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G. M. Fel′dman

American Mathematical Society

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Arithmetic of Probability Distributions, and Characterization Problems on Abelian Groups

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American Mathematical Society
Providence, Rhode Island

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АРИФМЕТИКА ВЕРОЯТНОСТНЫХ РАСПРЕДЕЛЕНИЙ И ХАРАКТЕРИЗАЦИОННЫЕ ЗАДАЧИ НА АБЕЛЕВЫХ ГРУППАХ

Translated from the Russian by Yu. Lyubarskiĭ
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ABSTRACT. The main problem of the arithmetic of probability distributions is the study of all possible representations of a given random variable as the sum of independent random variables. The book contains material concerning the applicability of that theory to random variables with values in a locally compact abelian group. The results on the arithmetic of probability distributions are then used to solve characteristic problems of mathematical statistics on abelian groups.

This book is intended for mathematician-specialists, graduate and advanced undergraduate students interested in probability, mathematical statistics and functional analysis.

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APPENDIX 1

Group Analogs of the Marcinkiewicz Theorem and the Lukacs Theorem

A. 1.1. Marcinkiewicz (see [LinO, Chapter II, §5]) proved the following theorem.

THEOREM. *Let $P(s)$ be a polynomial, $P(0) = 0$. If $f(s) = \exp\{P(s)\}$ is a characteristic function then $P(s) = -\sigma s^2 + i\beta s$, $\sigma \geq 0$, $\beta \in \mathbb{R}$, i.e., $f(s)$ is the characteristic function of a Gaussian distribution.*

This result was obtained by Marcinkiewicz as a consequence of his more general theorem that gave a necessary condition for an entire function of finite order to be characteristic.

We give a complete description of the class of those groups X for which a similar theorem takes place. After that we consider the group analog of the Lukacs theorem, which generalizes that of Marcinkiewicz.

A. 1.2. DEFINITION. A continuous function $\psi(y)$ on the group Y is called a polynomial if for some m,

$$\Delta_h^{m+1} \psi(y) = 0 \tag{i}$$

for all h, $y \in Y$ (here Δ_h is the finite difference operator: $\Delta_h \psi(y) = \psi(y+h) - \psi(y)$).

The degree of the polynomial $\psi(y)$ is the minimal m for which (i) is fulfilled.

EXAMPLES.

1. Let $l : Y \to \mathbb{C}$ be a nonzero continuous homomorphism. Then $l(y)$ is a polynomial of degree 1.

2. Let $\varphi(y)$ be a continuous function on Y satisfying equation 5.1(ii) ($\varphi(y) \not\equiv 0$). Then $\varphi(y)$ is a polynomial of degree 2.

Let us establish some properties of polynomials.

A. 1.3. LEMMA. *Let $\psi(y)$ be a polynomial on a group Y. Then*

$$\psi(y + \zeta) = \psi(y) \tag{i}$$

for all $y \in Y$, $\zeta \in Y_0$.

PROOF. Let $y \in Y$ and let $\zeta \in Y_0$ be fixed. Define the function $P(l)$ on the group $\mathbb{Z}$ by the relation $P(l) = \psi(y + l\zeta)$, $l \in \mathbb{Z}$. Suppose that the degree of the polynomial $\psi(y)$ is m. Then $\Delta_1^{m+1} P(l) = 0$, and therefore $P(l) = a_0 + a_1 + \cdots + a_m l^m$ (see [Ge, Chapter 5, §3]) where the coefficients a_j depend on y and ζ. Consider the compact subgroup M_ζ generated by the element ζ. The function $\psi(\eta)$ is continuous and bounded on the compact set $y + M_\zeta$. Therefore $P(l) = \text{const}$, i.e., $\psi(y+\zeta) = \psi(y)$. □

A. 1.4. Corollary. *Let Y be a compact group. If $\psi(y)$ is a polynomial on Y, then $\psi(y) = \text{const}$.*

A. 1.5. Remark. Let $\psi(y)$ be a polynomial on the group Y. By Lemma A.1.3 the function $\psi(y)$ is invariant with respect to the subgroup Y_0. Therefore $\psi(y)$ defines the polynomial $\tilde{\psi}[y] = \psi(y)$ on the factor-group Y/Y_0.

The following statement is a generalization of Proposition 5.4.

A. 1.6. Proposition. *Let $\mu \in \mathscr{M}^1(X)$ and let the characteristic function $\hat{\mu}(y)$ have the form*

$$\hat{\mu}(y) = \exp\{\psi(y)\}, \quad \psi(0) = 0, \tag{i}$$

where $\psi(y)$ is a polynomial. Then $\sigma(\mu) \subset C_X$.

Proof. Let L be a compact subgroup of the group Y. Corollary A.1.4 implies that $\psi(y) \equiv 0$ on L and so $\psi(y) \equiv 0$ on Y_0. Hence $\hat{\mu}(y) \equiv 1$ on Y_0. By Theorem 1.9 we have $A(X, Y_0) = C_X$ and Proposition 2.13 implies $\sigma(\mu) \subset C_X$. □

Let us prove a group analog of the Marcinkiewicz theorem.

A.1.7. Theorem. *Let a group X contain no subgroup topologically isomorphic to $\mathbb{T}$. Let $\mu \in \mathscr{M}^1(X)$, and let the characteristic function $\hat{\mu}(y)$ have form* A.1.6(i). *Then $\mu \in \Gamma(X)$.*

Proof. According to Proposition 1.6 one may assume that X is a connected group. By Theorem 1.15 we may then assume that $X = \mathbb{R}^n + K$, where $n \geq 0$ and K is a connected compact group. Let us prove the theorem for the case $X = K$. (The general case is similar.) By Corollary 1.10 $Y = D$ is a discrete torsion-free group. We also assume that $\dim X = \infty$; $\dim X < \infty$ only simplifies the argument.

Set $\varphi(y) = -\text{Re}\ \psi(y)$, $l(y) = \text{Im}\ \psi(y)$. Then $\varphi(y)$ and $l(y)$ also are polynomials. Let us prove that the function $\varphi(y)$ satisfies equation 5.1(ii) and

$$l(y_1 + y_2) = l(y_1) + l(y_2) \tag{1}$$

for all $y_1, y_2 \in D$. This would show that $\mu \in \Gamma(X)$.

Fix arbitrary elements $a_1, a_2 \in D$. Let $f : D \to \mathbb{R}_0^\infty$ be the continuous monomorphism constructed in 5.9. Since no subgroup of the group X is topologically isomorphic to $\mathbb{T}$, by Lemma 9.7 there exists a subgroup $B \subset D$ of a finite rank q such that $a_1, a_2 \in B$ and $\overline{f(B)} \approx \mathbb{R}^l$. Denote $G = f(B)$.

Let $\{b_j\}_{j=1}^q$ be a maximal independent system of elements of $f(B)$ and $\{e_j\}_{j=1}^q$ the standard basis in $\mathbb{R}^q$. Let the continuous monomorphism

$g : f(B) \to \mathbb{R}^q$ be constructed in the same way as the monomorphism f in 5.6. We have $g(b_j) = e_j$, $j = 1, \dots, q$, and g can be naturally extended to a continuous isomorphism of the groups G and $\mathbb{R}^q$. Let $h = g \circ f$, $A = h(B)$. Since $\overline{f(B)} = G$, then $\bar{A} = \mathbb{R}^q$.

Define the function $\zeta(a) = \psi(h^{-1}(a))$ on the group A. Obviously, it is a polynomial. Considering the group A in the discrete topology, by Theorem 2.8 $\exp\{\zeta(a)\}$ is a characteristic function. We have reduced the theorem to the following problem. Let A be a group such that $\mathbb{Z}^q \subset A \subset \mathbb{Q}^q \subset \mathbb{R}^q$, $\bar{A} = \mathbb{R}^q$, and $\zeta(a)$, $a \in A$, a polynomial such that $\exp\{\zeta(a)\}$ is a characteristic function. Then the functions $\zeta_1(a) = \operatorname{Re}\zeta(a)$ and $\zeta_2(a) = \operatorname{Im}\zeta(a)$ satisfy equations 5.1(ii) and (1), respectively.

Let $\eta(a)$ be a polynomial of degree m on the group A. Consider the restriction of the function $\eta(a)$ to the subgroup $\mathbb{Z}^q \subset A$. Since by hypothesis $\Delta_b^{m+1}\eta(a) = 0$ for all $a, b \in \mathbb{Z}^q$, then one can easily verify that the function $\eta(a)$ admits the representation

$$\eta(a) = \sum_{p=0}^{m} \sum_{\|k\|=p} C_k a^k, \tag{2}$$

$\mathbb{Z}^q$, where $k = (k_1, \dots, k_q)$, $a = (n_1, \dots, n_q) \in \mathbb{Z}^q$, $a^k = n_1^{k_1} \cdots n_q^{k_q}$. Hence if F is a subgroup of $\mathbb{Q}^q$ such that $F \approx \mathbb{Z}^q$ and $\mathbb{Z}^q \subset F \subset A$ and $\eta(a) = 0$ for all $a \in \mathbb{Z}^q$, then $\eta(a) = 0$ for all $a \in F$.

