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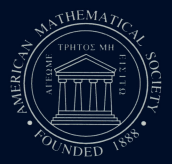
Volume 104

Wulff Construction

A Global Shape from Local Interaction

R. Dobrushin
R. Kotecký
S. Shlosman



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Providence, Rhode Island

Р. Л. ДОБРУШИН, Р. КОТЕЦКИ, С. Б. ШЛОСМАН

КОНСТРУКЦИЯ ВУЛФА: ФОРМА ОПРЕДЕЛЯЕТСЯ ЛОКАЛЬНЫМ ВЗАИМОДЕЙСТВИЕМ

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ABSTRACT. Crystal growth is a common physical phenomenon, and physicists could describe various aspects of this and similar processes rather well. However, until recently rigorous proof of various properties of these processes were not known. The present book fills one of these gaps. The authors consider the so-called Wulff construction (dated back to the beginning of the century), which predicts the final (equilibrium) shape of a crystal. From the mathematical point of view, the analysis of the Wulff construction is based on a careful study of the grand canonical ensemble for certain two-dimensional lattice spin models in statistical mechanics (the most famous among these models being the well-known Ising model).

To obtain the main results in the book (the theorem that predicts the final shape of the crystal from the optimality considerations for contours on the lattice), the authors develop, using mainly probabilistic technique, several interesting approaches, which may become useful in various problems of mathematical statistical mechanics.

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Preface

The present book contains, essentially, a proof of one theorem (Theorem 1.9) about the asymptotic shape of a droplet. That alone accounts for the long time it took to write down the proof: we found that when the length of a proof doubles, the time needed to write it down increases by a factor of four. The starting moment of this project took place in Třeboň, in September 1986, during the symposium "Statistical Mechanics of Phase Transitions—Mathematical and Physical Aspects", where Professor H. van Beijeren was talking about the Wulff construction. Another reason why it took so long to finish the manuscript was spatial separation between authors. But the travelling we underwent in the meantime improved (we hope) the quality of the book, since it was discussed a lot with many colleagues.

The authors would like to express their gratitude to various institutions where part of this work was done:

Rutgers University, CPT CNRS Marseille, Eidgenossische Technische Hochschule Zürich (all three authors), Cornell University (R. D. and S. S.), Harvard University (R. D. and R. K.), Rome University "La Sapienza" (S. S.) and "Tor Vergata" (R. D. and R. K.), École Polytechnique Fédérale Lausanne (R. K. and S. S.), Technion (R. D. and S. S.).

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Bibliography

[A] D. B. Abraham, *Surface structures and phase transitions — exact results*, Phase Transitions and Critical Phenomena (C. Domb and J. L. Lebowitz, eds.), vol. 10, Academic Press, 1987, pp. 1–74.

[ABSZ] J. E. Avron, H. van Beijeren, L. S. Schulman, and R. K. P. Zia, *Roughening transition, surface tension and equilibrium droplet shapes in a two-dimesional Ising system*, J. Phys. A **15** (1982), L 81–86.

[ACC] K. Alexander, J. T. Chayes, and L. Chayes, *The Wulff construction and asymptotics of the finite cluster distribution for two-dimensional Bernoulli percolation*, Comm. Math. Phys. **131** (1990), 1–50.

[AGL] M. Aizenman, S. Goldstein, and J. L. Lebowitz, *Conditional equilibrium and the equivalence of microcanonical and grandcanonical ensembles in the thermodynamic limit*, Comm. Math. Phys. **62** (1978), 279–302.

[AGM] D. B. Abraham, G. Gallavotti, and A. Martin-Löf, *Surface tension in the two-dimensional Ising model*, Physica **65** (1973), 73–88.

[Ai 1] M. Aizenman, *Instability of phase coexistence and translation invariance in two dimensions*, Phys. Rev. Lett. **43** (1979), 407–409.

[Ai 2] ———, *Translation invariance and instability of phase coexistence in the two-dimensional Ising system*, Comm. Math. Phys. **73** (1980), 83–94.

