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Typical Singularities of Differential 1-Forms and Pfaffian Equations

Michail Zhitomirskiĭ

American Mathematical Society

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Typical Singularities of Differential 1-Forms and Pfaffian Equations

Michail Zhitomirskiĭ

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МИХАИЛ ЖИТОМИРСКИЙ

ТИПИЧНЫЕ ОСОБЕННОСТИ ДИФФЕРЕНЦИАЛЬНЫХ 1-ФОРМ И УРАВНЕНИЙ ПФАФФА

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1991 *Mathematics Subject Classification.* Primary 58-02; Secondary 35F20, 53C15, 58A10, 58A17, 58A30, 58F36, 93B52.

ABSTRACT. This book is devoted to the problems formulated by J. Pfaff at the start of the 19th century: to what simplest form can a differential form be reduced by a change of coordinates (local classification of differential 1-forms) or by a change of coordinates and multiplication by a function (local classification of Pfaffian equations)? Answers to these classification problems are applied to basic questions related to the geometry of singularities, stable and finitely determined germs, and normal forms. Modern applications of these classification results include contact geometry, theory of differential equations, control theory, nonholonomic dynamics, and variational problems. Some of these applications are given in the first three appendices. In the other appendices, analytic and topological classification, the classification of distributions and modules of vector fields, and the classification of closed differential 2-forms are discussed. This book contains many fundamental concepts, techniques, and methods for the study of singularities that appear in any classification problem.

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To the Memory of
Professor Jean Martinet

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Contents

Introduction

Ordinarily in any local classification problem interest focuses on the generic case: classification of germs of a generic object on a manifold (generic function, vector field, differential form, etc.). Usually the orbit of the germ of a generic object at a generic point is an open and everywhere dense set in the space of germs. Nongeneric points (for which this property is violated) are said to be singular. When classifying germs of a generic object at singular points, we study typical singularities. Roughly speaking typical singularities are irremovable under a small perturbation while nontypical singularities may be eliminated by a suitable small perturbation (they decompose into typical ones).

In this monograph we deal with typical singularities of differential 1-forms and Pfaffian equations. Pfaffian equations in modern terms are modules of differential 1-forms generated by one differential 1-form. Local equivalence of differential 1-forms corresponds to the action of the group of local diffeomorphisms (reversible coordinate transformations). The classification of Pfaffian equations may be considered to be the classification of differential 1-forms: in addition to a change of coordinates we can multiply a 1-form by a nonvanishing function.

The problem of classifying differential 1-forms and Pfaffian equations was formulated by Pfaff at the start of the 19th century (in terms of reduction to "simplest" forms). The first basic step in this problem was made by Darboux, whose theorem can be formulated as follows: *for a generic differential 1-form (generic Pfaffian equation) on a manifold M there exists an open everywhere dense subset $\widetilde{M} \subset M$ such that all germs of the 1-form (Pfaffian equation) at points of $\widetilde{M}$ are equivalent to a standard germ.*

Martinet was the first to study singularities (classification of germs at points of $\widetilde{M}$) systematically. His results are collected in [Mar] which is both the starting point and a guide for other studies including the present one. Unfortunately I knew Professor Martinet only by correspondence. My wish to meet him personally will never come true.

Singularities and the classification of 1-forms and Pfaffian equations are interesting not only as a classical problem but also (and perhaps mainly) because of their applications (in contact geometry, the theory of partial differential equations, control theory, nonholonomic dynamics, and variational problems). Some important applications have appeared in the last 10–15 years. Most applications in contact geometry are due to the relative Darboux theorem, which was proved by Givental in 1982. This theorem states that *two submanifolds of a contact manifold are contactly locally equivalent* (i.e., their germs lie in the same orbit of the group of diffeomorphisms that preserve the germ of the contact structure) *if and only if the Pfaffian equations obtained by the restriction of the contact structure to the submanifolds are locally equivalent* (*with respect to the action of the complete group of diffeomorphisms*).

Classification results for Pfaffian equations can be reformulated as those for submanifolds of a contact manifold. The classification of Pfaffian equations also leads to the classification of first-order partial differential equations since the latter may be considered as hypersurfaces in a contact manifold of 1-jets.

An application to control theory is associated with the fact that a Pfaffian equation generated by a 1-form w defines a module of vector fields v such that $w(v) = 0$. On the other hand, a differential 1-form w defines an affine module of vector fields v such that $w(v) = 1$. Such modules may be interpreted as control systems, linear with respect to the control.

In this monograph we collect results on the geometry of singularities and classification of differential forms and Pfaffian equations. We also present applications and closely related classification problems. All the results are given with proofs. In the proofs we use the technique of jets on a manifold, the homotopy method and its modifications, the transversality theorem, the necessary and sufficient conditions for a germ's stability and finite determinacy, the relative Darboux theorem and related results, and theorems by Belitskii and Roussarie on the solvability of equations with respect to germs of flat functions. In Chapter II we collect the relevant material and the basics of singularity theory. In Chapters III–V we discuss differential 1-forms, and odd-dimensional and even-dimensional Pfaffian equations. At the end of each chapter we summarize the main results, tabulate the singularities, and list the normal forms. The main results of the book are also collected in Chapter I. In Appendices A, B, and C we apply the results respectively to the classification of the first-order partial differential equations, to the study of the geometry of submanifolds of a contact manifold, and to some problems of control theory. Our main results hold relative to the C^∞-equivalence or C^r-equivalence for arbitrary $r < \infty$. Nevertheless, we also dwell (on the level of formulation, conjecture, and brief discussion) on analytic and topological classification problems (Appendices D and F). We present some classification results for distributions, differential systems (modules of vector fields), and closed differential 2-forms (Appendices E and G).