Let the group A be represented as the union of an increasing sequence of subgroups A_j:

$$A = \bigcup_{j=1}^{\infty} A_j, \qquad A_1 = \mathbb{Z}^q, \qquad A_{j+1} \supset A_j, \qquad A_j \approx \mathbb{Z}^q, \qquad j = 1, 2, \dots.$$

Denote by $\eta(a)$ the restriction of the polynomial $\zeta(a)$ to the subgroup $\mathbb{Z}^q$. It was noted above that the function $\eta(a)$ may be represented in the form (2) on $\mathbb{Z}^q$. This relation extends $\eta(a)$ to a function on the group $\mathbb{R}^q$ and, in particular, on the group A. This extension will be also denoted by $\eta(a)$. By construction, the polynomial $\delta(a) = \zeta(a) - \eta(a)$, $a \in A$, vanishes on $\mathbb{Z}^q$. So $\delta(a) = 0$ when $a \in A_j$, $j = 1, 2, \dots$, and, hence, $\delta(a) \equiv 0$ on the group A. Therefore the function $\zeta(a)$ may be represented in the form

$$\zeta(a) = \sum_{p=0}^{m} \sum_{\|k\|=p} C_k a^k, \tag{3}$$

on the group A, where $k = (k_1, \dots, k_q)$, $a = (r_1, \dots, r_q) \in A$, $a^k = r_1^{k_1} \cdots r_q^{k_q}$.

Formula (3) extends $\zeta(a)$ from A to $\mathbb{R}^q$: this extension is also denoted by ζ. Since $\exp\{\zeta(a)\}$ is a positive definite function on the group A and A is dense in $\mathbb{R}^q$, the extended function $\exp\{\zeta(s)\}$ on the group $\mathbb{R}^q$ is positive definite as well. By Theorem 2.8 it is a characteristic function. From the Marcinkiewicz theorem A.1.1 one can now easily conclude that $\exp\{\zeta(s)\}$ is a characteristic function of a Gaussian distribution. Hence $\zeta_1(a)$ and $\zeta_2(a)$ satisfy equations 5.1(ii) and (1), respectively. □

A. 1.8. REMARK. It follows from Theorem A.1.7 that for every $t \geq 0$ the function $\exp\{t\psi(y)\}$ is the characteristic function of some distribution $\mu_t \in \Gamma(X)$. Moreover $\mu_0 = E_0$, $\mu_1 = \mu$, $\mu_{t+s} = \mu_t * \mu_s$ for all $t, s \geq 0$ and $\mu_t \Rightarrow E_0$ as $t \to 0$. Thus the distribution μ belongs to a continuous one-parameter semigroup of Gaussian distributions.

Theorem A.1.7 is sharp. Namely, the following theorem is true.

A. 1.9. THEOREM. *Let the group X contain a subgroup topologically isomorphic to $\mathbb{T}$. Then for every $m > 2$ there exists a distribution $\mu \notin \Gamma(X)$ with characteristic function of the form* A.1.6(i), *where $\psi(y)$ is a polynomial of degree m.*

PROOF. Clearly, it suffices to prove the theorem for the group $X = \mathbb{T}$. Consider the following polynomials of degree m on the group $\mathbb{Z}$,

$$\psi(n) = -n^2 + in^m, \quad n \in \mathbb{Z}, \text{ if } m = 2l + 1;$$
$$\psi(n) = -n^m, \quad n \in \mathbb{Z}, \text{ if } m = 2l.$$

Set $f(n) = \exp\{\psi(n)\}$, $n \in \mathbb{Z}$. As $\sum_{n\neq 0} f(n) < 1$, then the function

$$\rho(t) = \sum_{n=-\infty}^{\infty} f(n)\exp\{-int\} > 0.$$

Therefore the characteristic function of the distribution $\mu \in \mathscr{M}^1(\mathbb{T})$ with the density ρ with respect to m_T has the form $\hat{\mu}(n) = f(n)$. Obviously, $\mu \in \Gamma(\mathbb{T})$ and the characteristic function $\hat{\mu}(n)$ has the desired form. □

A. 1.10. REMARK. If the characteristic function of a distribution μ on the real axis is of the form A.1.6(i), where $\psi(y)$ is a polynomial of degree 2, then μ is Gaussian. This statement ceases to be true for general groups. Namely, on the group $X = \mathbb{T}^2$ there exists a distribution $\mu \notin \Gamma(X)$ with characteristic function of the form A.1.6(i) where $\psi(y)$ is a polynomial of degree 2.

Indeed, set

$$\psi(n, m) = -a(n^2 + m^2) + i\pi(n^2 + m^2 + nm), \qquad (n, m) \in \mathbb{Z}^2,$$

where $a > 0$ is such that inequality 9.9(1) is fulfilled. Consider the function

$$\rho(t, s) = \sum_{(n,m)\in\mathbb{Z}^2} \exp\{\psi(n, m) - i(tn + sm)\}$$

on the group $\mathbb{T}^2$. By construction $\rho(t, s) > 0$. Therefore the distribution $\mu \in \mathscr{M}^1(X)$ with density ρ with respect to m_X has the characteristic function $\hat{\mu}(y) = \hat{\mu}(n, m) = \exp\{\psi(n, m)\}$. Since $\exp\{i\pi(n^2 + m^2 + nm)\}$ is not a character of the group $\mathbb{Z}^2$, $\mu \notin \Gamma(X)$.

The Marcinkiewicz theorem was generalized by Lukacs, who proved the following assertion ([Lu, Chapter 7]).

A. 1.11. Theorem. *Let $P(s)$ be a polynomial, $P(0)=0$. If*

$$f(s)=\exp\{\lambda_1(e^{is}-1)+\lambda_2(e^{-is}-1)+P(s)\}$$

is a characteristic function, then $\lambda_j \geq 0$, $j=1,2$, and $P(s)=-\sigma s^2+i\beta s$, $\sigma \geq 0$, $\beta \in \mathbb{R}$.

Below (Theorem A.1.14) we completely describe the class of groups X for which a similar theorem is true. To do this, we need the following lemmas.

A. 1.12. Lemma. *Let $X=\mathbb{R}^n$, and let $P(s)$ be a polynomial defined on the group Y, $P(0)=0$. If*

$$f(s)=\exp\{\lambda_1(e^{i\langle t_0, s\rangle}-1)+\lambda_2(e^{-i\langle t_0, s\rangle}-1)+P(s)\}$$

is a characteristic function, then $\lambda_j \geq 0$, $j=1,2$, and $P(s)=-\langle As, s\rangle+i\langle\beta, s\rangle$, where A is a symmetric positive semidefinite matrix and $\beta \in \mathbb{R}^n$.

We omit the proof since it follows from Theorem A.1.11 in a standard way.

A. 1.13. Lemma. *Let $x_0 \in X$ be an element of infinite order, and let*

$$f(y)=\exp\{\lambda_1[(x_0, y)-1]+\lambda_2[(-x_0, y)-1]\} \qquad \text{(i)}$$

be a characteristic function. Then $\lambda_j \geq 0$, $j=1,2$. If x_0 is an element of order n, $n>2$, then there exists a characteristic function $f(y)$ of form (i) *with $\lambda_1>0$, $\lambda_2<0$.*

Proof. Consider the subgroup $F=\{\alpha \in \mathbb{T} : \alpha=(x_0, y), y \in Y\} \subset \mathbb{T}$. Since x_0 is an element of infinite order, then by Lemma 6.4 the subgroup F is dense in $\mathbb{T}$. Define the function $\Phi(\alpha)=f(y)$, $\alpha=(x_0, y)$, on F. This definition is meaningful because if $(x_0, y_1)=(x_0, y_2)$, then $f(y_1)=f(y_2)$. The relation

$$\Phi(\alpha)=\exp\{\lambda_1(\alpha-1)+\lambda_2(\bar{\alpha}-1)\}, \qquad \alpha \in F, \qquad (1)$$

extends the function Φ from F to the whole group $\mathbb{T}$. This extension is also denoted by Φ. Since the function $\Phi(\alpha)$ is positive definite on the group F and F is dense in $\mathbb{T}$, the extended function $\Phi(\xi)$ is positive definite on the group $\mathbb{T}$ as well. Therefore by Theorem 2.8 $\Phi(\xi)$ is a characteristic function on $\mathbb{T}$. This immediately implies $\lambda_j \geq 0$, $j=1,2$.

Let x_0 be an element of order n. Without loss of generality one may assume that $X=\mathbb{Z}(n)$. Let $\{1, \eta, \ldots, \eta^{n-1}\}$ be the elements of the group $\mathbb{Z}(n)$, $\eta=\exp\{\frac{2\pi i}{n}\}$, $\lambda \in \mathbb{R}$. Consider the charge $e(\lambda E_\eta)$ on the group $\mathbb{Z}(n)$. Since $e(E_\eta)(\{\eta^k\}) \geq \delta>0$, $k=0,1,\ldots,n-1$ (see Proposition 6.6) and $e(\lambda E_\eta) \Rightarrow E_1$ as $\lambda \to 0$, then for sufficiently small $\lambda_2<0$ the convolution $\nu = e(E_\eta) * e(\lambda_2 E_{\eta^{n-1}}) \in \mathscr{M}^1(\mathbb{Z}(n))$ and $\hat{\nu}(y) = f(y)$, where $\lambda_1 = 1$ and $\lambda_2 < 0$. □

We now prove the group analog of the Lukacs theorem.

A. 1.14. THEOREM. *Let a group X has no subgroup topologically isomorphic to $\mathbb{T}$, let $\mu \in \mathscr{M}^1(X)$, and let the characteristic function $\hat{\mu}(y)$ have the form*

$$\hat{\mu}(y) = \exp\{\lambda_1[(x_0, y] - 1] + \lambda_2[(-x_0, y) - 1] + \psi(y)\}, \qquad \psi(0) = 1, \quad \text{(i)}$$

where $\psi(y)$ is a polynomial. Then $\exp\{\psi(y)\}$ is the characteristic function of a Gaussian distribution. If x_0 is an element of infinite order, then $\lambda_j \geq 0$, $j = 1, 2$.

