Translations of
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Volume 114

Diffusion Equations

Seizô Itô

American Mathematical Society

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Volume 114

Diffusion Equations

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American Mathematical Society
Providence, Rhode Island

拡散方程式

KAKUSAN HŌTEISHIKI (Diffusion Equations)
by Seizô Itô

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ABSTRACT. In this book, the author presents diffusion equations with variable coefficients associated with boundary conditions and the corresponding elliptic boundary value problems. The fundamental solution of the initial-boundary value problem and Green function of the elliptic boundary value problem are constructed, and the formulae that express solutions of those problems by using the fundamental solution or Green function are presented. Several important properties of the solutions are also discussed.

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Preface to the English Edition

The present book is the English translation of my book originally published in Japanese by Kinokuniya Company Ltd. in its series "Kinokuniya Sūgaku Sōsho" (Kinokuniya Mathematical Series). As mentioned in the Preface to the Japanese edition, the aim of this book is to present a self-contained exposition, using as little knowledge of functional analysis as possible, of existence theorems and results on initial-boundary value problems for parabolic equations and elliptic boundary value problems described by second order elliptic partial differential operators with variable coefficients. The translation is faithful to the original. However, in the Supplementary Remarks and References at the end of this book, some of the books written in Japanese are replaced by those written in English.

More than ten years have passed since the original (Japanese) edition was published. Ordinarily, the translator (in this case the author of the original book) might add some remarks of criticism to such a classical treatment as appears in this book, but it is not always easy for an author to criticize his own work. There is an old Chinese proverb "Mountain dwellers cannot recognize the shape of the mountain." The author/translator would appreciate any helpful criticism by the readers regarding the contents of this book.

The translator wishes to express his appreciation to Professor Katsumi Nomizu of Brown University for his kind help in the publication of this translation and to the American Mathematical Society for their efficient handling of the publication. Finally it should be mentioned that the translator was partially supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan.

Seizô Itô
Tokyo
March 1992

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Preface to the Japanese Edition

The theory of partial differential equations is one of the fields of mathematics that have developed most successfully in the recent quarter century. This development is based on the fact that the theory has been constructed by using functional analysis and the theory of distributions. Thus, if one intends to write a book on the theory of partial differential equations or on a branch of this theory from a modern point of view; then one should expect the reader to have a strong background knowledge of functional analysis and distributions, or the writer should devote several pages to a summary of the necessary prerequisites. On the other hand, some conditions to be set and some results to be expected in classical problems cannot always be formulated in the framework of the theory established from a modern point of view (e.g. in the framework of the function space to which the theory is applicable).

In this book, the author takes diffusion equations as the main theme and derives the existence theorems and results on initial- and boundary-value problems for parabolic equations and elliptic boundary-value problems that are described by using second order elliptic partial differential operators with variable coefficients. This allows for a self-contained exposition by using as little knowledge of functional analysis as possible. Since both the differential operators and the boundary conditions treated in this book contain variable coefficients, it is necessary to carry out some complicated computations (even to prove some results corresponding to well-known facts in the classical case of differential operators with constant coefficients); in particular, the computation to construct the fundamental solution satisfying the boundary condition (§6) appears as a very tedious task to readers. However, even if one derived the similar result by means of a 'modern theory', one would have to carry out concrete computations in most cases to derive local properties such as boundary conditions in classical form (not in the abstract form described by using the terminology in functional analysis). Once we *toil* at the first step of the construction of the fundamental solution, we can set forward the arguments thereafter by means of a 'physically natural' method from the viewpoint of diffusion. Examples of such situations will be mentioned in the next paragraph.

The physical meaning of properties of the fundamental solution is mentioned in the Introduction. The dependence of solutions on the domain where the equation is considered (§11) means that, the nearer the absorbing barrier is located, the faster the density of the diffusing substance is diminished. The fundamental solutions in unbounded domains are constructed using solutions in bounded domains that are gradually enlarged. The existence of a fundamental solution in an unbounded domain as the limit function of fundamental solutions in bounded domains and the limit process of the semigroup property of fundamental solutions can readily be proved since the fundamental solution grows as the domain is enlarged. From the viewpoint that the elliptic equation describes the equilibrium state of a diffusion phenomenon, the Green function for the elliptic boundary value problem and the formula for the solution of the problem are derived from the limit of solutions of the diffusion equation as time t tends to infinity. In most books on partial differential equations, several expositions and results on equations of elliptic type are given before treating equations of parabolic type. This book follows the opposite direction by discussing diffusion equations as the main theme for the 'physical reason' mentioned above. (In this paragraph and also in the last paragraph, 'physical' means 'in the sense of physical phenomena' and not 'in the sense of physics'.)

