Translations of

MATHEMATICAL MONOGRAPHS

Volume 110

Introduction to Complex Analysis

Part II

Functions of Several Variables

B. V. Shabat

American Mathematical Society

Introduction to Complex Analysis

Part II

Functions of Several Variables

This page intentionally left blank

10.1090/mmono/110

TRANSLATIONS OF
MATHEMATICAL
MONOGRAPHS
VOLUME 110

B.V. Shabat

Introduction to Complex Analysis

Part II
Functions of Several Variables

American Mathematical Society

Б. В. ШАБАТ

ВВЕДЕНИЕ В КОМПЛЕКСНЫЙ АНАЛИЗ Часть II

ФУНКЦИИ НЕСКОЛЬКИХ ПЕРЕМЕННЫХ

ИЗДАНИЕ ТРЕТЬЕ

Translated from the Russian by J. S. Joel
Translation edited by Simeon Ivanov

1991 *Mathematics Subject Classification.* Primary 32-01, 32-02.

Abstract. This book is based on several courses of lectures given by the author at Moscow State University. The book contains a detailed exposition of the main notions and ideas in the theory of functions in several complex variables. The translation is made from the 3rd Russian edition, into which the author included certain modern developments of the theory and its applications to physics. The book can serve as a textbook in a graduate course on functions in several complex variables.

Library of Congress Cataloging-in-Publication Data

Shabat, B. V. (Boris Vladimirovich)
[Vvedenie v kompleksnyĭ analiz. English]
Introduction to complex analysis/B. V. Shabat; [translated from the Russian by J. S. Joel].
v. ⟨2 ⟩; cm.—(Translations of mathematical monographs; v. ⟨110⟩)
Vol. ⟨2 ⟩ includes index. Contents: —pt. 2. Functions of several variables.
ISBN 0-8218-4611-6 (pt. 2: alk. paper)
1. Functions of complex variables. 2. Mathematical analysis. I. Title. II. Series.
QA331.7.S5213 1992 92-19985
515′.94—dc20 CIP

Printed in the United States of America

The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability. ♾
This publication was typeset using AMS-TeX,
the American Mathematical Society's TeX macro system.

Visit the AMS home page at URL: http://www.ams.org/

10 9 8 7 6 5 4 3 2 11 10 09 08 07 06

Contents

This page intentionally left blank

Foreword to the Third Edition

This book is intended as a first study of higher-dimensional complex analysis, i.e., the theory of holomorphic functions of several complex variables, holomorphic mappings, and submanifolds of complex Euclidean space. It is a continuation of the author's book *Introduction to Complex Analysis. Part I*; (*) some ideas that were only mentioned there can be found in this volume in their natural setting.

Higher-dimensional complex analysis is rather young in comparison with the one-dimensional theory. If we exclude works of G. Jacobi (1830 and 1857) and of F. Didon (1873), in which functions of two complex variables and integrals of them appear, and also those of Ch. Hermite (1852) and J. Sylvester (1854 and 1857), devoted to the solution of systems of equations in several unknowns, the beginnings of the higher-dimensional theory goes back to 1879 and the appearance of K. Weierstrass's paper *Some theorems related to the theory of analytic functions of several variables.* Another creator of the higher-dimensional theory is Henri Poincaré (1854–1912). In 1883 he published a paper in which he proved that a locally rational function of two variables is a quotient of two entire functions (in 1895 this result was extended to an arbitrary number of variables by P. Cousin, a student of Poincaré). In the same year, in joint work with E. Picard, he began the study of algebraic submanifolds of complex space. In papers from 1886 and 1887 Poincaré extended Cauchy's theorem to functions of two variables and established the foundations of a higher-dimensional theory of residues.

The first period of flowering of higher-dimensional complex analysis goes back to the beginning of this century. In 1907 Poincaré published a paper that anticipated later investigations of biholomorphic mappings of domains of complex Euclidean space. At roughly the same time there appeared a large series of papers by F. Hartogs, devoted to the analytic continuation of functions of several variables, as well as the important work of E. Levi.