PROOF. If $C_X = \{0\}$, then by Corollary 1.10 $Y_0 = Y$. Then by Lemma A.1.3 $\psi(y) \equiv 0$ on Y and the assertion of the theorem follows from Lemma A.1.13.

Let $C_X \neq \{0\}$. By Theorem 1.15 $C_X \approx \mathbb{R}^n + K$, where $n \geq 0$ and K is a connected compact group. We shall confine ourselves to the case $C_X = K$. The general case is quite similar. Then by Corollary 1.10 $D = K^*$ is a discrete torsion-free group. Assume also that $\dim K = q < \infty$. Then, according to Theorem 1.13, $r(D) = q$. The case $\dim K = \infty$ is similar (compare with the proof of Theorem A.1.7).

Let $f : D \to \mathbb{R}^q$ be the continuous monomorphism constructed in 5.6. According to Remark A.1.5, the function $\psi(y)$ defines the function $\tilde{\psi}([y]) = \psi(y)$ on the factor-group Y/Y_0. By Theorems 1.6 and 1.9 we have $Y/Y_0 \approx D$. We shall keep the notation $[y]$ for elements of the group D and assume that the function $\tilde{\psi}([y])$ is defined on D. Let $A = f(D)$ and $\zeta(a) = \zeta(f([y])) = \tilde{\psi}([y])$, $a = f([y]) \in A$. The function $\zeta(a)$ is a polynomial on the group A. As was established in the proof of Theorem A.1.7, the function $\zeta(a)$ admits the representation A.1.7(3). This representation extends $\zeta(a)$ from the group A onto $\mathbb{R}^q$.

Since no subgroup of C_X is topologically isomorphic to $\mathbb{T}$, then, due to Remark 5.18, the subgroup A is dense in $\mathbb{R}^q$. Set $p = \tilde{f}$, $p : \mathbb{R}^q \to C_X$. Then by 1.20 p is a continuous monomorphism. Denote by π the embedding $\pi : C_X \to X$, i.e., $\pi(x) = x$, $x \in C_X$.

The following three cases are possible:

(1) For all $n \in \mathbb{Z}, n \neq 0$, we have $nx_0 \notin p(\mathbb{R}^q)$. In this case x_0 is an element of infinite order. Let us extend the monomorphism p to the continuous monomorphism $p_1 : \mathbb{R}^q + \mathbb{Z} \to X$ by setting $p_1(t, k) = \pi(p(t)) + kx_0$, $(t, k) \in \mathbb{R}^q + \mathbb{Z}$, and consider the adjoint homomorphism $\tilde{p}_1 : Y \to \mathbb{R}^q + \mathbb{T}$. We have

$$\begin{aligned}((t, k), \tilde{p}_1(y)) &= (p_1(t, k), y) = (\pi(p(t)) + kx_0, y) \\ &= (\pi(p(t)), y)(kx_0, y) = (t, f([y]))(x_0, y)^k.\end{aligned}$$

Therefore $\tilde{p}_1(y) = (f([y]), (x_0, y))$.

Observe now that if $\tilde{p}_1(y_1) = \tilde{p}_1(y_2)$, then $y_1 - y_2 \in Y_0$ and $(x_0, y_1) = (x_0, y_2)$. Therefore, by Lemma A.1.3 $\hat{\mu}(y_1) = \hat{\mu}(y_2)$. Define the function $\Phi(\tilde{p}_1(y)) = \hat{\mu}(y)$ on the group $\tilde{p}_1(Y)$. It follows from the above-said that this definition is meaningful. The relation

$$\Phi(a, \alpha) = \exp\{\lambda_1(\alpha - 1) + \lambda_2(\bar{\alpha} - 1) + \zeta(a)\}, \qquad (a, \alpha) = \tilde{p}_1(y), \qquad (1)$$

extends the function $\Phi(a, \alpha)$ from the subgroup $\tilde{p}_1(Y)$ to the whole group $\mathbb{R}^q + \mathbb{T}$. This extension also is denoted by Φ. The mapping $\tilde{p}_1$ is a monomorphism. Therefore, according to 1.20 the subgroup $\tilde{p}_1(Y)$ is dense in $\mathbb{R}^q + \mathbb{T}$. The function $\Phi(a, \alpha)$ is positive definite on the group $\tilde{p}_1(Y)$. Hence the extended function $\Phi(s, \xi)$, $(s, \xi) \in \mathbb{R}^q + \mathbb{T}$, is positive definite on the group $\mathbb{R}^q + \mathbb{T}$ and, due to Theorem 2.8, $\Phi(s, \xi)$ is a characteristic function. By 2.10(h), $\Phi(s, 1) = \exp\{\zeta(s)\}$ is a characteristic function on $\mathbb{R}^q$. Theorem 2.8 implies now that $\exp\{\psi(y)\}$ is a characteristic function and then, according to Theorem 1.7, $\exp\{\psi(y)\}$ is the characteristic function of a Gaussian distribution. Since $\Phi(0, \xi) = \exp\{\lambda_1(\xi - 1) + \lambda_2(\bar{\xi} - 1)\}$ is a characteristic function on $\mathbb{T}$, then $\lambda_j \geq 0$, $j = 1, 2$. We have proved the theorem for this case.

(2) Let, for some natural $n \geq 2$, the relations $x_0, \dots, (n-1)x_0 \notin p(\mathbb{R}^q)$ and $nx_0 \in p(\mathbb{R}^q)$ hold. Suppose $nx_0 = p(t_0)$. Set $x_1 = x_0 - \pi(p(\frac{t_0}{n}))$. Then x_1 is an element of order n. Let us extend the monomorphism p to a continuous monomorphism $p_1 : \mathbb{R}^q + \mathbb{Z}(n) \to X$ by setting $p_1(t, \eta^k) = \pi(p(t)) + kx_0$, $(t, \eta^k) \in \mathbb{R}^q + \mathbb{Z}(n)$. The elements of the group $(\mathbb{R}^q + \mathbb{Z}(n))^* \approx \mathbb{R}^q + \mathbb{Z}(n)$ will be denoted by (s, δ^l), where $\delta = \exp\{\frac{2\pi i}{n}\}$, $l = 0, 1, \dots, n-1$. One can easily see that the formula $\tilde{p}_1(y) = (f([y]), (x_1, y))$ defines the adjoint homomorphism $\tilde{p}_1 : Y \to \mathbb{R}^q + \mathbb{Z}(n)$. Note that $p_1(\frac{t_0}{n}, \eta) = x_0$. Let $\tilde{p}_1(y_1) = \tilde{p}_1(y_2)$. Then $y_1 - y_2 \in Y_0$ and $(x_1, y_1) = (x_1, y_2)$. Since by Theorem 1.9 we have $Y_0 = A(Y, C_X)$ and $p(\mathbb{R}^q) \subset C_X$, then $(\pi(p(t)), y_1) = (\pi(p(t)), y_2)$ for all $t \in \mathbb{R}^q$. Therefore

$$\begin{aligned}(x_0, y_1) &= \left(x_1 + \pi\left(p\left(\frac{t_0}{n}\right)\right), y_1\right) = (x_1, y_1)\left(\pi\left(p\left(\frac{t_0}{n}\right)\right), y_1\right) \\ &= (x_1, y_2)\left(\pi\left(p\left(\frac{t_0}{n}\right)\right), y_2\right) = \left(x_1 + \pi\left(p\left(\frac{t_0}{n}\right)\right), y_2\right) = (x_0, y_2).\end{aligned}$$

Therefore, by Lemma A.1.3 $\hat{\mu}(y_1) = \hat{\mu}(y_2)$. Define the function $\Phi(\tilde{p}_1(y)) = \hat{\mu}(y)$ on the group $\tilde{p}_1(Y)$. It follows from the arguments given above that this definition is meaningful. The relation

$$\begin{aligned}\Phi(a, \delta^l) = \exp\Bigg\{ &\lambda_1\left(\exp\left\{i\left(\frac{2\pi l}{n} + \left\langle\frac{t_0}{n}, a\right\rangle\right)\right\} - 1\right) \\ &+ \lambda_2\left(\exp\left\{-i\left(\frac{2\pi l}{n} + \left\langle\frac{t_0}{n}, a\right\rangle\right)\right\} - 1\right) + \zeta(a)\Bigg\},\end{aligned}$$

$$(a, \delta^l) = \tilde{p}_1(y) \qquad (2)$$

extends the function Φ from the subgroup $\tilde{p}_1(Y)$ to the group $\mathbb{R}^q + \mathbb{Z}(n)$. We shall keep the same notation Φ for the extended function. The mapping $\tilde{p}_1$ is a monomorphism. Therefore, according to §1.20, the subgroup $\tilde{p}_1(Y)$ is dense in $\mathbb{R}^q + \mathbb{Z}(n)$. The function $\Phi(a, \delta^l)$ is positive definite on $\tilde{p}_1(Y)$. Hence the extended function $\Phi(s, \delta^l)$, $(s, \delta^l) \in \mathbb{R}^q + \mathbb{Z}(n)$, is positive definite on the group $\mathbb{R}^q + \mathbb{Z}(n)$ and, by Theorem 2.8, Φ is a characteristic function. According to 2.10(h),

$$\Phi(s, 1) = \exp\left\{ \lambda_1 \left(\exp\left\{i\left\langle\frac{t_0}{n}, s\right\rangle\right\} - 1\right) + \lambda_2 \left(\exp\left\{-i\left\langle\frac{t_0}{n}, s\right\rangle\right\} - 1\right) + \zeta(s)\right\}$$

is a characteristic function on $\mathbb{R}^q$. According to Lemma A.1.11 this implies that, in particular, $\exp\{\zeta(s)\}$ is a characteristic function on $\mathbb{R}^q$. Then by Theorem 2.8 $\exp\{\psi(y)\}$ is a characteristic function and by virtue of Theorem A.1.7 $\exp\{\psi(y)\}$ is the characteristic function of a Gaussian distribution.