While this book is written with diffusion phenomena as a main theme, the contents are purely mathematical. Thus, not only the main results but also the preparatory propositions are given with proof. (This is a matter-of-course in a mathematical book; the author mentions it for emphasis.) Though the author intends to make as little use of functional analysis as possible, in order to clarify the mathematical argument, he partly applied some basic facts from the theory of the Lebesgue integral and some very elementary parts of the theory of function spaces and of integral equations (such as the definition of Hilbert space, orthogonality, Fourier series, and basic properties of integral equations with symmetric kernel). For these items the reader may, if necessary, refer to the books mentioned in the Supplementary Notes and References at the end of this book. Except for those elementary articles, proofs of all propositions are given. So it is not necessary for the reader to refer to more advanced books to understand proofs of important theorems in this book.

If one observes that, in this book, most of the main classical results on partial differential equations are generalized to the case of equations with variable coefficients by using elementary techniques; then one will have gained much information for applications in several directions.

The author would like to express his appreciation to both Mr. Ken-ichi Uzuoka and Mr. Hiroshi Mizuno of the Publication Division of Kinokuniya for their kind help in the preparation of this book.

Seizô Itô
In the height of summer 1979

Supplementary Notes and References

The following is a list of books and papers related to this book; it is not to be considered complete.

The prerequisite for reading this book is some basic knowledge of differential and integral calculus. For background information on function spaces used in §§14 and 15, it is sufficient for readers to refer to any one of the books [1–4]. For the theorems on integral equations quoted in §15, see [4] or [5].

1. S. Itô, *Introduction to Lebesgue integral*, Shokabo, 1963. (Japanese)
2. H. L. Royden, *Real analysis*, 2nd ed., Macmillan, New York, 1963.
3. W. Rudin, *Real and complex analysis*, 2nd ed., McGraw-Hill, New York, 1974.
4. A. N. Kolmogorov and S. V. Fomin, *Reelle Funktionen und Funktionalanalysis*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1975 (original in Russian, 1972).
5. K. Yosida, *Lectures on differential and integral equations*, Interscience, New York, 1960 (original in Japanese, 1950).

A brief exposition of the Bessel functions and Legendre polynomials that appear in the examples of eigenfunction expansions in §16 can be found in Chapter 1 of [5]; the book [6] is a handy and scrupulous primer to Bessel functions.

6. F. Bowman, *Introduction to Bessel functions*, Dover, New York, 1958 (Longmans, Green & Co., 1938).

In this book, in order to make the argument clear, we occasionally quote basic theorems in the theory of Lebesgue integrals . The reader may glance over such parts without caring about the conditions stated in integration theory and still understand the subsequent part, and may refer to any one of [1–4] as necessary.

Among the contents of this book, the "physical background of diffusion equations" in §0 was written by consulting Chapter 1 of

7. J. Crank, *The mathematics of diffusion*, 2nd ed., Oxford Univ. Press, London and New York, 1975.

However, in §1 and thereafter, all of the statements are set in the pure-mathematical theory. If one is interested in solutions of diffusion equations from the viewpoint of applied mathematics, see [7].

Most of this book is constructed by rearranging and elaborating the contents of the papers [20–23], and is illustrated mainly by classical methods—(partial) differential calculus, estimates of partial derivatives, and integration by parts, *etc.*—by using almost no knowledge of modern functional analysis. The author previously wrote the book

8. S. Itô, *Partial differential equations*, Baifukan, 1966. (Japanese)

In Chapters 2 and 3 of [8], the author treated the same results as those of the present work for the diffusion equation of the form $\partial u/\partial t = \Delta u - q(x)u$ in bounded domains, and, assuming the existence of a fundamental solution, mentioned the properties of solutions of the diffusion equation and related results. In the present work, the form of the partial differential operator (diffusion operator) and also the form of the space-domain are more generalized than those in [8], the proof of the existence of a fundamental solution is mentioned in detail, and several results (including those of [8]) are shown in greater depth.