However, later and for a rather long time, multidimensional problems of complex analysis remained in the background, and mathematicians con-

(*) *Editor's note.* "Nauka", Moscow, 1985; French transl., "Mir", Moscow, 1990. Throughout the text of this book there will be references to Part I.

cerned with them formed a small portion of the specialists in the theory of functions of one complex variable.

The situation changed rapidly in the 1960s, when higher-dimensional problems began to attract the attention of mathematicians with other specialties and theoretical physicists. One of the reasons for this was apparently the investigations begun in the 1930s by H. Cartan, K. Oka, and others, which connected problems of higher-dimensional complex analysis with algebra, topology, and algebraic geometry. Another reason was the discoveries made in the 1950s by N. N. Bogolyubov, V. S. Vladimirov, R. Jost, A. Wightman, and others, applying the theory of functions of several complex variables to quantum field theory.

There ensued the second period of flowering of higher-dimensional complex analysis, which has continued until now. Both classical and new results in this area have found numerous applications in analysis, differential and algebraic geometry, and, in particular, in contemporary mathematical physics. The mastery of the foundations of higher-dimensional complex analysis in many areas of modern mathematics has become necessary for any specialist.

In correspondence with the rapid development of higher-dimensional complex analysis in the last decade, this book has been significantly reworked in comparison with the previous edition (1976), and it does not make sense to describe the changes in detail. We mention only that as an example of applications we have included an elementary presentation of the twistor method for solving Maxwell's equations, proposed recently by R. Penrose.

This volume is based on a "special course" I presented at Moscow University a number of times under the auspices of the department of "Function theory and functional analysis". In the work on this edition I have widely used suggestions by my friends and students, to all of whom I express my deep appreciation.

B. V. Shabat

APPENDIX

Complex Potential Theory

Here we shall give of necessity a cursory survey of several themes that complement the main presentation. These themes are mostly concerned with higher-dimensional generalizations of harmonic functions related to the complex structure of $\mathbb{C}^n$.

1. Plurisubharmonic measure. In a number of questions of the function theory of one complex variables an important role is played by the *harmonic measure* of a subset E of the closure of some domain $D \subset \mathbb{C}$. This measure is a function $\omega_D(z, E)$, which at each point z is equal to the supremum of the values at this point of functions u belonging to the class $\mathrm{sh}(D)$ of subharmonic functions in D, which are nonpositive in D and do not exceed -1 on E. The function $\omega_D(z, E)$ turns out to be harmonic in $D \setminus \overline{E}$ (this is what justifies the name); to learn about its properties, see, for example, the book by G. M. GOLUZIN, *Geometric theory of functions of a complex variable*, 2nd ed., "Nauka", Moscow, 1966; English transl., Transl. Math. Monographs, vol. 26, Amer. Math. Soc., Providence, RI, 1969. (We remark that the function ω_D considered here is expressed using the classical ω via the formula $\omega_D = 1 - \omega$; this is due to the fact that we prefer to deal with subharmonic functions, not with superharmonic ones.)

When we pass to domains $D \subset \mathbb{C}^n$, $n > 1$, it is natural to replace $\mathrm{sh}(D)$ by the class $\mathrm{psh}(D)$ of plurisubharmonic functions in D:

$$\omega_D(z, E) = \sup\{u(z) \colon u \in \mathrm{psh}(D),\ u \le 0,\ u|_E \le -1\}. \tag{1}$$

However, for $n > 1$ it is not only not harmonic, but not even upper semicontinuous. Therefore, instead of ω_D we consider its so-called UPPER REGULARIZATION

$$\omega_D^*(z, E) = \limsup_{z' \to z} \omega_D(z', E), \tag{2}$$

which is already upper semicontinuous and, hence, plurisubharmonic in $D \setminus E$. This function is called the *plurisubharmonic measure*, or, more briefly, the *p-measure* of a set relative to the domain D.[1]

Although the unregularized function ω_D is equal to -1 on the set E, under regularization a jump upwards can occur at some points of E. A