Let x_0 be an element of infinite order. Then $t_0 \neq 0$. Applying Lemma A.1.12 to the characteristic function $\Phi(s, 1)$, we obtain that $\lambda_j \geq 0$, $j = 1, 2$. We have proved the theorem for the case 2 as well.

It should be noted that when x_0 is an element of finite order $n > 2$, it follows from Lemma A.1.13 that, in general, λ_j may be negative. But if $n = 2$, then $x_0 = -x_0$ and the initial characteristic function has the form

$$\hat{\mu}(y) = \exp\{\lambda[(x_0, y) - 1] + \psi(y)\}, \qquad \lambda = \lambda_1 + \lambda_2.$$

In this case the fact that

$$\Phi(0, (-1)^l) = \exp\{\lambda(\exp\{i\pi l\} - 1)\}$$

is a characteristic function on the group $\mathbb{Z}(2)$ implies $\lambda \geq 0$.

(3) $x_0 \in p(\mathbb{R}^q)$. In this case x_0 is an element of infinite order. Extend the monomorphism p to the continuous monomorphism $p_1 : \mathbb{R}^q \to X$ setting $p_1(t) = \pi(p(t))$. The adjoint monomorphism $\tilde{p}_1 : Y \to \mathbb{R}^q$ has the form $\tilde{p}_1(y) = f([y])$. As in the cases 1 and 2 one can verify that if $\tilde{p}_1(y_1) = \tilde{p}_1(y_2)$, then $\hat{\mu}(y_1) = \hat{\mu}(y_2)$. This permits the definition of the function $\Phi(\tilde{p}_1(y)) = \hat{\mu}(y)$ on the group $\tilde{p}(Y)$. The rest of the argument is the same as in the cases (1) and (2). □

APPENDIX 2

On Decomposition Stability of Distributions

A.2.1. Let d be some metric on $\mathscr{M}^1(X)$, and $F(\mu)$ the set of all divisors of a distribution μ. The distribution μ is said to possess decomposition stability with respect to the metric d if for any sequence of distributions $\{\mu_n\}$, $\mu_n \in \mathscr{M}^1(X)$, such that $d(\mu_n, \mu) \to 0$ as $n \to \infty$, and for every sequence $\{\nu_n\}$, $\nu_n \in F(\mu_n)$, $n = 1, 2, 3, \ldots$, there exists a sequence $\{\gamma_n\}$, $\gamma_n \in F(\mu)$, such that $d(\nu_n, \gamma_n) \to 0$ as $n \to \infty$.

To measure the decomposition stability of distributions with respect to the metric d one may use the following quantity, introduced by Zolotarev [Z2]:

$$\beta_d(\mu, \varepsilon) = \sup_{\nu \in B_\varepsilon(\mu)} \sup_{\alpha \in F(\nu)} \inf_{\beta \in F(\mu)} d(\alpha, \beta),$$

where $B_\varepsilon(\mu) = \{\nu \in \mathscr{M}^1(X) : d(\mu, \nu) \leq \varepsilon\}$ is the ball of the radius ε centered at the point μ.

One can easily see that the distribution μ possesses decomposition stability with respect to the metric d if and only if $\beta_d(\mu, \varepsilon) \to 0$ as $\varepsilon \to 0$.

Suppose that X is a complete separable metric abelian group. Let us verify that any distribution on X is stable with respect to any metric realizing the weak topology.

Denote by π the Lévy-Prokhorov metric in the space $\mathscr{M}^1(X)$:

$$\pi(\mu, \nu) = \inf_{A \in \mathscr{B}(X)} \{\varepsilon : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon, \ \nu(A) \leq \mu(A^\varepsilon) + \varepsilon\},$$

where $A^\varepsilon = \{x : \rho(x, y) < \varepsilon, \ y \in A\}$, ρ is the metric in the group X.

A.2.2. THEOREM. *Let X be a complete separable metric abelian group. Then for every $\mu \in \mathscr{M}^1(X)$,*

$$\beta_\pi(\mu, \varepsilon) \to 0 \quad \textit{as } \varepsilon \to 0.$$

PROOF. To prove the theorem it suffices to establish the following. Let $\{\mu_n\}$ be a sequence of distributions, $\mu_n \in \mathscr{M}^1(X)$, $\mu_n = \mu_{n1} * \mu_{n2}$, $n = 1, 2, 3, \ldots$, and let $\pi(\mu_n, \mu) \to 0$, $n \to \infty$. Then:

(a) the sequences $\{\mu_{n1}\}$ and $\{\mu_{n2}\}$ are shift-compact,

(b) the limit distributions corresponding to these sequences are divisors of the distribution μ.

Statement (a) immediately follows from Theorem 2.2. The proof of statement (b) is similar to that in the one-dimensional case. We shall assume that all distributions μ_{n1} and μ_{n2} are already shifted in such a way that the sequences $\{\mu_{n1}\}$ and $\{\mu_{n2}\}$ are compact. Let

$$\pi(\mu_{n_k, i}, \nu_i) \to 0 \quad \text{as } k \to \infty, \; i = 1, 2.$$

We show that $\nu_1, \nu_2 \in F(\mu)$. Indeed, accounting for the weak regularity of the metric π, we obtain

$$\begin{aligned}\pi(\nu_1 * \nu_2, \mu) &\le \pi(\nu_1 * \nu_2, \mu_{n_k}) + \pi(\mu_{n_k}, \mu) \\ &\le \pi(\nu_1, \mu_{n_k, 1}) + \pi(\nu_2, \mu_{n_k, 2}) + \pi(\mu_{n_k}, \mu) \to 0\end{aligned}$$

as $k \to \infty$, i.e. $\nu_1 * \nu_2 = \mu$. □

Let d_1 and d_2 be metrics in $\mathscr{M}^1(X)$. In some cases to measure the decomposition stability of a distribution μ it is convenient to use another quantity, also introduced by Zolotarev [Z2]:

$$\beta_{d_1, d_2}(\mu, \varepsilon) = \sup_{\nu \in B_\varepsilon(\mu)} \sup_{\alpha \in F(\nu)} \inf_{\beta \in F(\mu)} d_2(\alpha, \beta),$$

where $B_\varepsilon(\mu) = \{\nu \in \mathscr{M}^1(X) : d_1(\mu, \nu) \le \varepsilon\}$.

Let X be a locally compact abelian group satisfying the second axiom of countability. Choose the metric

$$\sigma(\mu, \nu) = \sup_{A \in \mathscr{B}(X)} \{|\mu(A) - \nu(A)|\}$$

as the metric d_1 and the metric

$$\chi_0(\mu, \nu) = \sup_{y \in Y} \{| \hat{\mu}(y) - \hat{\nu}(y) |\}$$

as the metric d_2.

Our aim is to give an estimate for $\beta_{\sigma, \chi_0}(e(\Phi), \varepsilon)$, where the measure $\Phi \in \mathscr{M}^1(X)$ satisfies the conditions of Theorem 6.13. The following theorem plays the main role in this respect.

A.2.3. THEOREM. *Let $\Phi \in \mathscr{M}_+(X)$ and let the measures Φ^{*n} and Φ^{*m} be pairwise singular for all natural $n, m, n \neq m$. Let distributions $\mu_j, j = 1, 2,$ be such that*

$$\sigma(\mu_1 * \mu_2, e(\Phi)) \le \varepsilon, \quad 0 < \varepsilon < e^{-2}. \tag{1}$$

Then there exist elements $x_1, x_2 \in X$, $x_1 + x_2 = 0$, depending on the distributions μ_j only and such that $\mu_j(\{x_j\}) > c_0$, $j = 1, 2$ and

$$\chi_0(\mu_j, e(\Phi_j) * E_{x_j}) \le c_1 \frac{\ln \ln(\frac{1}{\varepsilon})}{\ln(\frac{1}{\varepsilon})}, \qquad j = 1, 2,$$

where Φ_j *is the restriction of the measure* $(\mu_j(\{x_j\}))^{-1}(\mu_j * E_{-x_j})$ *to the set* A_1 *on which the measure* Φ *is concentrated and the constants* $c_0 > 0$, $c_1 > 0$ *depend on the distribution* $e(\Phi)$ *only.*

PROOF. Without loss of generality we assume that $\varepsilon > 0$ ($\varepsilon \leq \varepsilon(\Phi)$) is sufficiently small. In what follows all positive constants that depend on the distribution $e(\Phi)$ only are denoted by c. Estimate (1) obviously yields

$$(\mu_1 * \mu_2)(\{0\}) \geq \tfrac{1}{2}\exp\{-\Phi(X)\} = \tilde{c}_0.$$

Since the number of points $x \in X$ for which

$$\max(\mu_1(\{x\}), \mu_2(\{-x\})) \geq \tfrac{1}{4}\tilde{c}_0$$

is finite and depends on the measure Φ, one can easily verify that there exists an element $x_0 \in X$ such that $\mu_1(\{x_0\}) \geq c$, $\mu_2(\{-x_0\}) \geq c$. Without loss of generality we assume that $x_0 = 0$; otherwise we go over to the shifts $\mu_1 * E_{-x_0}$, $\mu_2 * E_{x_0}$. Using estimate (1), we obtain the relation

$$c\mu_j(A) \leq e(\Phi)(A) + \varepsilon, \qquad j = 1, 2, \tag{2}$$

for any $A \in B(X)$ (compare with Lemma 4.8). Let the measures Φ^{*n} be concentrated on the sets A_n, $n = 1, 2, 3, \ldots$. By hypothesis these sets may be chosen to be pairwise disjoint. Denote the restriction of the distribution μ_j to A_n by ν_{nj}, $n = 1, 2, 3, \ldots$; $j = 1, 2$.