[AR] D. B. Abraham and P. Reed, *Diagonal interface in the two-dimensional Ising ferromagnet*, J. Phys. A **10** (1977), L 121–123.

[B] H. van Beijeren, *Exactly solvable model for the roughening transition of a crystal surface*, Phys. Rev. Lett. **38** (1977), 993–996.

[BC 1] S. Balibar and B. Castaign, *Possible observation of the roughening transition in helium*, J. Physique **41** (1980), L 329–332.

[BC 2] ———, *Helium: solid-liquid interfaces*, Surface Sci. Rep. **5** (1985), 87–143.

[BCF] W. K. Burton, N. Cabrera, and F. C. Frank, *The growth of crystals and the equilibrium structure of their surfaces*, Philos. Trans. Roy. Soc. London Ser. A **243** (1951), 40–359.

[BF 1] J. Bricmont and J. Fröhlich, *Statistical mechanical methods in particle structure analysis of lattice field theories* II: *Scalar and surface models*, Comm. Math. Phys. **98** (1985), 553–578.

[BF 2] ———, *Statistical mechanical methods in particle structure analysis of lattice field theories* III. *Confinement and bound states in gauge theories*, Nuclear Phys. B **280 [FS 18]** (1987), 385–444.

[BI] C. Borgs and J. Imbrie, *A unified approach to phase diagrams in field theory and statistical mechanics*, Comm. Math. Phys. **123** (1989), 305–328.

[BK] C. Borgs and R. Kotecký, *A rigorous theory of finite-size scaling at first-order phase transitions*, J. Statist. Phys. **61** (1990), 79–119.

[BLP] J. Bricmont, J. L. Lebowitz, and C. E. Pfister, *On the surface tension of lattice systems*, Ann. New York Acad. Sci. **337** (1980), 214–223.

[BN] H. van Beijeren and I. Nolden, *The roughening transition*, Topics in Current Physics (W. Schommers and P. von Blackenhagen, eds.), vol. 43, Springer-Verlag, 1987, pp. 259–300.

[BZ] Yu. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Grundlehren Math. Wiss., vol. 285, Springer-Verlag, New York, 1988.

[CDR] J. De Coninck, F. Dunlop, and V. Rivasseau, *On the microscopic validity of the Wulff construction and of the generalized Young equation*, Comm. Math. Phys. **121** (1989), 401–419.

[D 1] R. L. Dobrushin, *Gibbs state describing coexistence of phases for a three-dimensional Ising model*, Teor. Veroyatnost. i Primenen. **17** (1972), 619–639; English transl., Theory Probab. Appl. **17** (1972), 582–600.

[D 2] ———, *Induction in volume and no cluster expansion*, Proc. of the VIIIth International Congress of Mathematical Physics (Marseille, 1986; M. Mebkhout and R. Sénéor, eds.), World Scientific, Singapore, 1987, pp. 73–91.

[D 3] ———, *Asymptotic behaviour of Gibbsian distributions for lattice systems and its dependence on the form of the volume*, Teoret. Mat. Fiz. **12** (1972), 115–134; English transl., Theoret. and Math. Phys. **12** (1972), 699–711.

[Din] A. Dinghas, *Über einen geometrischen Satz von Wulff für die Gleichtgewichts-form von Kristallen*, Z. Krist. **105** (1944), 301–314.

[Dis] V. I. Diskant, *A generalization of Bonnesen's inequalities*, Dokl. Akad. Nauk SSSR **213** (1973), 519–521; English transl., Soviet Math. Dokl. **14** (1973), 1728–1731.

[DKS] R. L. Dobrushin, R. Kotecký, and S. B. Shlosman, *Equilibrium crystal shapes — a microscopic proof of the Wulff construction*, Proc. of the XXIVth Karpacz Winter School Stochastic Methods in Mathematics and Physics (Karpacz, 1988; R. Gielerak and W. Karwowski, eds.), World Scientific, Singapore, 1989, pp. 221–229.