Some of the results in the book are due to Martinet (degenerations of codimension 1: first occurring singularities of 1-forms and odd-dimensional Pfaffian equations [Mar]), some to Lychagin (point singularities in the even-dimensional case associated with the vanishing of 1-forms; such singularities correspond to first occurring singularities of first-order partial differential equations [L1]), other results were obtained by the author (degenerations of codimension $\geqslant 2$, in particular, first occurring singularities of even-dimensional Pfaffian equations [Z3, Z4, Z8, Z9]). Some of the results are published for the first time (not taking into account the preprint [Z14]). The results in this monograph give complete answers to the principal questions of the problems of local classification.

I would like to express my profound gratitude to Professor V. Arnold, to my teacher Professor G. Belitskii, to Professors Yu. Il'yashenko and V. Lychagin, with whom I discussed both some of the concrete results and the book as a whole. I am very thankful to Professor D. Leites who published the preliminary text [Z14] in the transactions of his Seminar on Supermanifolds and called my attention to related classification questions in superanalysis in his introduction to the text.

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APPENDIX A

Local Classification of First-Order Partial Differential Equations

1. In this appendix we give some results of [L1] where first-order partial differential equations are considered as hypersurfaces in the contact space of 1-jets. These results can be obtained as corollaries of the classification of 1-forms and Pfaffian equations.

2. A germ of a first-order partial differential equation

$$F\left(x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0$$

is treated as a germ of the hypersurface E: $\{F(x_1, \dots, x_n, u, y_1, \dots, y_n) = 0\}$ in the contact space $(\mathbb{R}^{2n+1}, \omega)$, where $\omega = du - \sum y_i\, dx_i$. Two germs E_1 and E_2 are said to be equivalent if there exists a germ of a contactomorphism sending E_1 to E_2. By Theorem 7.1 they are equivalent if and only if the germs $\{\omega|_{E_1} = 0\} \in PW(2n)$ and $\{\omega|_{E_2} = 0\} \in PW(2n)$ are equivalent.

3. Assume that the plane of ω at $0 \in \mathbb{R}^{2n+1}$ is transverse to an equation $E \subset (\mathbb{R}^{2n+1}, \omega)$, i.e., $(dF \wedge \omega)|_0 \neq 0$. Then $\mathrm{cl}\{\omega|_E = 0\} = 2n - 1$, and the germ $\{\omega|_E = 0\}$ is equivalent to the germ $du - y_2\, dx_2 - y_3\, dx_3 - \cdots - y_n\, dx_n$ (in the coordinates $u, x_1, x_2, y_2, x_3, y_3, \dots, x_n, y_n$). Therefore, by Theorem 7.1 the germ E is reducible to the normal form

$$y_1 = 0 \quad \text{or} \quad \frac{\partial u}{\partial x_1} = 0. \tag{A.1}$$

4. Singularities in the problem occur at points where $dF \wedge \omega = 0$, or, equivalently, where the restriction $\widetilde{\omega} = \omega|_E$ vanishes. The germ $\{\widetilde{\omega} = 0\}$ at other points satisfies the Darboux condition $\widetilde{\omega} \wedge (d\widetilde{\omega})^{n-1} \neq 0$.

Let $\widetilde{\omega}|_0 = 0$. Then the germ of $\widetilde{\omega}$ at 0 can be an arbitrary germ of a 1-form belonging to W^0_{2n} and satisfying the condition $\operatorname{rank} d\widetilde{\omega}|_0 = 2n$. Thus, the classification of singularities of first-order partial differential equations is reduced to the classification of the germs $\{\omega = 0\} \in PW^0_{2n}$, such that $\operatorname{rank} d\omega|_0 = 2n$. The following theorem is a corollary of the results in §13, and §29.

Theorem A.1. *A generic germ at $0 \in \mathbb{R}^{2n+1}$ of a first-order partial differential equation $F(x_1, \dots, x_n, u, \partial u/\partial x_1, \dots, \partial u/\partial x_n) = 0$ satisfying the condition $dF|_0 \wedge du = 0$ is reducible to the form*

$$u - \sum_1^n \lambda_i x_i y_i = 0 \qquad \left(u = \sum_1^n \lambda_i x_i \frac{\partial u}{\partial x_i}\right) \tag{A.2}$$

(generally speaking, in complex coordinates, see § 13*). Here λ_i are moduli*([1])*, i.e., two different normal forms* (A.2) *are not equivalent.*

Notice that $\lambda_i = 1/2 + \mu_i$ (see Corollary 13.1).

Since $\operatorname{codim} PW_{2n}^0 = 2n$, we can formulate the following result.