One can easily see that inequality (2) and the fact that Φ^{*n} are pairwise singular yield that the measures ν_{nj} possess the property

$$c\nu_{nj}(X) \leq \exp\{-\Phi(X)\}\frac{(\Phi(X))^n}{n!} + \varepsilon, \qquad n = 1, 2, 3, \ldots. \tag{3}$$

Consider the measures

$$\mu_j^* = \sum_{n=0}^{T} \nu_{nj}, \qquad \nu_{0j} = \mu_j(\{0\})E_0, \qquad T = \left[\frac{1}{2}\frac{\ln(\frac{1}{\varepsilon})}{\ln\ln(\frac{1}{\varepsilon})}\right], \qquad j = 1, 2.$$

Set

$$B_T = \bigcup_{n=0}^{T} A_n.$$

By inequality (2) we obtain

$$\begin{aligned}\sigma(\mu_j, \mu_j^*) &\leq \mu_j(B_T) \leq c(e(\Phi)(B_T) + \varepsilon) \\ &\leq c\left(\sum_{n=T+1}^{\infty} \exp\{-\Phi(X)\}\frac{(\Phi(X))^n}{n!} + \varepsilon\right) \leq \varepsilon^{\frac{1}{4}}, \qquad j = 1, 2.\end{aligned} \tag{4}$$

Consider the functions

$$\varphi_j(z, y) = \sum_{n=0}^{T} z^n \hat{\nu}_{nj}(y), \qquad z \in \mathbb{C},\ y \in Y,\ j = 1, 2. \tag{5}$$

Inequalities (3) yield the estimate

$$|\varphi_j(z,y)| \le \exp(c|z|), \qquad y\in Y,\ j=1,2 \tag{6}$$

in the disk $|z| \le \frac{T}{2}$. So we have

$$\varphi_1(z,y)\varphi_2(z,y)=\sum_{n=0}^{2T} z^n b_n(y), \qquad z\in\mathbb{C},\ y\in Y.$$

The quantities $b_n(y)$, $n=0,1,\dots,2T$ satisfy the inequalities

$$q_n(y)=\left|b_n(y)-\exp\{-\Phi(X)\}\frac{(\hat\Phi(y))^n}{n!}\right| \le \sum_{m=0}^{n}\hat\nu_{m1}(y)\hat\nu_{n-m,2}(y)$$
$$-\exp\{-\Phi\}(X)\frac{(\hat\Phi(y))^n}{n!}+\sum_{\{0\le m<n-T\}\cup\{T<m\le n\}}|\hat\nu_{m1}(y)||\hat\nu_{n-m,2}(y)|.$$

From this, with the aid of the inequalities (1)–(3) and taking into consideration the mutual singularity of the measures Φ^{*n}, we obtain

$$q_n(y)\le c\varepsilon + c\sum_{\{0\le m<n-T\}\cup\{T<m\le n\}}\left(\varepsilon+\frac{(\Phi(X))^m}{m!}\right)\left(\varepsilon+\frac{(\Phi(X))^{n-m}}{(n-m)!}\right)$$
$$\le c(\Phi(X)+1)^n\left(T\varepsilon+\frac{\gamma(n)}{n!}\right),$$

where $\gamma(n)=0$ if $n\le T$ and $\gamma(n)=2^n$ if $T<n\le 2T$.

From these inequalities for $\alpha=(e^5(F(X))+1)^{-1}$, $y\in Y$, we obtain

$$\sum_{n=0}^{T}(\alpha T)^n q_n(y)\le cT(\alpha(\Phi(X)+1)T)^{2T}\varepsilon+\sum_{n=T}^{2T}\frac{(2\alpha T(\Phi(X)+1))^n}{n!}$$
$$\le \exp\{-3T\}. \tag{7}$$

We will also need the following inequality (it may be checked directly)

$$\sum_{n=2T+1}^{\infty}(\alpha T)^n\frac{(\Phi(X))^n}{n!}\le\exp\{-3T\}. \tag{8}$$

From inequalities (7) and (8) we obtain the estimate

$$|\varphi_1(z,y)\varphi_2(z,y)-\exp\{z\hat\Phi(y)-\hat\Phi(0)\}|$$
$$\le\sum_{n=0}^{2T}|z|^n q_n(y)+\sum_{n=2T+1}^{\infty}\exp\{-\Phi(X)\}\frac{(\Phi(X))^n}{n!}|z|^n \le 2\exp\{-3T\}$$

for all $|z|\le\alpha T$, $y\in Y$. For these values of z and y we also have the obvious lower estimate

$$|\exp\{z\hat\Phi(y)-\hat\Phi(0)\}|\ge\exp\{-2\hat\Phi(0)\alpha T\}\ge\exp\{-T\}.$$

Comparing the two last estimates, we obtain

$$|\varphi_1(z,y)\varphi_2(z,y)-\exp\{z\hat\Phi(y)-\hat\Phi(0)\}|\le\tfrac12|\exp\{z\hat\Phi(y)-\hat\Phi(0)\}|$$

for all z, y under consideration. Therefore

$$\tfrac{1}{2}|\exp\{z\hat{\Phi}(y)-\hat{\Phi}(0)\}|\le|\varphi_1(z,y)\varphi_2(z,y)|\le\tfrac{3}{2}|\exp\{z\hat{\Phi}(y)-\hat{\Phi}(0)\}| \quad (9)$$

in the disk $|z| < \alpha T$.

For any fixed character $y \in Y$ the functions $\varphi_j(z, y)$, $j = 1, 2$, are analytic with respect to z in the whole complex z-plane and in view of (9) they do not vanish in the disk $|z| < \alpha T$. Therefore for $|z| < \alpha T$ the functions $\varphi_j(z, y)$ admit the representation

$$\varphi_j(z, y) = \exp\{f_j(z, y)\}, \quad f_d(0, y) < 0, \qquad j = 1, 2,$$

where the function $f_j(z, y)$ is analytic in the disk $|z| < \alpha T$. We expand $f_j(z, y)$ into a Taylor series

$$f_j(z, y) = \sum_{n=0}^{\infty} a_{nj}(y) z^n, \qquad j = 1, 2$$

in this disk. By estimate (6) we have

$$\operatorname{Re} f_j(z, y) \le c(|z| + 1), \qquad |z| < \alpha T, \; y \in Y, \; j = 1, 2.$$

Let us apply the following analytical result (see [LinO, Chapter VIII,§2]):

If a function

$$f(z) = \sum_{n=1}^{\infty} b_n z^n$$

is analytic in the disk $|z| \le R$ and $\operatorname{Re} f(z) \le A$, $|z| \le R$, then

$$|b_n| \le \frac{2A}{R^n}, \qquad n = 1, 2, 3, \ldots.$$

Setting $f(z) = f_j(z, y) - a_{0j}(y)$, $R = \alpha\frac{T}{2}$, $A = cT$ we obtain

$$|a_{nj}(y)| \le \left(\frac{c}{T}\right)^{n-1}, \qquad y \in Y, \; n = 2, 3, \ldots.$$

These estimates imply the following representation for the functions $\varphi_j(z, y)$:

$$\varphi_j(z, y) = \exp\{a_{0j}(y) + a_{1j}(y)z\}(1 + H_j(z, y)), \qquad j = 1, 2, \qquad (10)$$

for $|z| \le cT$, $y \in Y$. Here the functions $H_j(z, y)$ $(H_j(0, y) = H_j'(0, y) = 0)$ are analytic in the disk $|z| \le cT$ for every fixed $y \in Y$ and admit the estimate

$$|H_j(1, y)| \le \frac{c}{T}, \qquad y \in Y.$$

Comparing relations (5) and (10) as functions of z we conclude that for all $y \in Y$,

$$\exp\{a_{0j}(y)\} = \mu_j(\{0\}), \quad \exp\{a_{0j}(y)\} \cdot a_{1j}(y) = \hat{\nu}_{1j}(y). \qquad (11)$$

By estimate (4) we have

$$|\varphi_j(1, 0) - 1| \le c\varepsilon^{\frac{1}{4}};$$

therefore relations (10) for $z = 1$, $y = 0$, and (11) yield

$$|\hat{\nu}_{1j}(0) + \mu_j(\{0\}) \ln \mu_j(\{0\})| \le \frac{c}{T}. \tag{12}$$

Now relations (4) and (10)-(12) lead to the inequalities

$$\left| \hat{\mu}_j(y) - \exp\left\{ \frac{\hat{\nu}_{1j}(y) - \hat{\nu}_{1j}(0)}{\mu_j(\{0\})} \right\} \right| \le \frac{c}{T}, \qquad y \in Y, \ j = 1, 2,$$

that prove the theorem. □

A.2.4. Corollary. *Let $\Phi \in \mathscr{M}_+(X)$ and let the measure Φ satisfy the conditions of Theorem* A.2.3. *Then the following estimate is valid*:

$$\beta_{\sigma, \chi_0}(e(\Phi), \varepsilon) \le c \frac{\ln \ln(\frac{1}{\varepsilon})}{\ln(\frac{1}{\varepsilon})}, \tag{13}$$

where the constant c depends on the distribution $e(\Phi)$ only.

