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Introduction to the General Theory of Singular Perturbations

S. A. Lomov

American Mathematical Society

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ВВЕДЕНИЕ В ОБЩУЮ ТЕОРИЮ СИНГУЛЯРНЫХ ВОЗМУЩЕНИЙ

Translated from the Russian by J. R. Schulenberger
Translation edited by Simeon Ivanov

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Abstract. The book presents in a systematic manner for the first time a general approach to the integration of singularly perturbed differential equations describing nonuniform transitions such as the occurrence of a boundary layer, discontinuities, boundary effects, etc. The method of regularization of singular perturbations presented in the book is applied to the asymptotic integration of systems of ordinary differential equations (linear and nonlinear) and linear partial differential equations.

The book is intended for physicists, mathematicians, engineers, and students who come in contact with applied mathematics.

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Preface to the English Edition

The theory of singular perturbations is not at present a settled direction in mathematics, and the path of its development is, in our view, a dramatic one. On one hand, many practical problems, such as the mathematical boundary layer theory or approximation of solutions of various problems described by differential equations involving large or small parameters, certainly are in need of creation of a general theory. From the purely theoretical stand such a theory is necessary to at least achieve a deeper understanding of how smoothness is preserved in the presence of degeneracy. On the other hand, if one looks at the plans of mathematical development, one is left with the impression that such a problem did not exist—it is absent from those plans. I have in mind, primarily, the plans in our country—the USSR. But I have the impression that the situation in other countries is not much different—judging by the published papers and monographs on the subject.

It is known that mathematicians, physicists, mechanics have developed a multitude of various asymptotic methods which, at present, constitute the basis of the theory of singular perturbations. In 1984, however, on the 80th anniversary of L. Prandtl's boundary layer theory, the well-known researcher K. Nickel said the following in his review article [1*] † : "Many fundamental problems (of existence, stability and instability, etc.) have been solved in the mathematical theory of boundary layer. However, there is no satisfactory theory whatever of the phenomenon of boundary layer."

This circumstance is explained, in our view, by the incorrect extension of the notion of Poincaré asymptotic series to functions with dual dependence on the variable. And the functions describing boundary effects and generally-nonuniform transfers are of this type.

At the basis of our book we put a new notion of asymptotic series for solutions of singularly perturbed problems—for solutions that depend dually on the perturbation: regularly and singularly, which is well illustrated by the following elementary example:

$$\varepsilon \dot{y} + e^{t} y = e^{2t}, \qquad y(0, \varepsilon) = y^{0}, \ \varepsilon \to 0, \tag{0.1}$$

† *Note*: An asterisk following a reference number indicates a supplementary reference, to be found at the end of the References chapter.

a solution of which is the function

$$y(t, \varepsilon) = e^{(1-e^t)/\varepsilon}[y^0 - 1 + \varepsilon] + e^t - \varepsilon. \tag{0.2}$$

Here ε appears singularly in the first exponent (the latter does not exist at $\varepsilon = 0$), and in the rest the dependence on ε is regular. Investigations have shown that there is a unique possible description of the singular dependence on the perturbation (in this case on ε), in which the sum of the asymptotic series in powers of ε may potentially coincide with the exact solution. Such series are called regularized in the book.

With the earlier notion of asymptotic series the phenomenon of coincidence of the sum with this exact solution was lost, the description of the singular dependence was non-single-valued, a consequence of which was the development of many asymptotic methods. The functions describing the boundary layer are pseudoanalytic in ε (in "viscosity", as one would say in hydrodynamics), i.e., they are analytic with respect to the regular dependence on ε for fixed ε in the singular dependence. In the above example the function (0.2) is a pseudo-entire function of ε, since the expression

$$e^{(1-e^t)/\varepsilon^*}[y^0 - 1 + \varepsilon] + e^t - \varepsilon$$

is an entire function with respect to ε. Such are the specifics of the solutions in the presence of a singular point $\varepsilon = 0$ in equations similar to the equation (0.1). Investigations have also shown that the uniquely possible description of the singular dependence is determined by the spectrum of a pencil of operators corresponding to the singularly perturbed problem considered. The variable spectrum of bounded and unbounded operators causes additional difficulties in developing a general theory. However, these difficulties have become tractable; after the first publication of this book a theory of asymptotic integration in the presence of certain spectral singularities has been developed (see [2*]). This refines Sections 1 and 2 of Chapter 6 of the book and is a development of the theory of inner boundary layer. In the extension of the method to noncompact domains, where the additional singular point $t = \infty$ appears, a certain development of the theory has also been obtained (see [3*]).

New problems with continuous spectrum of the pencil of operators (see [4*]) and problems with periodic solutions that are analytic with respect to the perturbation (see [5*]) have been solved. The theory of analytic and pseudo-analytic solutions has been developed more fully (see [6*], [7*]), and a new concept of analytic integration has been formulated (see [8*]). In the survey article [9*] the regularization method for bounded operators of simple structure, and also for operator with Jordan structure, is presented. The existence of pseudo-analytic solutions also for nonlinear singularly perturbed ordinary differential equations was proved in [10*], where the method of normal forms for asymptotic integration of the corresponding nonlinear problems, both in the resonance and resonance-free cases, was developed. However, there remain, as always, many unsolved problems.