Theorem A.2. *A germ of a generic first-order partial differential equation on a manifold M (considered as a hypersurface in the set of 1-jets on M) is either stable (at generic points) and equivalent to the normal form* (A.1) *or unstable but 2-determined (at isolated points) and equivalent to the normal form* (A.2).

The case of nongeneric germs from W_{2n}^0 and nontypical singularities of first-order partial differential equations is considered in [Z1, Z5, Z6].

APPENDIX B

Classification of Submanifolds of a Contact Manifold

Appendix A is concerned with the classification of hypersurfaces in a contact space. In this appendix we consider submanifolds of codimension ≥ 2. The notion of equivalence of submanifolds of a contact manifold is given in §7.

The following theorem shows that singularities from PW_{2p+1}^{2p-1} and PW_{2p}^{2p-3} are realized as restrictions of a contact structure on submanifolds in a contact space.

THEOREM B.1. *For an arbitrary germ* $e \in PW_{2p+1}^{2p-1}$ $(e \in PW_{2p}^{2p-3})$ *and an arbitrary contact* 1*-form* $\omega \in W(2n+1)$, $n > p$, *there exists a manifold* $E \subset \mathbb{R}^{2n+1}$ *such that the germ of the restriction* $\{\omega = 0\}|_E$ *is equivalent to the germ* e.

PROOF. We can assume that $\omega = dz + x_1\,dy_1 + \cdots + x_n\,dy_n$. By Theorem 16.3 the germ $e \in PW_{2p+1}^{2p-1}$ is equivalent to the germ

$$e_1\colon \{dz + x_1\,dy_1 + \cdots + x_{p-1}\,dy_{p-1} + f\,dy_p = 0\}, \qquad j^1 f = 0$$

(in the coordinates $z, x_1, y_1, \ldots, x_p, y_p$). By Theorem 24.1 the germ $e \in PW_{2p}^{2p-3}$ is equivalent to the germ

$$e_2\colon \{dz + x_1\,dy_1 + \cdots + x_{p-2}\,dy_{p-2} + g_1\,dy_{p-1} + g_2\,dy_p = 0\}, \qquad j^1 g_i = 0$$

(coordinates $z, x_1, y_1, \ldots, x_{p-1}, y_{p-1}, y_p$).

Let E_1 be a manifold given by the equations

$$\{x_{p+2} = y_{p+2} = \cdots = x_n = y_n = 0,\ x_{p+1} = f - x_p,\ y_{p+1} = y_p\}$$

and let E_2 be a manifold given by

$$\{x_{p+2} = y_{p+2} = \cdots = x_n = y_n = 0,\ x_{p+1} = g_1 - x_{p-1},\ x_p = g_2,\ y_{p+1} = y_{p-1}\}.$$

Then $\{\omega|_{E_1} = 0\} = e_1$, $\{\omega|_{E_2} = 0\} = e_2$. Q.E.D.

The following statements are corollaries of Theorem B.1, Theorem 7.1, and classification results in Chapters IV and V.

THEOREM B.2. *Let $l \geq 2$, $n > l$. The germ of a generic $(2l+1)$-dimensional submanifold E of the contact manifold $(\mathbb{R}^{2n+1}, dz + x_1\,dy_1 + \cdots + x_n\,dy_n)$ at a generic point $\alpha \in E$ is stable and equivalent to the germ of one of the following manifolds:*

(1) $\{x_{l+1} = y_{l+1} = \cdots = x_n = y_n = 0\}$;

(2) $\{x_{l+2} = y_{l+2} = \cdots = x_n = y_n = 0,\ x_{l+1} = x_l^2 - x_l,\ y_{l+1} = y_l\}$;

(3) $\{x_{l+2} = y_{l+2} = \cdots = x_n = y_n = 0,\ x_{l+1} = x_l y_1 + x_l^2 y_l - x_l,\ y_{l+1} = y_l\}$;

(4) $\{x_{l+2} = y_{l+2} = \cdots = x_n = y_n = 0,\ x_{l+1} = x_l y_1 + x_l^3/3 + x_l y_l^2 - x_l,\ y_{l+1} = y_l\}$.

Nongeneric points form a set of codimension ≥ 4 in E.

THEOREM B.3. *Let $l \geq 3$, $n > l$. The germ of a generic $2p$-dimensional submanifold of the contact manifold $(\mathbb{R}^{2n+1}, dz + x_1\,dy_1 + \cdots + x_n\,dy_n)$ at a generic point of this submanifold is stable (relative to C^r-equivalence for an arbitrary $r < \infty$) and equivalent to the germ of one of the following manifolds:*

(1) $\{x_p = x_{p+1} = y_{p+1} = \cdots = x_n = y_n = 0\}$;

(2) $\{x_{p+2} = y_{p+2} = \cdots = x_n = y_n = 0,$
$x_{p+1} = x_{p-1}y_p - x_{p-1},\ y_{p+1} = y_{p-1},$
$x_p = x_{p-1}y_{p-1}(1 + \lambda + x_1)\}$ $(\lambda \in (0, 1))$;

(3) $\{x_{p+2} = y_{p+2} = \cdots = x_n = y_n = 0,$
$x_{p+1} = x_{p-1}y_p + x_{p-1}y_{p-1}(2\lambda + x_1) - x_{p-1},$
$x_p = x_{p-1}y_p(2\lambda + x_1) - x_{p-1}y_{p-1},\ y_{p+1} = y_{p-1}\}$ $(\lambda \in (0, \infty))$.