Proof. We use the notation of Theorem A.2.3. As in the proof of Theorem A.2.3, positive constants depending on the distribution $e(\Phi)$ only will be denoted by a single letter c, regardless of their magnitude. To prove inequality (13) it suffices to verify that

$$\Phi_j(A) \le \Phi(A) + c\varepsilon, \qquad j = 1, 2, \tag{14}$$

for all $A \in \mathfrak{B}(X)$. Arguing as in the proof of Theorem A.2.3, we may assume $\mu_j(\{0\}) > c$, $j = 1, 2$, without loss of generality. From the definition of the measures ν_{nj}, $n = 1, 2, 3, \ldots, j = 1, 2$, the pairwise singularity of the measures Φ^{*n} for different n and inequalities (1) and (2) we easily obtain

$$\mu_2(\{0\})\nu_{11}(A) + \mu_1(\{0\})\nu_{12}(A) \le \exp\{-\Phi(X)\}\Phi(A) + c\varepsilon$$

for all $A \in \mathfrak{B}$ and also

$$|\mu_1(\{0\})\mu_2(\{0\}) - \exp\{-\Phi(X)\}| \le c\varepsilon.$$

These two inequalities imply (14) and hence also (13). □

A.2.5. Remark. Estimate (13) is sharp in the following sense. Let $X = \mathbb{R}$, and let Φ be a degenerated distribution concentrated at the point $x = 1$. In [Ch3] sequences of distributions $\mu_{nj} \in \mathscr{M}^1(\mathbb{R})$, $n = 1, 2, 3, \ldots, j = 1, 2$, are constructed such that

$$\sigma(\mu_{n1} * \mu_{n2}, e(\Phi)) \le \exp\{-c_1 n \ln n\}, \tag{15}$$

where the constant c_1 depends on the measure Φ only and is independent of n. The characteristic functions of the distributions μ_{n1} have the form

$$\hat{\mu}_{n1}(s) = \exp\left\{ \frac{1}{2}\lambda(e^{is} - 1) + \frac{\delta(\lambda)}{n}(e^{2is} - 1) \right\}, \qquad s \in \mathbb{R},$$

where $\lambda = \Phi(\mathbb{R})$ and $\delta(\lambda) > 0$ is a sufficiently small constant that depends on λ only. Theorem 6.5 implies that the characteristic function of an arbitrary divisor ν of the distribution $e(\Phi)$ has the form

$$\hat{\nu}(s) = \exp\{\lambda_0(e^{is} - 1) + i\beta_0 s\}, \qquad 0 \le \lambda_0 \le \lambda, \ -\infty < \beta < \infty.$$

Clearly the characteristic functions $\hat{\mu}_{n1}(s)$ and $\hat{\nu}(s)$ satisfy the inequalities

$$\inf_{\nu \in F(e(\Phi))} \sup_{s \in \mathbb{R}} |\hat{\mu}_{n1}(s) - \hat{\nu}(s)| \ge \frac{c_2}{n}, \qquad n = 1, 2, 3, \dots, \tag{16}$$

where the constant $c_2 > 0$ depends on the measure Φ only and is independent of n. Comparing the estimates (15) and (16) we obtain the lower estimate

$$\beta_{\sigma, \chi_0}(e(\Phi), \varepsilon) \ge c_3 \frac{\ln \ln(\frac{1}{\varepsilon})}{\ln(\frac{1}{\varepsilon})},$$

where $c_3 > 0$ is a constant depending on the measure Φ only and is independent of n. This proves that the estimate (13) is sharp in the class of all locally compact abelian groups.

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APPENDIX 3

Structure of Infinitely Divisible Poisson Distributions

A.3.1. Let X be a nondiscrete group and $\mu \in \mathscr{M}^1(X)$ be an arbitrary distribution. It will be convenient for us to write the decomposition 5.13(i) of μ in the form

$$\mu = \alpha_1 \mu^{(1)} + \alpha_2 \mu^{(2)} + \alpha_3 \mu^{(3)}, \tag{1}$$

where $\alpha_1 \mu^{(1)} = \mu_{ac}$, $\alpha_2 \mu^{(2)} = \mu_s$, $\alpha_3 \mu^{(3)}$, $\mu^{(j)} \in \mathscr{M}^1(X)$. Therefore $\alpha_j \geq 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$. We introduce the following notation.

$\mathscr{L} = \mathscr{L}(X)$ — the set of all Lévy measures on the group X (see 2.21);

$\mathscr{L}_1 = \mathscr{L}_1(X)$ — the set of all absolutely continuous Lévy measures on the group X;

$\mathscr{D} = \mathscr{D}(X)$ — the set of all discrete Lévy measures on the group X.

The family of singular Lévy measures on the group X will be subjected to further, more detailed, classification. To this end denote $U_m = \{x : \rho(x, 0) \geq \frac{1}{m}\}$, where ρ is the distance on the group X. Let $\Phi \in \mathscr{L}$. Set $\Phi_m(A) = \Phi(A \cap U_m) < \infty$. We have $\Phi_m(A) < \infty$ for every $A \in \mathscr{B}(X)$, and the measures Φ_m, $m = 1, 2, \ldots$, weakly converge to Φ on every set $A \in \mathscr{B}(X)$ with boundary of Φ-measure zero.

A measure $\Phi \in \mathscr{L}$ belongs to the class $\mathscr{L}_{\frac{1}{n}} = \mathscr{L}_{\frac{1}{n}}(X)$, $n \geq 2$, if the $(n-1)$-fold convolution of each measure Φ_m is singular, but for some m the n-fold convolution of Φ_m contains a nonzero absolutely continuous component in its representation of type 5.13(i).

We denote by $\mathscr{L}_0 = \mathscr{L}_0(X)$ the class of all singular measures $\Phi \in \mathscr{L}$ such that any convolution power of Φ_m is also singular.

It should be noted that in the case $\Phi \in \mathscr{M}_+(X)$ we may replace Φ_m by Φ in definition of the classes $\mathscr{L}_{\frac{1}{n}}$ and $\mathscr{L}_0$.

We have

$$\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_{\frac{1}{2}} \cup \cdots \cup \mathscr{L}_0 \cup \mathscr{D}.$$

Consider an infinitely divisible Poisson distribution μ, i.e., distribution with characteristic function of the form

$$\hat{\mu}(y) = \exp\left\{ \int_{X \setminus \{0\}} [(x, y) - 1 - ig(x, y)] d\Phi(x) \right\}$$

(see 2.21). In this case we use the notation $\mu = \Pi(\Phi)$. Our purpose is to describe the possible structure (i.e., decomposition (1)) of the distribution μ subject to the properties of the Lévy measure Φ.

We say that a product $\prod_{j=1}^{\infty} g_j$ of complex numbers g_j, $j = 1, 2, 3, \dots$, strictly converges if for some natural n the product $\prod_{j=n}^{\infty} g_j$ converges to a nonzero value.

A.3.2. Lemma ([Par]). *Let ξ_j be independent random variables with values in a group X and with distributions μ_j. The series $\sum_{j=1}^{\infty} \xi_j$ converges with probability 1 if and only if the product $\prod_{j=1}^{\infty} \hat{\mu}_j(y)$ strictly converges for every character $y \in Y$.*

A.3.3. Lemma. *Let a distribution $\mu \in \mathcal{M}^1(X)$ be a convolution of infinitely many discrete distributions*

$$\mu = \mathop{*}_{j=1}^{\infty} \mu_j.$$

Then μ may only be either absolutely continuous or singular or discrete.

Proof. We split the proof into several steps.

(a) Suppose that some characteristic function $f(y)$ is a product of infinitely many characteristic functions $f_j(y)$, $j = 1, 2, \dots$, i.e.,

$$f(y) = \prod_{j=1}^{\infty} f_j(y).$$

Denote by Y_1 the set of characters $y \in Y$ on which the product $\prod_{j=1}^{\infty} f_j(y)$ strictly converges. We shall prove that Y_1 is an open subgroup of Y. Let $y_1, y_2 \in Y_1$. Then there exist numbers n_1 and n_2 such that

$$\prod_{j=n_i}^{\infty} f_j(y_i) \neq 0, \qquad i = 1, 2.$$

Let $n \geq \max\{n_1, n_2\}$. Set $\hat{\mu}(y) = |f_j(y)|^2$ in inequality 2.13(1). We have

$$1 - |f_j(y_1 + y_2)|^2 \leq 2[(1 - |f_j(y_1)|^2) + (1 - |f_j(y_2)|^2)].$$

This implies that

$$\sum_{j=n}^{\infty}(1 - |f_j(y_1 + y_2)|^2) \leq 2\sum_{j=n}^{\infty}[(1 - |f_j(y_1)|^2) + (1 - |f_j(y_2)|^2)].$$

So $y_1 + y_2 \in Y_1$. Since the products $\prod_{j=n}^{\infty} f_j(y)$, $n = 1, 2, 3, \dots$, are continuous, Y_1 is an open set.