[1] See the paper by A. SADULLAEV, *Plurisubharmonic measures and capacities on complex manifolds*, Uspekhi Mat. Nauk **36** (1981), no. 4, 53–105; English transl., Russian Math. Surveys **36** (1981), no. 4, 61–119. See also the author's book, cited at the end of §20.

closed set $E \subset \overline{D}$ is said to be *pluriregular* if such jumps do not arise, i.e., $\omega_D^*|_E = -1$. A domain D is said to be *strongly Pseudoconvex* if it is bounded and if its defining function belongs to $C(\overline{D}) \cap \mathrm{psh}(\overline{D})$. If a domain D is strongly pseudoconvex and $E \subset\subset D$ is a pluriregular set, then V. P. ZAKHARYUTA showed in 1976 that the function $\omega_D^*(z, E)$ is continuous in D.

Just as for $n = 1$ the function ω_D satisfies the Laplace equation in $D \setminus E$, its higher-dimensional analogue ω_D^* is related to a nonlinear partial differential equation, the so-called complex Monge-Ampère equation. In fact this equation appears in a number of problems as a natural higher-dimensional analogue of the Laplace equation, reflecting (in contrast to the higher-dimensional Laplace equation) the complex structure of $\mathbb{C}^n$.

For functions of class C^2 in a domain $D \subset \mathbb{C}^n$ the complex Monge-Ampère operator is defined as the nth exterior power of the operator $dd^c = \frac{i}{2}\partial\overline{\partial}$ (see subsection 18):

$$(dd^c u)^n = n! \det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) \prod_{j=1}^{n} \frac{i}{2} dz_j \wedge d\overline{z}_k \tag{3}$$

and the *complex Monge-Ampère equation* in $\mathbb{C}^n$ has the form(2)

$$(dd^c u)^n = 0 \Leftrightarrow \det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) = 0. \tag{4}$$

For $n = 1$ this equation is the same as the Laplace equation.

If u is a plurisubharmonic function of class C^2, then $dd^c u \geq 0$ (see subsection 38). Using the theory of so-called positive currents, BEDFORD and TAYLOR(3) defined a Monge-Ampère operator in the generalized sense for arbitrary continuous and even for locally bounded plurisubharmonic functions.

The Dirichlet problem for the Monge-Ampère equation in a domain $D \subset \mathbb{C}^n$ consists in finding a solution of this equation which takes given values φ on the boundary ∂D. For $n = 1$ one of the methods of solving this problem is PERRON'S METHOD; extending it to domains $D \subset \mathbb{C}^n$, $n > 1$, we consider the function

$$\omega(z) = \sup\{u(z) \colon u \in \mathrm{psh}(D),\ u|_{\partial D} \leq \varphi\}; \tag{5}$$

its upper regularization $\omega^*(z) = \limsup_{z' \to z} \omega(z') \in \mathrm{psh}(D)$. If in addition D is strictly pseudoconvex and $\varphi \in C(\partial D)$, then Bedford and Taylor proved that ω^* is a distributional solution of the Dirichlet problem. These same authors proved a *maximality property* for continuous plurisubharmonic solutions of the Monge-Ampère equations: if $u \in C(\overline{D})$ is such a solution, and

(2)The real Monge-Ampère equation in domains $D \subset \mathbb{R}^n$ is the equation $\det(\frac{\partial^2 u}{\partial x_j \partial x_k}) = 0$; it plays an important role in geometry.

(3)E. BEDFORD AND B. A. TAYLOR, *The Dirichlet theorem for a complex Monge-Ampère equation*, Invent. Math. 37 (1976), 1–44.

$v \in \text{psh}(D) \cap C(\overline{D})$ is an arbitrary function, then

$$u|_{\partial D} \geq v|_{\partial D} \Rightarrow u(z) \geq v(z), \qquad z \in D. \tag{6}$$

Under the conditions mentioned above this property obviously guarantees the uniqueness of the solution of the Dirichlet problem for the Monge-Ampère equation.