[DN] R. L. Dobrushin and B. S. Nakhapetyan, *Strong convexity of the pressure for lattice systems of classical statistical physics*, Teoret. Mat. Fiz. **20** (1974), 223–234; English transl., Theoret. and Math. Phys. **20** (1974), 782–790.

[DS 1] R. L. Dobrushin and S. B. Shlosman, *The problem of translation invariance of Gibbs fields at low temperatures*, Sov. Sci. Rev. Sect. C: Math. Phys. Rev. **5** (1985), 53–185.

[DS 2] ———, *Thermodynamic inequalities for the surface tension and the geometry of the Wulff construction*, Ideas and Methods in Mathematical Analysis, Stochastics, and Applications, vol. 2 (S. Albeverio, J. F. Fenstad, H. Holden, T. Lindstrom, eds.), Cambridge University Press, Cambridge, 1992.

[DS 3] ———, *Large and moderate deviations in the Ising model and droplet condensation*, (in preparation).

[DS 4] ———, *Large deviation behaviour of the models of statistical mechanics in the multiphase regime*, (in preparation).

[DT] R. L. Dobrushin and B. Tirozzi, *The central limit theorem and the problem of equivalence of ensembles*, Comm. Math. Phys. **54** (1977), 173–192.

[DW] K. Druhl and H. Wagner, *Algebraic formulation of duality transformations for abelian models*, Ann. Physics **141** (1982), 225–253.

[FO] H. Föllmer and M. Ort, *Large deviations and surface entropy for Markov fields*, Astérisque **157–158** (1988), 173–190.

[Fe] H. Federer, *Geometric measure theory*, Grundlehren Math. Wiss., vol. 153, Springer-Verlag, 1969.

[Ga 1] G. Gallavotti, *Instabilities and phase transitions in the Ising model. A review*, Riv. Nuovo Cimento **2** (1972), 133–169.

[Ga 2] ———, *The phase separation line in the two-dimensional Ising model*, Comm. Math. Phys. **27** (1972), 103–136.

[Ge 1] H.-O. Georgii, *Canonical Gibbs measures*, Lecture Notes in Math., vol. 760, Springer-Verlag, 1979.

[Ge 2] ———, *Gibbs measures and phase transitions*, de Gruyter Stud. Math., vol. 9, de Gruyter, Berlin, 1988.

[GHM] C. Gruber, A. Hinterman, and D. Merlini, *Group analysis of classical lattice systems*, Lecture Notes in Phys., vol. 60, Springer-Verlag, 1977.

[GK] C. Gruber and H. Kunz, *General properties of polymer systems*, Comm. Math. Phys. **22** (1971), 133–161.

[GKK] K. Gawędzki, R. Kotecký, and A. Kupiainen, *Coarse-graining approach to first-order phase transitions*, J. Statist. Phys. **47** (1987), 701–724.

[Gne] B. V. Gnedenko, *The theory of probability*, Chelsea, New York, 1962.

[Gro] H. Groemer, *Stability properties of geometric inequalities*, Amer. Math. Monthly **97** (1990), 382–394.

[Ha] K. M. Halfina, *The limiting equivalence of the canonical and grand canonical ensemble (low density case)*, Mat. Sb. **80(122)** (1969), 3–51; English transl., Math. USSR-Sb. **9** (1969), 1–52.

[He] C. Herring, *Some theorems on the free energies of crystal surfaces*, Phys. Rev. **82** (1951), 87–93.

[Hi 1] Y. Higuchi, *On some limit theorems related to the phase line in the two-dimensional Ising model*, Z. Wahrsch. Verw. Gebiete **50** (1979), 287–315.

[Hi 2] ———, *On the absence of non-translationally invariant Gibbs states for the two-dimensional Ising system*, Rigorous Results in Statistical Mechanics and Quantum Field Theory (Proc. of the Colloqium on Random Fields, Esztergom, 1979; J. Fritz, J. L. Lebowitz, and D. Szász, eds.), Colloq. Math. Soc. János Bolyai, vol. 27, North-Holland, Amsterdam, 1982, pp. 517–534.