Nongeneric points form a set of codimension ≥ 4.

We can formulate other corollaries, for example, the normal forms of 3- and 4-dimensional submanifolds of a contact manifold can be distinguished.

APPENDIX C

Feedback Equivalence of Control Systems

Consider a system of ordinary differential equations

$$\dot{x} = F(x) + \sum_{1}^{k} u_i(x) G_i(x), \tag{C.1}$$

where x, $F(x)$, $G_i(x) \in \mathbb{R}^n$, $u_1, \dots, u_k$ are scalar functions (controls) the choice of which must ensure some properties of system (C.1). When studying qualitative local properties of a control system it is expedient to replace it by an equivalent simpler system. The introduction of new controls $v_1, \dots, v_k$, i.e.,

$$u_i = \beta_i(x) + \sum_{j=1}^{k} h_{ij}(x) v_i, \quad i = 1, \dots, k, \qquad \det(h_{ij}) \neq 0, \tag{C.2}$$

and new coordinates

$$x = \Phi(y), \qquad \Phi'(0) \neq 0, \tag{C.3}$$

leads to another system

$$\dot{y} = \widetilde{F}(y) + \sum_{1}^{k} v_i(y) \widetilde{G}_i(y) \tag{C.4}$$

that is also linear with respect to controls.

Control systems (C.1) and (C.4) are said to be *feedback equivalent* if there exist transformations of the form (C.2), (C.3) sending (C.1) to (C.4).

This definition can be simplified by passing to invariant terms. A control system of form (C.1) defines an affine module of vector fields

$$V = \left\{ f + \sum_{1}^{k} u_i g_i, \; u_i \in C^\infty(n) \right\}, \tag{C.5}$$

where the field f corresponds to the system $\dot{x} = F(x)$ and the fields g_i correspond to the systems $\dot{x} = G_i(x)$, $i = 1, \dots, k$.

THEOREM C.1 [J]. *Two control systems of the form* (C.1) *are locally feedback equivalent if and only if the corresponding affine modules of vector fields* *are equivalent, i.e., there exists a diffeomorphism sending each vector field of the first module into a vector field of the second one.*

Using this theorem we can apply classification results for differential 1-forms to control systems of the form (C.1) for the case $k = n - 1$. Assume that $k = n - 1$ and a system of form (C.1) is nondegenerate, i.e.,

$$\dim(F(x), G_1(x), \dots, G_{n-1}(x)) = n, \quad x \in \mathbb{R}^n.$$

An affine module (C.5) corresponding to such system can be defined as

$$V = \{v \in \mathrm{Vect}(n) | \omega(v) \equiv 1\},$$

where ω is a differential 1-form such that $\omega(f) \equiv 1$, $\omega(g_i) \equiv 0$, $i = 1, \dots, n-1$. There exists a unique 1-form ω satisfying these properties. Now, using Theorem (C.1) and results of Chapter III we obtain the following result.

THEOREM C.2. *Let $k = n - 1$. A germ of a nondegenerate control system* (C.1) *at a generic point $x \in \mathbb{R}^n$ is stable and feedback equivalent to one of the germs*

$$\{\dot{x}_i = u_i, \dot{y}_i = \widetilde{u}_i (i = 1, \dots, k), \ \dot{z} = 1 - x_1\widetilde{u}_1 - \cdots - x_k\widetilde{u}_k\} \qquad (n = 2k+1);$$

$$\{\dot{x}_j = u_j, \dot{y}_i = (1 + x_1)\widetilde{u}_i \ (j = 1, \dots, k, i = 2, \dots, k), \ \dot{z} = (1 + x_1)u, \ \dot{y}_1 = 1/(1 + x_1) - x_2\widetilde{u}_2 - \cdots - x_k\widetilde{u}_k \pm zu\} \qquad (n = 2k+1);$$

$$\{\dot{x}_j = u_j, \dot{y}_i = (1 + x_1)\widetilde{u}_i \ (j = 1, \dots, k, i = 2, \dots, k), \ \dot{y}_1 = 1/(1 + x_1) - x_2\widetilde{u}_2 - \cdots - x_k\widetilde{u}_k\} \qquad (n = 2k);$$

$$\{\dot{x}_i = u_i \ (i = 1, \dots, k), \dot{y}_j = (1 \pm x_1^2)\widetilde{u}_j \ (j = 2, \dots k), \ \dot{y}_1 = 1/(1 \pm x_1^2) - x_2\widetilde{u}_2 - \cdots - x_k\widetilde{u}_k\} \qquad (n = 2k);$$

where u, u_i, $\widetilde{u}_i$ are functional control parameters (functions in n variables).

Nongeneric points form a set of codimension 2. Each stable (with respect to the feedback equivalence) germ of a nondegenerate control system is feedback equivalent to one of the germs given above.