THEOREM 1. *If D is a pseudoconvex domain and $E \subset\subset D$ is a pluriregular set, then the p-measure $\omega_D^*(z, E)$ is a continuous solution of the Monge-Ampère equation everywhere in $D \setminus E$.*

◀ Let $B \subset\subset D \setminus E$ be an arbitrary ball and u_B the solution of the Dirichlet problem $(dd^c u)^n = 0$, $u|_{\partial B} = \omega_D^*|_{\partial B}$; under the hypotheses $\omega_D^* \in C(\overline{D})$, and the solution of this problem exists and is continuous in $\overline{B}$. In view of the maximality property (6) we have $u_B(z) \geq \omega_D^*(z, E)$ everywhere in B. Therefore the function

$$v(z) = \begin{cases} u_B(z) & \text{if } z \in \overline{B}, \\ \omega_D^*(z, E) & \text{if } z \in D \setminus \overline{B} \end{cases}$$

belongs to psh(D) as it is the upper envelope of two plurisubharmonic functions. It is nonpositive in D (since it could take positive values only in B, and then it would attain its maximum there, which contradicts the plurisubharmonicity) and is equal to -1 on E. Thus, by the definition of p-measure $v(z) \leq \omega_D^*(z, E)$ in D and, in particular, $u_B(z) \leq \omega_D^*(z, E)$ in B. Combining this with the opposite equality obtained above, we find that ω_D^* in B coincides with the solution of the Monge-Ampère equation. ▶

REMARK. From this proof we see that any continuous plurisubharmonic function in a domain D which possesses the maximality property in subdomains $G \subset\subset D$ is a distributional solution of the Monge-Ampère equation in D.

We shall indicate the connection between the p-measure and thin sets (i.e., subsets of D on which some function $\not\equiv -\infty$ from psh(D) is equal to $-\infty$; see page 354 above); such subsets are termed *pluripolar.* By the maximum principle the p-measure $\omega_D^*(z, E)$ is either nowhere vanishing in D or $\equiv 0$. A. SADULLAEV proved (see the article cited at the beginning of this appendix) that the latter case characterizes pluripolar sets.

THEOREM 2. *If a domain D is strongly pseudoconvex, then a set $E \subset D$ is pluripolar if and only if $\omega_D^*(z, E) \equiv 0$.*

We note another generalization of the TWO-CONSTANT theorem, which follows simply from the definition of p-measure.

THEOREM 3. *Let u be a plurisubharmonic function in a domain $D \subset \mathbb{C}^n$ with $u \leq m$ on some set $E \subset \overline{D}$ and not exceeding M anywhere in D. Then for all $z \in D$*

$$u(z) \leq M(1 + \omega_D^*(z, E)) - m\omega_D^*(z, E). \tag{7}$$

For other applications of p-measure see, for example, the author's book cited at the end of §20.

2. Invariant Green's function. Green's function of a domain $D \subset \mathbb{C}$ is the fundamental solution of the Laplace equation with a singularity of type $\ln|z-w|$ at a fixed point $w \in D$, and is equal to 0 on ∂D. We shall describe a higher-dimensional generalization of this, due to E. A. POLETSKIĬ, which reflects the complex structure; the role of the Laplace equation here is again taken by the Monge-Ampère equation.

In a domain $D \subset \mathbb{C}^n$ we fix two distinct points z and w, we consider a holomorphic mapping f of the unit disc $U \subset \mathbb{C}$ into D such that $f(0) = z$, and we write

$$u_f(z, w) = \sum_{\zeta_\nu \in f^{-1}(w)} k_\nu \ln|\zeta_\nu|, \tag{1}$$

where the sum is taken over all inverse images $\zeta_\nu \in U$ of the point w, and k_ν is the multiplicity of f at the point ζ_ν (see Problem 1 to Chapter III); if there are no such inverse images, we set $u_f = 0$. We shall call the function

$$g_D(z, w) = \inf u_f(z, w), \tag{2}$$

where the infimum is taken over all holomorphic mappings $f: U \to D$, $f(0) = z$, the *invariant Green's function* of the domain D with a singularity at the point w.