[HKZ] P. Holický, R. Kotecký, and M. Zahradník, *Rigid interfaces for lattice models at low temperatures*, J. Statist. Phys. **50** (1988), 755–812.

[HM 1] J. C. Heyraud and J. J. Métois, *Establishment of the equilibrium shape of metal crystallites on a foreign substrate: gold on graphite*, J. Cryst. Growth **50** (1980), 571–574.

[HM 2] ———, *Equilibrium shape and temperature; lead on graphite*, Surface. Sci. **128** (1983), 334–342.

[K] R. Kotecký, *Statistical mechanics of interfaces and equilibrium crystal shapes*, Proc. of the IXth International Congress of Mathematical Physics (Swansea, 1988; B. Simon, A. Truman, and I. M. Davies, eds.), Adam Hilger, Bristol, 1989, pp. 148–163.

[KP 1] R. Kotecký and D. Preiss, *An inductive approach to the Pirogov-Sinai theory*, Rend. Circ. Mat. Palermo (2) Suppl. **3** (1984), 161–164.

[KP 2] ———, *Cluster expansion for abstract polymer models*, Comm. Math. Phys. **103** (1986), 491–498.

[KPB] K. O. Keshishev, A. Ya. Parshin, and A. V. Babkin, *Crystallization waves in* He^4, Zh. Èksper. Teoret. Fiz. **80** (1981), 716–72; English transl., Soviet. Phys. JETP **53** (1981), 362–369.

[KPf] R. Kotecký and C. Pfister, (in preparation).

[KW] H. A. Kramers and G. H. Wannier, *Statistics of the two-dimensional ferromagnet* I, II, Phys. Rev. **60** (1941), 252–276.

[KZ] R. Kotecký and A. Ziermann, (in preparation).

[LLMBE] J. Landau, S. G. Lipson, L. M. Määttänen, L. S. Balfour, and D. O. Edwards, *Interface between superfluid and solid* ^{4}He, Phys. Rev. Lett. **45** (1980), 31–35.

[M] V. A. Malyshev, *Cluster expansions in lattice models of statistical physics and the quantum field theory*, Uspekhi. Mat. Nauk **35** (1980), no. 2, 3–52; English transl., Russian Math. Surveys **35** (1980), no. 2, 1–62.

[MH] R. A. Minlos and A. Haitov, *Limiting equivalence of thermodynamic ensembles in case of one-dimensional system*, Trudy Moskov. Mat. Obshch. **32** (1975), 147–186; English transl., Trans. Moscow Math. Soc. **32** (1975), 143–180.

[MKM] K. Matthes, J. Kerstan and J. Mecke, *Infinitely divisible point processes*, Wiley, 1978.

[ML] A. Martin-Löf and J. L. Lebowitz, *On the uniqueness of the equilibrium state for Ising spin systems*, Comm. Math. Phys. **25** (1972), 276–282.

[MM] V. A. Malyshev and R. A. Minlos, *Gibbs states; cluster expansions*, Kluwer, Dordrecht, 1991.

[MMR] A. Messager, S. Miracle-Solé, and J. Ruiz, *Convexity of the surface tension and equilibrium crystal* (to appear).

[MS 1] R. A. Minlos and Ya. G Sinaĭ, *The phenomenon of "phase separation" at low temperatures in some lattice models of a gas* I, Mat. Sb. **73** (1967), 375–448; English transl., Math. USSR-Sb. **2** (1967), 335–395.

[MS 2] ———, *The phenomenon of "phase separation" at low temperatures in some lattice models of a gas* II, Trudy Moskov. Mat. Obshch. **19** (1968), 113–178; English transl., Trans. Moscow Math. Soc. **19** (1968), 121–196.

[O 1] R. Osserman, *The isoperimetric inequality*, Bull. Amer. Math. Soc. **84** (1978), 1182–1238.