Moscow, December 1991

S. Lomov

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Preface

In the intensive development of science and technology mathematical models of the real world become more complex, and, therefore, in their analysis it is natural to use asymptotic methods. However, the asymptotic analysis for differential operators has a developed theory mainly for the case of regular perturbations, when the perturbations carry a subordinate character with respect to the unperturbed operator. As concerns singularly perturbed problems, i.e., problems with perturbations of the principal parts of the operator or, in another terminology, problems with a small parameter at the highest derivatives, until recently methods of asymptotic integration of them have been worked out individually for different classes of problems.

There is no doubt as to the importance of results of investigations of singularly perturbed problems. Such problems arise naturally where there are nonuniform transitions from certain physical characteristics to others. It is known, for example, that in problems connected with the solution of the Navier-Stokes equations for a small viscosity these nonuniformities create a boundary layer zone. Without a thorough asymptotic analysis it is difficult to create a mathematical theory of the boundary layer or to carry out numerical computations of singularly perturbed problems. Many problems of the theory of nonlinear oscillations, the theory of automatic control, and the theory of gyroscopes can be described by means of differential equations containing a small parameter at the highest derivatives. Therefore, the theory of asymptotic analysis is of major importance both for the development of fundamental investigations and for the solution of concrete practical problems.

In the present monograph a general approach is developed for solution of singularly perturbed problems. This approach is based on a general method proposed by the author of regularizing singularly perturbed problems by means of passing to a space of higher dimension, which is induced by the original problem. This induced space is determined by the spectral characteristics of the original operator, which affords the possibility of using the spectral theory of operators. The last circumstance allowed the author to classify singularly perturbed problems into discrete and continuous types and to study them by a unified scheme. Singularity of the original problem calls forth the occurrence of a nonzero kernel of the leading operator of the induced problem. A correct choice of the space of solutions of the induced problem makes it possible to use methods of functional analysis to describe both normal and unique solvability of such problems.

This approach enabled S. A. Lomov to create the foundations of general asymptotic analysis of singularly perturbed problems. The book is devoted to a description of this approach and its application to a broad class of various problems, both linear and nonlinear. This class of problems includes the Cauchy problem and a boundary value problem for ordinary differential equations and systems of equations, for partial differential equations, as well as the general case of abstract operator equations in Hilbert space. This class of problems also includes operators with continuous spectrum. In spite of the multitude of problems different in nature, the method of solving them remains the same. Among the solved examples we note the complete solution of the one-dimensional linearized problem of the Navier-Stokes equations in an interior domain and the solution of an inhomogeneous equation with a turning point.

There is no doubt that the method presented will receive further development and will encompass a broad circle of still unsolved problems.

A. N. Tikhonov

Author's Preface

Many physical processes connected with nonuniform transitions are described by differential equations with large or small parameters. If, in problems arising in this manner, the role of the perturbation is played by the leading terms of the operator (or a part of them), then the problem is called a singularly perturbed problem. An example where a small parameter occurs in a natural manner in front of the leading part of the differential operator is the Schrödinger equation in which the quantity of a quantum of action occurs as the small parameter. If this quantity tends to zero, then certain laws of quantum mechanics go over into laws of classical mechanics. For the approximate solution of singularly perturbed problems there is a large collection of asymptotic methods, each of which makes it possible to solve a particular circle of problems.

At the present time on the basis of the regularization method, which makes it possible to reduce a singularly perturbed problem to a regularly perturbed problem, it is possible to develop the foundations of a general theory of singular perturbations. This book is devoted to an exposition of these foundations. The first chapter is, on the one hand, a survey and brief history of the development of the theory of singular perturbations and, on the other hand, an exposition of the basic ideas of the regularization method on this background. Chapters 2–7 contain the foundations of the asymptotic integration of various problems for ordinary differential equations, both linear (Chapters 2–6) and nonlinear (Chapter 7). For partial differential equations the theory of asymptotic integration in the case of discrete spectrum of the limit operator is presented in Chapters 8 and 9. Some singularly perturbed evolution equations, both with discrete and continuous spectrum of the limit operator, are studied in Chapter 10. The content of the book is reflected in more detail in the table of contents.

The major part of the results presented in this book have not been published in a detailed exposition. Some results are presented for the first time. In writing this book the author strove that it be accessible not only to mathematicians and physicists, but also to engineers, students, and all those who use applied mathematics in their work.

The author is deeply thankful to A. N. Tikhonov for useful advice and expresses his sincere gratitude to Professors E. A. Grebennikov and M. M. Khapaev, whose critical remarks contributed to the improvement of the book. M. A. Valiev, Yu. P. Gubin, A. G. Eliseev, A. D. Ryzhikh, V. F. Safonov, and A. S. Yudina helped me with the work in separate chapters of the book. The attentative and qualified editing of A. S. Chistopol′skiĭ assisted in eliminating from the book all that should have been eliminated. The author thanks them all for their help. I am also obliged to those who helped me in the technical organization of the manuscript.

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S. Lomov

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