It is obvious that $g_D(z, w) \le 0$, and for bounded domains in a neighborhood of w we have $g_D(z, w) = \ln|z-w| + O(1)$, where $O(1)$ is a bounded function. It is also obvious that g_D is invariant relative to biholomorphic mappings, and this is what justifies the name.

Proofs of the following results can be found in the paper by E. A. POLETSKIĬ and the author, *Invariant metrics*, Itogi Nauki i Tekhniki: Sovremennye Problemy Mat.: Fundamental'nye Napravleniya, vol. 9, VINITI, Moscow, 1986, pp. 73–127; English transl. in Encyclopedia of Math. Sci., vol. 9 (Several Complex Variables. III), Springer-Verlag, Berlin, 1989.

THEOREM 1. *For any fixed $w \in D$ the function g_D is plurisubharmonic with respect to z in the domain D.*

THEOREM 2. *The function $g_D(z, w)$ can be defined by the relation*

$$g_D(z, w) = \sup v_w(z), \tag{3}$$

where the supremum is taken over all nonpositive functions $v_w \in \operatorname{psh}(D)$ in D, which in a neighborhood of a point $w \in D$ have the form $\ln|z-w| + O(1)$.

THEOREM 3. *If $D \subset \mathbb{C}^n$ is a strongly pseudoconvex domain, then the function $g_D(z, w)$ satisfies the complex Monge-Ampère equation with respect to z in D.*

COROLLARY. *For $n = 1$ the function $g_D(z, w)$ coincides with the classical Green's function.*

The invariant Green's function is related to the invariant metrics of Carathéodory and Kobayashi. We shall call the function

$$c_1(z, w) = \inf \ln(1/|f_w(z)|), \tag{4}$$

where the infimum is taken over holomorphic functions $f_w: D \to U$, $f_w(w) = 0$, the *Carathéodory function* of the domain $D \subset \mathbb{C}^n$. The *Kobayashi function* of the domain D is defined as

$$k_1(z, w) = \sup(1/|\zeta|), \tag{5}$$

where the supremum is taken over all holomorphic curves $f_z: U \to D$, $f_z(0) = z$, and ζ is an inverse image of the point $w \in D \setminus \{z\}$ with smallest modulus. These functions are connected by a simple relation, respectively, with the Carathéodory distance $c_D(z, w)$ and with the "one-link distance" of Kobayashi $\tilde{k}(z, w)$ (see the exercise following the proof of Theorem 1 of subsection 58).

Both functions c_1 and k_1 are symmetric relative to their arguments and are lower semicontinuous with respect to each of them. For bounded domains D in a neighborhood of the point w they both have the form $-\ln|z - w| + O(1)$. It is not hard to see that these functions are related to the invariant Green's function of the domain D via the two-sided inequality

$$k_1(z, w) \le -g_D(z, w) \le c_1(z, w). \tag{6}$$

The Carathéodory function $c_1(z, w)$ is plurisuperharmonic with respect to each argument when the other one is fixed, but the Kobayashi function does not always possess this property. If it is plurisuperharmonic, then $-k_1(z, w) \in \text{psh}(D \times D)$, and, in view of (6), $-k_1(z, w) \ge g_D(z, w)$; Considering also that $-k_1$ has a singularity of the necessary form, by Theorem 2 we conclude that in this case $k_1(z, w) \equiv -g_D(z, w)$. If for some domain D the Carathéodory distance c_D and the Kobayashi distance $\tilde{k}_D$ coincide, then $k_1 \equiv c_1$ for this domain; in view of (6) these functions then coincide with $-g_D$, and, hence, are plurisuperharmonic and satisfy the Monge-Ampère equation. L. Lempert proved (see the reference at the beginning of subsection 62) that these three functions coincide for all geometrically convex domains $D \subset \mathbb{C}^n$.