[O 2] ———, *Bonnesen-style isoperimetric inequalities*, Amer. Math. Monthly **86** (1979), 1–29.

[P] V. Privman, *Finite-size properties of the angle-dependent surface tension of rough interfaces*, Phys. Rev. Lett. **61** (1988), 183–186.

[Pf 1] C. E. Pfister, *Interfqce and surface tension in Ising model*, Scaling and Self-Similarity in Physics (J. Fröhlich, ed.), Birkhäuser, Basel, 1983, pp. 139–161.

[Pf 2] ———, *Large deviations and phase separation in the two-dimensional Ising model* (to appear).

[PN 1] A. Pavlovska and D. Nenow, *Experimental investigation of the surface melting of equilibrium form faces of diphenyl*, Surface. Sci. **27** (1971), 211–217.

[PN 2] ———, *Les surfaces non-singulières sur la forme d'équilibre du naphtalène*, J. Cryst. Growth **12** (1971), 9–12.

[PN 3] ———, *Experimental study of the surface melting of tetrabrommethane*, J. Cryst. Growth **39** (1977), 346–352.

[RW] C. Rottman and M. Wortis, *Statistical mechanics of equilibrium crystal shapes: Interfacial phase diagrams and phase transitions*, Phys. Rep. **103** (1984), 59–79.

[R] D. Ruelle, *Statistical mechanics: Rigorous results*, Benjamin, New York, 1969.

[S 1] S. B. Shlosman, *The droplet in the tube: A case of phase transition in the canonical ensemble*, Comm. Math. Phys. **125** (1989), 81–90.

[S 2] ———, *Wulff construction justified*, Proc. of the IXth International Congress of Mathematical Physics (Swansea, 1988; B. Simon, A. Truman, and I. M. Davies, eds.), Adam Hilger, Bristol, 1989, pp. 384–387.

[Sch] R. H. Schonman, *Second order large deviation estimates for ferromagnetic systems in the phase coexistence region*, Comm. Math. Phys. **112** (1987), 409–422.

[Si] Ya. G. Sinaĭ, *Theory of phase transitions: Rigorous results*, Pergamon Press, 1982.

[ŠPA] N. M. Švrakić, V. Privman, and D. B. Abraham, *Finite-size corrections for inclined interfaces in two-dimensions: Exact results for Ising and solid-on-solid models*, J. Statist. Phys. **53** (1988), 1041–1059.

[T 1] J. E. Taylor, *Unique structure of solutions to a class of nonelliptic variational problems*, Proc. Sympos. Pure Math. **27** (1975), 419–427.

[T 2] ———, *Some crystalline variational techniques and results*, Astérisque **154–155** (1987), 307–320.

[Th] R. L. Thompson, *Equilibrium states in thin energy shells*, Mem. Amer. Math. Soc. **150** (1974).

[W] G. Wulff, *Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Krystallflachen*, Z. Kryst. **34** (1901), 449–530.

[WBG] P. E. Wolf, S. Balibar, and F. Gallet, *Experimental observation of a third roughening transition on hcp* ^{4}He *crystals*, Phys. Rev. Lett. **51** (1983), 1366–1369.

[We] F. J. Wegner, *Duality in generalized Ising models and phase transitions without local order parameters*, J. Math. Phys. **12** (1971), 2259–2272.

[Z] R. K. P. Zia, *An isotropic surface tension and equilibrium crystal shapes*, Progress in Statistical Mechanics (C. K. Hu, ed.), World Scientific, Singapore, 1988, pp. 303–357.

[Za 1] M. Zahradník, *An alternate version of Pirogov-Sinai theory*, Comm. Math. Phys. **93** (1984), 559–581.

[Za 2] ———, *Analyticity of low-temperature phase diagrams of lattice spin models*, J. Statist. Phys. **47** (1988), 725–755.

[Zin] Yu. M. Zinoviev, *Duality in the abelian gauge lattice theories*, Teoret. and Mat. Fiz. **43** (1980), 309–322; English transl., Theoret. and Math. Phys. **43** (1980), 309–322.

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