudodifferential equations was studied in [93], in particular, for singular integral equations it was considered in [91], [92], for hyperbolic equations in [89]. Two other examples are considered in the last chapters of the present book.

Chapter V

In the case where the initial data decay at infinity faster than any polynomial, a problem which is slightly more general than (0.1), (0.2) was studied in [50]. In this paper the method of averaging [9] was used, which yields only the first terms of the asymptotics. Earlier, the method of averaging was applied to the analysis of the asymptotics of solutions for similar problems with periodic initial data (see [52], [54], [105], [112]).

For solutions exhibiting asymptotic stability at infinity, the method of matched asymptotic expansion in the form demonstrated in the present chapter proved more convenient. A detailed presentation of these results for general hyperbolic systems is given in [51], for other problems in [53]. These papers include the assumption that initial functions tend to their limits faster than any polynomial. The analysis of the asymptotics in Chapter V in the case where initial functions tend to their limits at infinity as powers is due to L. A. Kalyakin and is published here for the first time.

Chapter VI

The problem formulated in this chapter attracted mathematicians for a long time (see [34]). The most comprehensive study of the Cauchy problem for the limit equation in the most general case was made in the works by O. A. Oleĭnik ([96], [98]). Here the limit transition for the solutions of perturbed equations for $\varepsilon \to 0$ was considered and justified. However, the construction of the complete asymptotic expansion of the solution requires stronger restrictions on the initial function. This problem has been considered from different standpoints in the papers [6], [46], [104], [114], [8] etc.). The asymptotics in the vicinity of the discontinuity line of the limit equation, as well as the so-called soliton-like solutions, was recently studied for a wide range of problems ([74]–[76], [18], et al.). These problems are more difficult than the one considered in Chapter VI, still more so because one usually considers the effects of small dispersion instead of small dissipation. However, the analysis of the behavior of the solution in a neighborhood of the "gradient catastrophe" point is much more complicated, and has not been done yet.

§**1.** The results of this section are well known and are of an auxiliary nature.

§**2.** The problem presented in this section was published in [46], [88].

§**3–6.** These sections present the results of [39].

§**7.** The results of these section were obtained jointly with S. I. Khudyayev [41].

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