3. Pseudoconcave sets. A closed subset Σ of a domain $D \subset \mathbb{C}^n$ is said to be *Pseudoconcave* if its complement $D \backslash \Sigma$ is pseudoconvex. Following E. M. Chirka, we consider some results connected with such sets. These results are centered around Hartogs's theorem of the analyticity of the set of singularities from subsection 42. Since the set of singularities of a holomorphic function is pseudoconcave, the Hartogs's theorem can be reformulated as follows: if the pseduconcave set $\Sigma = \{(z, w): z \in D, w \in \mathbb{C}\}$ is the graph of some function g, then this function is holomorphic in D.

We need a local maximum principle for plurisubharmonic functions.

THEOREM 1. *Let Σ be a pseudoconcave subset of a domain $D \subset \mathbb{C}^n$ and let $u \in \operatorname{psh}(D)$. Then for any domain $G \subset\subset D$*

$$\max_{\Sigma\cap\overline{G}} u = \max_{\Sigma\cap\partial G} u. \tag{1}$$

◄ Suppose that $M = \max_{\Sigma\cap\overline{G}} u$ is attained at an interior point $z^0 \in G$, and $\max_{\Sigma\cap\partial G} u < M$. Without loss of generality the function u can be assumed to be strictly plurisubharmonic, and that the maximum is strict (see subsection 38). Then the real hypersurface $S = \{z \in G : u(z) = M\}$ lies entirely, except for the point z^0, outside of Σ. Since the set where $u(z) > M$ is not a domain of holomorphy, then there is a family of holomorphic discs S_t, $0 < t < 1$, lying outside of Σ, which tends as $t \to 0$ to the disc S_0, where ∂S_0 lies outside of Σ, but $S_0 \ni z^0$. According to the continuity principle (subsection 36), the complement to Σ then cannot be a domain of holomorphy. ►

For what follows we assume that $\Sigma \subset D \times \mathbb{C}$, where $D \subset \mathbb{C}^n$, and the projection $\pi: \Sigma \to D$ is a proper mapping.([4]) For $z \in D$ we write $\Sigma_z = \{w \in \mathbb{C} : (z, w) \in \Sigma\}$.

We need the following assertion: an upper semicontinuous function $\varphi > 0$ in $D \subset \mathbb{C}$ has a subharmonic logarithm if the maximum principle holds for $|e^p|\varphi$, where $p = p(z)$ is an arbitrary polynomial. The idea of the proof is based on the fact that if $\operatorname{Re} p + \ln\varphi$ possesses this property, then $h + \ln\varphi$ also possesses it, where h is an arbitrary harmonic function, and the subharmonicity of $\ln\varphi$ is easy to derive from this.

LEMMA 1. *If D is a domain in $\mathbb{C}^n$, $\Sigma \subset D \times \mathbb{C}$ is a pseudoconcave set, and $u \in \operatorname{psh}(D \times \mathbb{C})$, then the function*

$$v(z) = \max_{w\in\Sigma_z} u(z, w) \in \operatorname{psh}(D). \tag{2}$$

◄ The upper semicontinuity of v is obvious, and we need only check that its restriction to any complex line l is subharmonic in $l \cap D$. Therefore without loss of generality we may assume that $D \subset \mathbb{C}$. It suffices to prove that the maximum principle holds for the function $|e^p|e^v$, where p is an arbitrary polynomial.

Let $U \subset\subset D$ be the disc and suppose that the maximum is attained in it at a point z_0, where $|e^{p(z_0)}|e^{v(z_0)} = \max_{w\in\Sigma_{z_0}} |e^{p(z_0)}|e^{u(z_0, w)}$. But the function $|e^{p(z)}|e^{u(z,w)} \in \operatorname{psh}(D \times \mathbb{C})$, and the set Σ is pseudoconcave, so that by Theorem 1 we have

$$|e^{p(z_0)}|e^{v(z_0)} \le \max_{z\in\partial U} \max_{w\in\Sigma_z} |e^{p(z)}|e^{u(z,w)} = \max_{z\in\partial U} |e^{p(z)}|e^{v(z)},$$

and the maximum principle holds. ►

([4])We can arrive at this situation by a linear-fractional transformation, considering the intersections of $\Sigma \subset \mathbb{C}^{n+1}$ with complex lines passing through a fixed point $a \in \mathbb{C}^{n+1} \setminus \Sigma$.