APPENDIX D

Analytic Classification of Differential Forms and Pfaffian Equations

1. In this appendix we consider germs of real-analytic and holomorphic 1-forms and Pfaffian equations, i.e., germs of the form

$$\omega = \sum a_i(x)\, dx_i, \qquad e: \left\{ \sum a_i(x) dx_i = 0 \right\},$$

where the $a_i(x)$ are germs of functions in n variables $x = (x_1, \dots, x_n)$ that are real-analytic at 0 (of functions in n complex variables, holomorphic at 0).

Two analytic (holomorphic) germs of 1-forms are said to be equivalent if there exists a germ at 0 of an analytic (holomorphic) diffeomorphism, sending one germ into another. Two analytic (holomorphic) germs of Pfaffian equations $\{\omega_1 = 0\}$, $\{\omega_2 = 0\}$ are said to be equivalent if there exists a germ H at 0 of an analytic (holomorphic) function such that the forms $H\omega_1$ and ω_2 are equivalent.

For the analytic case all the singularity classes considered in Chapters III–V are defined. For the holomorphic case the classes $PW_{2k+1}^{2k-1,1,h}$ and $PW_{2k+1}^{2k-1,1,e}$ coincide.

2. The Darboux and Martinet analytic classifications coincide with the smooth classification. We mean that Theorems 12.1, 12.2, 17.2, and Darboux theorems (§§10, 16, 24) hold also in the analytic category. The holomorphic classification is the same, but the normal form ω^+ is equivalent to the normal form ω^- (see Theorems 12.1 and 12.2).

The results on the preliminary normal forms (Theorems 10.2, 16.3 and 24.1) hold for analytic and holomorphic cases as well.

Proofs of all these statements are exactly the same as for the smooth case.

3. Analytic and holomorphic classifications of singularities from the classes $PW_{2k+1}^{2k-1,1}$ and PW_{2k}^{2k-3} differ essentially from the smooth classification. First let us consider the singularity class $PW_3^{1,1}$.

THEOREM D.1. *For an arbitrary holomorphic germ X of a vector field on a plane with 3-jet of the form*

$$(ix + x^2 y)\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y}, \tag{D.1}$$

there exists a germ of a holomorphic Pfaffian equation $e \in PW_3^{1,1}$ such that $X_e = X$ (to within orbital equivalence).

Theorem D.1 shows that the holomorphic classification of singularities from $PW_3^{1,1}$ includes the holomorphic orbital classification of germs of resonance vector fields on a plane with a 3-jet of form (D.1). Such classification was obtained in [MarRa], [EI]. It follows from these works that the orbits of the holomorphic classification are parametrized not only by a real number (the only invariant of the formal and smooth classifications), but also by functional moduli. We do not know if these functional moduli, together with the mentioned invariant of the smooth classification, give a complete system of holomorphic invariants of germs of Pfaffian equations belonging to $PW_3^{1,1}$; supposedly, this is true.

We have obtained the following result.

THEOREM D.2. *None of the germs from the singularity class $PW_3^{1,1}$ is finitely determined (in the holomorphic category).*

This statement holds in the real-analytic category as well.

4. Let us consider the singularity class $PW_{2k+1}^{2k-1,1}$, $k \geq 2$. For the smooth classification, there exist stable normal forms (18.1), (18.2). We announce the following result.

THEOREM D.3. *None of the germs from the singularity class $PW_{2k+1}^{2k-1,1}$ is finitely determined (in the analytic or holomorphic category).*

The scheme of the proof is as follows.

(1) If a germ $e \in PW_{2k+1}^{2k-1,1}$ is finitely determined in the analytic (holomorphic) category, then by Theorem 18.1 the germ (18.1) or the germ (18.2) is 3-determined in the analytic (holomorphic) category.

(2) Let $\{\widehat{\omega} = 0\}$ be the normal form (18.1) or (18.2). If the germ $\{\widehat{\omega} = 0\}$ is 3-determined in the analytic (holomorphic) category, then the infinitesimal equation (6.2) with $\omega = \widehat{\omega}$ has an analytic (holomorphic) solution (h, v) for each right-hand side with vanishing 3-jet.

Arguments proving this statement are close to those used in [Il].

(3) The infinitesimal equation (6.2) with $\omega = \widehat{\omega}$ is unsolvable in the analytic (holomorphic) category.

To prove statement (3) we reduce the infinitesimal equation to an equation of the form $Xu + bu = \tau$, where X is a vector field, b and τ are function-germs, and u is the unknown function-germ. Such a reduction was realized in §18

It is noteworthy that for a generic germ $e \in PW_{2k+1}^{2k-1,1}$ the corresponding vector field X_e in suitable coordinates looks like

$$x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + x_1x_2\frac{\partial}{\partial x_3} + \cdots \quad \text{or} \quad x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} + (x_1^2+x_2^2)\frac{\partial}{\partial x_3} + \cdots, \tag{D.2}$$

where $\cdots$ denotes terms of the form $x_1^\alpha z^\beta f(x_1, \ldots, x_{2k})$, $\alpha+\beta \geq 3$. These terms can be "killed" in the smooth category; but for the analytic or holomorphic classification of germs (D.2), one can point out functional moduli close to those obtained in [MarRa], [EI], [She] (S. M. Voronin, private communication). This can be done by reduction to the problem that was solved in [Vo].