(b) Let $\mu_j \in \mathcal{M}^1(X)$, $j = 1, 2, 3, \dots$, be an arbitrary sequence of distributions and suppose that the infinite convolution

$$\mu = \mathop{*}_{j=1}^{\infty} \mu_j$$

exists. Suppose also that ξ_j, $j = 1, 2, 3, \ldots$, are X-valued independent random variables with distributions μ_j. Let us prove that the sequence of the partial sums $\zeta_n = \xi_1 + \cdots + \xi_n$, $n = 1, 2, 3, \ldots$, is compact with the probability 1.

Denote the characteristic functions of the distributions μ and μ_j by $f(y)$ and $f_j(y)$, respectively. According to part (a) the set of characters

$$Y_1 = \left\{ y : \prod_{j=1}^{\infty} f_j(y) \text{ strictly converges} \right\}$$

is an open subgroup of Y. Therefore the factor-group Y/Y_1 is discrete. Set $X_1 = A(X, Y_1)$. By Theorem 1.6, $X_1 \approx (Y/Y_1)^*$. Hence by Theorem 1.7 X_1 is a compact group. According to Theorem 1.6 $Y_1 \approx (X/X_1)^*$. If $\hat{x} = x + X_1 \in X/X_1$ and $y \in Y_1$, then $(\hat{x}, y) = (x, y)$. The relations $\hat{\xi}_n = \xi_n + X_1$, $n = 1, 2, \ldots$, define a sequence of independent random variables with values in the factor-group X/X_1. Observe that $\mathbf{E}(\hat{\xi}_n, y) = \mathbf{E}(\xi_n, y)$, $y \in Y_1$. Hence the product $\prod_{j=1}^{\infty} \mathbf{E}(\hat{\xi}_n, y)$, $y \in Y_1$ of characteristic functions strictly converges. By Lemma A.3.2, the sequence of the random variables $\hat{\zeta}_n = \hat{\xi}_1 + \cdots + \hat{\xi}_n$ converges with probability 1 to some random variable $\hat{\xi} \in X/X_1$.

Let U be some neighborhood of zero in the group X with compact closure. We have

$$\mathbf{P}\left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{\hat{\zeta}_n - \hat{\xi} \in \widehat{U}\} \right) = 1, \qquad \widehat{U} = \{x + X_1, x \in U\}.$$

The latter means that for almost every elementary event $\omega \in \Omega$ there exists a number $N = N(\omega)$ and elements $b_n = b_n(\omega) \in a(\omega) + X_1$, $n \geq N$, such that $\zeta_n(\omega) + b_n(\omega) \in U$ for $n \geq N$. Since X_1 is a compact group and U has compact closure, the sequence $\{\zeta_n\}$ is compact with the probability 1.

(c) Suppose now that the set of values of each random variable ξ_n is at most countable. According to part (b) there exists a sequence $\zeta'_n = \xi_1 + \cdots + \xi_{m_n}$, $m_n < m_{n+1}$, converging with probability 1 to a random variable ξ with distribution μ. Let M be the algebraic group generated by the elements ξ_n, $n = 1, 2, 3, \ldots$. Clearly, the group M is at most countable. Let $E \in \mathscr{B}(X)$ be any fixed set. One can easily verify that the sets $A_k = \{\sum_{j=k}^{\infty}(\zeta'_{j+1} - \zeta'_j)$ converges and $\sum_{j=k}^{\infty}(\zeta'_{j+1} - \zeta'_j) \in E + M\}$, $k = 1, 2, 3, \ldots$, coincide. According to the zero-one criterion, the measure of the set $A_1 = A_2 = \ldots$ is either zero or one. Therefore if $\mu(E) > 0$, then $\mu(E + M) = 1$. This implies the assertion of the lemma. □

A.3.4. THEOREM. *Let $\Phi \in \mathscr{L}(X)$, $V = \Phi(X)$. Then the structure of the distribution $\mu = \Pi(\Phi)$ (see in A.3.1(1)) depends on the properties of the measure Φ in the following way.*

<table>
<tr><th></th><th>$\Phi \in \mathscr{L}_1$</th><th>$\Phi \in \mathscr{L}_{1/n}$, $n \geq 2$</th><th>$\Phi \in \mathscr{L}_0$</th><th>$\Phi \in \mathscr{D}$</th></tr>
<tr><td rowspan="2">$V < \infty$</td><td>$\alpha_1 = 1 - \alpha_3$
$\alpha_2 = 0$</td><td>$\alpha_1 > 0$
$\alpha_2 \geq \exp\{-V\} \sum_{k=1}^{n-1} \frac{V^k}{k!}$</td><td>$\alpha_1 = 0$
$\alpha_2 = 1 - \alpha_3$</td><td rowspan="2">$\alpha_3 = 1$</td></tr>
<tr><td colspan="3">$\alpha_3 = \exp\{-V\}$</td></tr>
<tr><td>$V = \infty$</td><td>$\alpha_1 = 1$</td><td colspan="2">$\alpha_3 = 0$</td><td>$(1 - \alpha_1)$
$\cdot(1 - \alpha_2) = 0$</td></tr>
</table>

PROOF. 1. The case $V < \infty$. Considering the function $g(x, y)$, one can ensure that there exists an element $z_0 \in X$ such that

$$(z_0, y) = \exp\left\{-i \int_{X \setminus \{0\}} g(x, y) d\Phi(x)\right\}.$$

Hence

$$\mu = \Pi(\Phi) = \exp\{-V\} E_{z_0} * \left[E_0 + \Phi + \frac{\Phi^{*2}}{2!} + \cdots + \frac{\Phi^{*n}}{n!} + \cdots\right].$$

Since a convolution of an absolutely continuous distribution with an arbitrary one is absolutely continuous, we obviously have:

if $\Phi \in \mathscr{L}_1$, then

$$\alpha_1 = 1 - \exp\{-V\}, \qquad \alpha_2 = 0, \qquad \alpha_3 = \exp\{-V\};$$

if $\Phi \in \mathscr{L}_{\frac{1}{n}}$, $n \geq 2$, then

$$\alpha_1 > 0, \qquad \alpha_2 \geq \exp\{-V\} \sum_{k=1}^{n-1} \frac{V^k}{k!}, \qquad \alpha_3 = \exp\{-V\};$$

if $\Phi \in \mathscr{L}_0$, then

$$\alpha_1 = 0, \qquad \alpha_2 = 1 - \exp\{-V\}, \qquad \alpha_3 = \exp\{-V\};$$

if $\Phi \in \mathscr{D}$, then

$$\alpha_1 = \alpha_2 = 0, \qquad \alpha_3 = 1.$$

2. The case $V = \infty$;

(a) $\Phi \in \mathscr{L}_1$. Let U be a neighborhood of zero in the group X such that the boundary of U has zero Φ-measure. Let us represent $\mu = \Pi(\Phi)$ as the convolution

$$\Pi(\Phi) = \nu_1 * \nu_2,$$

where the distribution ν_j are defined by their characteristic functions

$$\hat{\nu}_1(y) = \exp\left\{\int_{U \setminus \{0\}} [(x, y) - 1 - ig(x, y)] d\Phi(x)\right\},$$

$$\hat{\nu}_2(y) = (z_0, y) \exp\left\{\int_{X \setminus U} [(x, y) - 1] d\Phi(x)\right\},$$

$$(z_0, y) = \exp\left\{-i\int_{X\setminus U} g(x, y)d\Phi(x)\right\}.$$

Let us prove that if $\mu \in \mathscr{L}_1$, then $\mu = \Pi(\Phi)$ is absolutely continuous. Indeed, denote $\tilde{V} = \Phi(X\setminus U)$, and define the distribution $F_0 \in \mathscr{M}_1(X)$ by $F_0(A) = \tilde{V}^{-1}\Phi(A \cap (X\setminus U))$, $A \in \mathscr{B}(X)$, and verify that $\alpha_2 = \alpha_3 = 0$ in expansion A.3.1(1) of the distribution $\mu = \prod(\Phi)$. We have

$$\nu_2 = \exp\{-\tilde{V}\}E_{z_0} * \left[E_0 + \tilde{V}F_0 + \frac{\tilde{V}^2F_0^{*2}}{2!} + \cdots + \frac{\tilde{V}^nF_0^{*n}}{n!} + \cdots\right]. \quad (1)$$

Consider expansion A.3.1(1) for ν_2:

$$\nu_2 = \beta_1\nu_2^{(1)} + \beta_3\nu_2^{(3)}.$$

It follows from (1) that $\beta = \exp\{-\tilde{V}\}$. From the "smoothness preservation" principle it follows (after measure convolution is performed) that $\alpha_1 \geq \beta_1$. Since $\tilde{V} \to \infty$ as U shrinks to zero, we obtain $\alpha_1 = 1$.

(b) $\Phi \in \mathscr{L}_0$. The only assertion here is that $\alpha_3 = 0$. To prove this, consider expansion A.3.1(1) for ν_2,

$$\nu_2 = \beta_1\nu_2^{(1)} + \beta_2\nu_2^{(2)} + \beta\nu_2^{(3)}$$

where $\beta_j \geq 0$, $\beta_1 + \beta_2 + \beta_3 = 1$. It follows from (1) that $\beta_3 = \exp\{-\tilde{V}\}$. From this, by the "smoothness preserving" principle, under convolution of measures, it follows that $\alpha_3 \leq \exp\{-\tilde{V}\}$ and hence $\alpha_3 = 0$.