From this lemma by induction on k we obtain the following:

LEMMA 2. *Let D be a domain in $\mathbb{C}^n$ and suppose that $u(z, w)$, where $z \in D$ and $w = (w_1, \dots, w_k) \in \mathbb{C}^k$, is a plurisubharmonic function in $D \times \mathbb{C}^k$. If $\Sigma \subset D \times \mathbb{C}$ is a pseudoconcave set and $\Sigma_z^k = \Sigma_z \times \cdots \times \Sigma_z$ (k times), then the function*

$$v(z) = \max_{w \in \Sigma_z^k} u(z, w) \in \operatorname{psh}(D). \tag{3}$$

A generalization of Hartogs's theorem was obtained by K. OKA in 1934, but became accessible thanks to a 1962 paper by T. NISHINO. The latter proof is based on his theorem concerning the plurisubharmonicity of the function $\ln \operatorname{diam} \Sigma_z$, where Σ is a pseudoconcave set. We give a stronger variant of this theorem, in which the diameter of the set Σ_z is replaced by its capacity. We recall that the *capacity* of a closed set $E \subset \mathbb{C}$ is the quantity

$$\operatorname{cap} E = \lim_{k \to \infty} \left(\max \prod_{1 \le i < j \le k} |w_i - w_j| \right)^{2/(k(k-1))}, \tag{4}$$

where the maximum is taken over all possible arrangements of the points $w_1, \dots, w_k \in E$; the limit here exists, since the sequence decreases (see Goluzin's book, cited at the beginning of this appendix).

THEOREM 2. *If D is a domain in $\mathbb{C}^n$, and $\Sigma \subset D \times \mathbb{C}$ is a pseudoconcave set, then $\operatorname{cap} \Sigma_z$ is logarithmically plurisubharmonic in D.*

◀ If $w_1, \dots, w_k$ are distinct points of the set Σ_z, then the function $\ln \prod |w_i - w_j| = \sum \ln |w_i - w_j|$, where the product and the sum are taken over all sets $1 \le i < j \le k$, is plurisubharmonic with respect to $w = (w_1, \dots, w_k)$ in $\mathbb{C}^k$. Then by Lemma 2 we also have

$$\delta_k(z) = \frac{2}{k(k-1)} \max_{w \in \Sigma_z^k} \ln \prod |w_i - w_j| \in \operatorname{psh}(D) \tag{5}$$

for any k. Since $\ln \operatorname{cap} \Sigma_z = \lim_{k \to \infty} \delta_k(z)$ and $\delta_k(z)$ decreases as k grows, then $\ln \operatorname{cap} \Sigma_z \in \operatorname{psh}(D)$. ▶

Using Theorem 2 we can give a significantly simpler proof of OKA'S THEOREM than Nishino's proof.

THEOREM 3. *Let D be a domain in $\mathbb{C}^n$ and let $\Sigma \subset D \times \mathbb{C}$ be a pseudoconcave set. If for each point z belonging to a nonpluripolar set $M \subset D$ the fiber Σ_z consists of a finite number of points, then Σ is an analytic set.*

◀ We write $M_j = \{z \in M : \operatorname{card} \Sigma_z \le j\}$, where card is the number of points; this is an increasing sequence of sets, exhausting M. By the properties of the p-measure at least one of these sets, say M_k, is nonpluripolar. Then the function $\delta_{k+1}|_{M_k} = -\infty$, and since it is plurisubharmonic in D

and M_k is not pluripolar, then $\delta_{k+1}(z) \equiv -\infty$ in D. From this it follows that $\operatorname{card}\Sigma_z \leq k$ for all $z \in D$, and by Hartogs's theorem (at the very end of Chapter III), Σ is an analytic set. ▶

Finally, we observe that in a cycle of papers Sadullaev has found substantial applications of the concepts of pseudoconcavity in problems of the approximation of functions of several complex variables by rational functions.

Subject Index

ISBN 0-8218-1975-5

9 780821 819753

MMONO/110.S