5. Now consider the case of 1-forms ω, $\omega|_0 = 0$. Using the notation of §13 we state the following result.

THEOREM D.4. *Suppose that for some constants $C, \nu > 0$ the invariants $\lambda_i = \frac{1}{2} + \mu_i$, $i = 1, \ldots, k$, and $\lambda_i = \frac{1}{2} - \mu_i$, $i = k+1, \ldots, 2k$, of the 1-jet of an analytic (holomorphic) 1-form $\omega \in W_{2k}^0$ satisfy the estimate*

$$(\alpha_1\lambda_1 + \cdots + \alpha_{2k}\lambda_{2k} - 1) \geq C|\alpha|^{-\nu} \tag{D.3}$$

for all integers $\alpha_i \geq 0$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_{2k} \geq 3$. Then ω is equivalent to the normal form (13.1) (for the holomorphic case ω is equivalent to the normal form (13.3)).

This result was announced in [Z4], where the case $\omega \in W_{2k+1}^0$ was considered as well. A similar result in terms of 1-order partial differential equations was proved by Webster [W]: if the invariants $\{\lambda_1, \ldots, \lambda_{2k}\} = \{1/2 \pm \mu_i, i = 1, \ldots, k\}$ of a 2-jet of a holomorphic partial differential equation

$$F(x_1, \ldots, x_k, u, \partial u/\partial x_1, \ldots, \partial u/\partial x_k) = 0$$

(see Appendix A) satisfy the estimates (D.3) for some $C, \nu > 0$, then the equation is reducible to the form $u = \lambda_1 x_1\, \partial u/\partial x_1 + \cdots + \lambda_k x_k\, \partial u/\partial x_k$.

APPENDIX E

Distributions and Differential Systems

Let $w \in \Lambda^1(M)$ be a 1-form which does not vanish at any point of M. Then a germ of a Pfaffian equation $\{\omega = 0\}$ can be treated as a germ of an $(n-1)$-distribution in $\mathbb{R}^n$ $(n = \dim M)$ [VG1]. An $(n-1)$-distribution in $\mathbb{R}^n$ can be given by $(n-1)$ vector fields $X_1, \dots, X_{n-1}$ in $\mathbb{R}^n$ such that $\omega(X_i) = 0$ and $\operatorname{rank}(X_1|_\alpha, \dots, X_{n-1}|_\alpha) = n-1$ for each point α. For example, one can treat the contact structure $(\mathbb{R}^{2k+1}, dz + x_1\,dy_1 + \cdots + x_k\,dy_k)$ as a modulus of vector fields over a ring of smooth functions; this modulus is generated by the $2k$ vector fields

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_1} - x_1\frac{\partial}{\partial z}, \dots, \frac{\partial}{\partial y_k} - x_k\frac{\partial}{\partial z}.$$

Similarly, the quasicontact structure $(\mathbb{R}^{2k}, dy_1 + x_2\,dy_2 + \cdots + x_k\,dy_k)$ is the distribution generated by

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_2} - x_2\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_k} - x_k\frac{\partial}{\partial y_1},$$

and the normal form (18.1) can be treated as a distribution generated by

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_2} - x_2\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_k} - x_k\frac{\partial}{\partial y_1}, \frac{\partial}{\partial z} - (x_1y_2 + x_1^2 z)\frac{\partial}{\partial y_1}.$$

For $n = 3$ and $n = 4$ the Darboux condition $(\operatorname{cl}\{\omega = 0\} = \text{maximum} = 3)$ can be formulated in terms of vector fields. Let $X_1, \dots, X_{n-1}$ be an arbitrary system of vector fields such that $\omega(X_i) = 0$, $i = 1, \dots, n-1$, and $\operatorname{rank}(X_1|_\alpha, \dots, X_{n-1}|_\alpha) = n-1$. Let us denote by $V_{\{\omega=0\}} = V$ the modulus consisting of germs of vector fields of the form $\sum \beta_i X_i$ where β_i are arbitrary functions. Then $V = \{V|_\alpha\}$ where $V_\alpha \subset T_\alpha\mathbb{R}^n$ is a hyperplane. We use $[V, V]$ to denote the modulus generated by vector fields $[v_1, v_2]$, $v_1, v_2 \in V$. For $n = 3$ and $n = 4$ the Darboux condition can be formulated as

$$\dim[V, V]|_0 = n.$$

In terms of vector fields we can formulate other conditions for a germ $\{\omega = 0\}$ to belong to various singularity classes:

$$\{\omega = 0\} \in PW_3^{1,0} \quad \text{if } S(\omega) \text{ is transversal to } V_{\{\omega=0\}} \text{ at } 0,$$

$$\{\omega = 0\} \in PW^1_{4,\mathbb{R}e} \cup PW^1_{4,\mathrm{Im}} \quad \text{if } D(\omega) \text{ is transversal to } V_{\{\omega=0\}} \text{ at } 0.$$

Here $S(\omega)$ and $D(\omega)$ are the first degeneration manifolds.([1])

At the same time, for $n \geq 5$ even the Darboux condition cannot be formulated in terms of the growth vector of a distribution [VG1].