The theory of the general equation of evolution, including the case of diffusion equations, is discussed by the method of functional analysis in many books, *e.g.*

9. K. Masuda, *Equation of evolution*, Kinokuniya, Tokyo, 1975. (Japanese)
10. H. Tanabe, *Equations of evolution*, Pitman, New York and London, 1979 (original in Japanese, 1975).

The theory of nonlinear evolution equations is treated in *e.g.*

11. I. Miyadera, *Nonlinear semigroups*, Transl. Math. Monographs, vol. 109, Amer. Math. Soc., Providence, RI, 1992 (original in Japanese, 1977).

There are several references in which the theory of elliptic and/or parabolic equations (and also equations of the other types) is fully discussed, though not necessarily in relation to diffusion equations directly. Here we list some of them.

12. N. Shimakura, *Partial differential operators of elliptic type*, Transl. Math. Monographs, vol. 99, Amer. Math. Soc., Providence, RI, 1992 (original in Japanese, 1978).
13. H. Kumano-go, *Partial differential equations*, Kyōritsu-Shuppan, Tokyo, 1978 (Japanese).
14. I. G. Petrovskiĭ, *Lectures on partial differential equations*, Interscience, New York, 1954 (original in Russian, 1953).
15. A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, NJ, 1964.
16. S. Mizohata, *Theory of partial differential equations*, Cambridge Univ. Press, London and New York, 1973 (original in Japanese, 1965).
17. O. A. Ladyzhenskaia, V. A. Solonnikov, and N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monographs, vol. 23, Amer. Math. Soc., Providence, RI, 1968 (original in Russian, 1967).

For the ergodic theory related to the contents of §20 of this book, readers may refer to:

18. H. Totoki, *Introduction to ergodic theory*, Kyōritsu-Shuppan, Tokyo, 1971. (Japanese)
19. Y. Ito and T. Hamachi, *Ergodic theory and von Neumann algebras*, Kinokuniya, Tokyo, 1992. (Japanese)

Chapter 1 of this book is due to [20] and [21]; most of Chapters 2, 3, and 4 are written following [22] and [23].

20. W. Feller, *Zur Theorie der stochastischen Prozesse*, Math. Ann. **113** (1936), 113–160.
21. F. G. Dressel, *The fundamental solution of the parabolic equation*, I, Duke Math. J. **7** (1940), 186–203; II, Duke Math. J **13** (1946), 61–70.
22. S. Itô, *Fundamental solutions of parabolic differential equations and boundary value problems*, Japan. J. Math. **27** (1957), 55–102;
23. S. Itô, *On Neumann problem for nonsymmetric second order partial differential equations of elliptic type*, J. Fac. Sci., Univ. Tokyo, Sec. I, **10** (1963), 20–28.

The Strong maximum principle for diffusion equations mentioned in §10 was originally proved (by an entirely different method) in

24. L. Nirenberg, *A strong maximum principle for parabolic equations*, Comm. Pure Appl. Math. **6** (1953), 167–177.

The author intended to add some topics on superharmonic functions and the unique continuation theorem for solutions of elliptic equations to the contents of this book (the original Japanese edition), but he had to give this up for want of space. Some basic properties of superharmonic functions are treated in:

25. S. Itô, *Superharmonic functions and ideal boundaries*, Kinokuniya, Tokyo, 1988. (Japanese)

The book [25] was written just after the publication of the original (Japanese) edition of the present work. For the unique continuation theorem, readers may refer to §5.6 of [13] or any one of the following papers.

26. N. Aronszajn, *A unique continuation theorem for solutions of elliptic equations or inequalities of second order*, J. Math. Pures Appl. **36** (1957), 235–249.
27. H. O. Cordes, *Über die eindeutige Bestimmtheit der Lösungen elliptischer Differentialgleichungen durch Anfangsvorgaben*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. IIa, **11** (1956), 239–258.

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