It is also interesting to consider differential systems $V = (X_1, \dots, X_{n-1})$ which differ from distributions only by the possibility of linear dependence of $X_1|_\alpha, \dots, X_{n-1}|_\alpha$ at some points $\alpha \in M$. Various questions connected with singularities of differential systems were studied in [JP2, Mor1–Mor3, MorRo]. A 2-tuple of vector fields in $\mathbb{R}^3$ and 3-tuple of vector fields in $\mathbb{R}^4$ were studied in detail. Here we formulate two classification results from the mentioned works. Let S_n be the set of germs at $0 \in \mathbb{R}^n$ of differential systems $V = (X_1, \dots, X_{n-1})$ such that $\operatorname{rank}(X_1|_0, \dots, X_{n-1}|_0) = n-2$.

THEOREM E.1 [JP2]. *A generic germ* $V \in S_3$ *is reducible to the normal form* $(\partial/\partial x, x\,\partial/\partial z + z\,\partial/\partial y)$.

THEOREM E.2 [MorRo]. *A generic germ* $V \in S_4$ *is reducible to the normal form* $(\partial/\partial x, \partial/\partial y, x\,\partial/\partial z + y\,\partial/\partial u)$.

The problem of the classification of germs of k-distributions in $\mathbb{R}^n$ is not "wild" only for the following three cases: $k = 1$ (a direction field), $k = n-1$ (Darboux's case), and $k = 2$, $n = 4$. The normal form of generic germs of 2-distributions in $\mathbb{R}^4$ was obtained by Engel in 1889 [En] (see also [VG2]). The work [Z10] is devoted to the classification of singularities of 2-distributions in $\mathbb{R}^4$. If $2 \leq k \leq n-2$ and $(k, n) \neq (2, 4)$, then the orbit of a generic germ of a k-distribution in $\mathbb{R}^n$ has an infinite codimension.([2]) Preliminary normal forms with functional moduli can be found in [Z11]. The classification of germs of regular distributions with a fixed (but not generic) growth vector [VG1] can be found in [KR], [Z12]. For the growth vector $(k, k+1, k+2, \dots, n)$, $n - k \leq 3$, a germ is reducible to a stable normal form. If the growth vector has another form, then the classification problem remains "wild".

Singularities of differential systems $(X_1, \dots, X_k)$ in $\mathbb{R}^n$ were considered in [JP2].

([1])i.e. the manifolds $\{\alpha \in M | \dim [V_{\{\omega=0\}}, V_{\{\omega=0\}}]|_\alpha = n-1\}$.

([2])This fact was proved in [JP2], then also in [VG2]; moduli appear not only in formal series but also in the C^1-classification, see [Va].

APPENDIX F

Topological Classification of Distributions

Consider a k-distribution E in $\mathbb{R}^n$ (it can be given either by $(n-k)$ differential forms $w_1, \ldots, w_{n-k}$ or by a k-tuple of vector fields $V = (X_1, \ldots, X_k)$). A differentiable curve $\gamma = \gamma(t)$ such that $\dot{\gamma}(t) \in E$ (i.e., $\dot{\gamma}(t) \in \operatorname{Ker} \omega_i|_{\gamma(t)} \Leftrightarrow \dot{\gamma}(t) \in \operatorname{span} V|_{\gamma(t)}$) is said to be a *trajectory of* E. Two germs E_1, E_2 of k-distributions are said to be *topologically equivalent* if a germ of a homeomorphism maps the germs of the trajectories of E_1 into the germs of the trajectories of E_2 [JP2].

In [JP1,JP2] the classes of the topological equivalence were discussed. We announce the result for 2-distributions in $\mathbb{R}^3$.

THEOREM F.1. *The germ at each point of a generic 2-distribution on a 3-manifold is topologically stable and topologically equivalent to one and only one of the germs*

(1) $\{dz + x\,dy = 0\} \Leftrightarrow \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} - x\dfrac{\partial}{\partial z}\right)$;

(2) $\{dy + x^2\,dz = 0\} \Leftrightarrow \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial z} - x^2\dfrac{\partial}{\partial y}\right)$;

(3) $\{dy + (xy + x^2 z)\,dz = 0\} \Leftrightarrow \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial z} - (xy + x^2 z)\dfrac{\partial}{\partial y}\right)$;

(4) $\left\{dy + \left(xy + \dfrac{x^3}{3} + xz^2\right) dz = 0\right\} \Leftrightarrow \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial z} - \left(xy + \dfrac{x^3}{3} + xz^2\right)\dfrac{\partial}{\partial y}\right)$.

In order to prove Theorem F.1 it suffices to prove the topological equivalence of germ (19.1) to germ (3) and of germ (19.2) to germ (4) (for an arbitrary $b \in \mathbb{R}$).

As for the global classification of 2-distributions on a 3-manifold, it was shown in [Ben] that there exists a contact structure $(\mathbb{R}^3, w)$ which is not equivalent to the standard contact structure $(\mathbb{R}^3, dz + x\,dy)$.

APPENDIX G

Degenerations of Closed 2-Forms in $\mathbb{R}^{2k}$

In this appendix we give the list of normal forms of closed 2-forms in $\mathbb{R}^{2k}$, $k \geq 2$. The normal form of first occurring singularities

$$x_1\, dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_k \wedge dy_k \tag{G.1}$$

was obtained by Martinet [Mar] who also announced the normal forms for the next degeneration (of codimension 3):

$$d\left(x_1 - \frac{x_2^2}{2}\right) \wedge dy_1 + d\left(x_1x_2 \pm y_1y_2 - \frac{x_2^3}{3}\right) \wedge dy_2 + \sum_{i=3}^{k} dx_i \wedge dy_i. \tag{G.2}$$

The theorem on reduction to these normal forms was proved by Roussarie [Ro]. The germs, corresponding to the next degeneration (of codimension 4) are unstable [GT2] and, supposedly, they are not finitely determined.

At generic points of a $2k$-dimensional manifold a germ of a generic closed 2-form is stable and equivalent either to the standard Darboux model $dx_1 \wedge dy_1 + \cdots + x_k\, dy_k$, or to the germ (G.1), or to the germ (G.2); nongeneric points form a set of codimension ≥ 4.

The description of degenerations can be found in [AG, Mar, Ro, GT2]. The Darboux and Martinet normal forms hold in the analytic (holomorphic) category as well. At the same time, analytic germs that are C^∞-equivalent to the germ (G.2) are not (generally speaking) analytically equivalent to (G.2) (arguments are similar to those used in Appendix D).

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Subject Index

List of Symbols

$\Lambda^1(M)$	the space of external differential 1-forms on a manifold M
$P\Lambda^1(M)$	the set of Pfaffian equations on a manifold M
$W(n)$	the set of germs at $0 \in \mathbb{R}^n$ of 1-forms in $\mathbb{R}^n$
$PW(n)$	the set of germs at $0 \in \mathbb{R}^n$ of Pfaffian equations in $\mathbb{R}^n$
$C^\infty(n)$	the space of function-germs at $0 \in \mathbb{R}^n$
$\mathrm{Vect}\,(n)$	the space of germs of vector fields at $0 \in \mathbb{R}^n$
$\mathrm{Diff}\,(n)$	the group of germs at $0 \in \mathbb{R}^n$ of diffeomorphisms $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$
π_α	translation of a germ's source point α into $0 \in \mathbb{R}^n$
$\mathfrak{M}_S$	the set of germs of functions vanishing at each point of a manifold S or the set of germs of vector fields (differential 1-forms) of the form $$v = \sum f_i v_i, \qquad v_i \in \mathrm{Vect}\,(n),\ f_i \in \mathfrak{M}_S$$ $$\left(\omega = \sum f_i \omega_i, \qquad \omega_i \in W(n),\ f_i \in \mathfrak{M}_S\right)$$
$\mathfrak{M}_S^k$	the kth power of $\mathfrak{M}_S$ or the set of germs of vector fields (differential 1-forms) of the form $$v = \sum f_i v_i, \qquad v_i \in \mathrm{Vect}\,(n),\ f_i \in \mathfrak{M}_S^k$$ $$\left(\omega = \sum f_i \omega_i, \qquad \omega_i \in W(n),\ f_i \in \mathfrak{M}_S^k\right)$$
J_S^k	the set of k-jets on a manifold S
J_S^k	$= C^\infty(n)/\mathfrak{M}_S^{k-1}$, $J_S^k = \mathrm{Vect}\,(n)/\mathfrak{M}_S^{k-1}$, $J_S^k = W(n)/\mathfrak{M}_S^{k-1}$
j_S^k	a map from $C^\infty(n)$, $\mathrm{Vect}\,(n)$, $W(n)$ into J_S^k
$\mathfrak{M}^k, J^k, j^k$	$= \mathfrak{M}^k_{\{0\}}, J^k_{\{0\}}, j^k_{\{0\}}$ (resp.)

$S(w)$ $(S(E))$	the set of points at which the germ of a differential form w (Pfaffian equation E) belongs to the singularity class S
$S(\omega)$ $(S(e))$	the germ of $S(w)$ $(S(E))$ at $0 \in \mathbb{R}^n$, where w is a 1-form (E is a Pfaffian equation) with the germ $\omega(e)$ at $0 \in \mathbb{R}^n$
cl	the class of a 1-form or Pfaffian equation
W_n^j (PW_n^j)	the singularity class consisting of a germs of 1-forms (Pfaffian equations) of the class j
$W_n^{n-1,0}$, $PW_{2k+1}^{2k,1,1}$, *etc.*	various singularity classes of 1-forms and Pfaffian equations
X_ω, X_e	the field of directions invariantly connected with singularities $\omega \in W(n)$, $e \in PW(n)$
$\lrcorner$	if W is a k-form and X is a vector field, then $X \lrcorner W$ is a $(k-1)$-form $\widetilde{W}$ such that $\widetilde{W}(Y_1, \dots, Y_{k-1}) = W(X, Y_1, \dots, Y_{k-1})$ for arbitrary $Y_1, \dots, Y_{k-1}$; if $W \in \Lambda^1(M)$, then $X \lrcorner W = W(X)$ is a function

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