(c) $\Phi \in \mathscr{D}$. Let the measure Φ be concentrated on the set $S = \{a_1, a_2, a_3, \ldots\}$. Let us choose a zero neighborhood U whose boundary is free of points of S. Define the measures Φ_1 and Φ_2 by the condition: $\Phi_2(\{a\}) = \Phi(\{a\})$ if $\Phi(\{a\}) \leq 1$, and $\Phi_2(\{a\}) = 1$ if $\Phi(\{a\}) > 1$, $a \in S$, and $\Phi_1 = \Phi - \Phi_2$. In its turn, the measure Φ_2 may be represented as a sum of the two measures $\Phi_2 = \Phi_3 + \Phi_4$ where $\Phi_3(A) = \Phi_2(A \cap U)$, $\Phi_4(A) = \Phi_2(A \cap (X\setminus U))$, $A \in \mathscr{B}(X)$. Obviously,

$$\Pi(\Phi) = \Pi(\Phi_1) * \Pi(\Phi_3) * \Pi(\Phi_4).$$

Denote $v = \Phi_2(X\setminus U)$ and define a distribution $G \in \mathscr{M}^1(X)$ by setting $G(A) = v^{-1}\Phi_4(A)$, $A \in \mathscr{B}(X)$. Let us show that

$$v^nG^{*n}(\{a\}) \leq v^{n-1}, \qquad n = 2, 3, 4, \ldots$$

for every point $a \in X$. Indeed, in the case $n = 2$ we have

$$v^2G^{*2}(\{a\}) = \int_X vG(a-u)d((vG)(x)) \leq v,$$

since, by the construction of G, $vG(a-u) \leq 1$. The rest is by induction. Applying representation (1) to the distribution $\Pi(\Phi_4)$, we obtain the resulting inequality

$$\Pi(\Phi_4)(\{a\}) \leq e^{-v}\left[1 + \sum_{n=1}^{\infty}\frac{v^{n-1}}{n!}\right] \leq e^{-v} + \frac{1}{v}.$$

Since the measure Φ_2 is unbounded in a neighborhood of zero, one may contract the chosen neighborhood U to make $e^{-v} + \frac{1}{v}$ as small as desired. Hence $\alpha_3 = 0$. On the other hand, denoting $G_k(A) = \Phi(A \cap \{a_k\})$, $A \in \mathscr{B}(X)$, $k = 1, 2, 3, \ldots$, we have

$$\Pi(\Phi) = \Pi(G_1) * \Pi(G_2) * \cdots .$$

Each distribution $\Pi(G_k)$ is discrete. According to Lemma A.3.4 the distribution $\Pi(\Phi)$ is either absolutely continuous or singular. □

A.3.5. COROLLARY. (a) *For the distribution* $\Pi(\Phi)$ *to be continuous it is necessary and sufficient that* $V = \infty$.

(b) *For the distribution* $\Pi(\Phi)$ *to be discrete it is necessary and sufficient that the Lévy measure of* Φ *be discrete and* $V < \infty$.

(c) *If the Lévy measure of* Φ *is absolutely continuous and* $V = \infty$, *then* $\Pi(\Phi)$ *is absolutely continuous.*

APPENDIX 4

On Distributions with Mutually Singular Powers

Let X be a nondiscrete locally compact abelian group, satisfying the second axiom of countability. In this Appendix we give a proof of the Lin-Saeki theorem (Theorem 6.15) on the existence of a distribution with mutually singular powers. We used this result in the study of the problem of whether a generalized Poisson distribution belongs to the class I_0 (Theorem 6.13 and Lemma 7.9) and in the proof that the class I_0 is dense in the class of all infinitely divisible distributions on a nondiscrete group (Proposition 8.3). Some related subjects are considered as well.

A.4.1. Let $\mathscr{M}(X)$ denote the convolution algebra of all finite regular Borel complex-valued measures on the group X. We shall use a description of the maximal ideal space $\Delta(\mathscr{M}(X))$ of $\mathscr{M}(X)$ in terms of generalized characters [Sr]. Define a generalized character of $\mathscr{M}(X)$ to be an element $f = (f_\mu)_{\mu\in\mathscr{M}(X)}$ of the product space $\prod_{\mu\in\mathscr{M}(X)} L^\infty(X, \mu)$ satisfying the following conditions:

(i) if ν is absolutely continuous with respect to μ, then $f_\nu(x) = f_\mu(x)$ (ν-a.e.),

(ii) $f_\mu(u+v) = f_\mu(u)f_\mu(v)$ ($\mu\times\mu$-a.e.),

(iii) $\sup_{\mu\in\mathscr{M}(X)} \| f_\mu \|_\infty = 1$.

Any complex homomorphism of $\mathscr{M}(X)$ has the form $\mu \to \int_X f_\mu(x)d\mu(x)$, where (f_μ) is a generalized character of $\mathscr{M}(X)$, and conversely, any generalized character of $\mathscr{M}(X)$ generates a complex homomorphism defined by this relation. The Gelfand topology on the maximal ideal space $\Delta(\mathscr{M}(X))$ corresponds to the topology induced on the set of generalized characters by the product of the $\sigma(L^\infty(X, \mu), L^1(X, \mu))$ topologies ($\mu \in \mathscr{M}(X)$).

A local base $\{U_n\}_{n=1}^\infty$ at zero in X is called admissible if (1) each U_n is a compact neighborhood of zero and (2) $U_{n+1} \subset U_n$ for all n. A sequence $\{(a_n, b_n, c_n)\}_{n=1}^\infty$ of triples of nonnegative real numbers is called admissible if $a_n + b_n + c_n = 1$ for all n.

In what follows, we fix an arbitrary admissible local base $\{U_n\}_{n=1}^\infty$ at zero in X and an arbitrary admissible sequence $\{(a_n, b_n, c_n)\}_{n=1}^\infty$. Let $U_n = \prod_{n=1}^\infty U_n$, and let L denote the set of all limit points of $\{(a_n, b_n, c_n)\}_{n=1}^\infty$ in

$[0, 1]^3$. For each $x = (x_1, x_2, \dots) \in U$, we write

$$\nu_n = a_n E_0 + b_n E_{x_n} + c_n E_{-x_n}.$$

For each $\tilde{x} \in U$ the convolution

$$\nu_{\tilde{x}} = \nu(\tilde{x}) = \mathop{*}_{n=1}^{\infty} \nu_n$$

converges in $\mathscr{M}(X)$, as will be shown in Lemma A.4.2. We define $\overline{Y}_{\tilde{x}}$ to be the weak* closure of Y in $L^\infty(X, \nu_{\tilde{x}})$. The set of all constant functions in $\overline{Y}_{\tilde{x}}$ is denoted by $\overline{S}_{\tilde{x}}$.

The group X is called an I-group if every neighborhood of zero contains an element of infinite order. Let $\mathscr{D}$ be the closed unit disk $\mathscr{D} = \{z \in \mathbb{C} : |z| \le 1\}$.

We shall say that some property holds for quasi-all $\tilde{x} \in U$ if this property holds everywhere on U except, maybe, a set of the first category.

To prove the main result we need the following assertions.

A.4.2. Lemma. *For any given $\tilde{x} \in U$, the convolution product $*_{n=1}^{\infty} \nu_n$ converges to some $\nu_{\tilde{x}} \in \mathscr{M}(X)$. Moreover, the mapping $(\tilde{x}, y) \to \hat{\nu}_{\tilde{x}}(y)$ is a continuous function on $U \times Y$.*

Proof. Let $x \in U$, and $y \in Y$. Given natural numbers r, p, $r > p$, we have

$$\begin{aligned}\left| \left(\mathop{*}_{n=1}^{r} \nu_n \right)^{\wedge} (y) - \left(\mathop{*}_{n=1}^{p} \nu_n \right)^{\wedge} (y) \right| &= \left| \prod_{n=1}^{r} \hat{\nu}_n(y) - \prod_{n=1}^{p} \hat{\nu}_n(y) \right| \le \left| \prod_{n=p+1}^{r} \hat{\nu}_n(y) - 1 \right| \\ &= \left| \int_X ((x, y) - 1) d \left(\mathop{*}_{n=p+1}^{r} \nu_n \right)(x) \right| \le \sup_{x \in U_p - U_p} |(x, y) - 1|\end{aligned}$$

because (2) $U_{n+1} \subset U_n$ for all $n \ge 1$. Since $\{U_n\}_{n=1}^{\infty}$ is a local base at zero in X, it follows that the sequence $(*_{n=1}^{p} \hat{\nu}_n)(y)$ converges uniformly with respect to $(\tilde{x}, y) \in U \times A$ for any compact subset A of Y. Therefore the product $*_{n=1}^{\infty} \nu_n$ converges to some $\nu_{\tilde{x}} \in \mathscr{M}(X)$ for any $\tilde{x} \in U$ (notice that all the measures under consideration are concentrated on the compact set $2U_1 - 2U_1$). The second assertion of our lemma is obvious by the above arguments. □

Let us choose and fix an arbitrary countable dense subset $\{\Psi_k\}_{k=1}^{\infty}$ of Y.

A.4.3. Lemma. *Let $\alpha \in \mathscr{D}$ and $\nu \in \mathscr{M}(X)$. Suppose that for each $N \ge 1$ there is an $y_N \in Y$ such that $|\alpha\hat{\nu}(\Psi_k) - \hat{\nu}(y_N + \Psi_k)| < \frac{1}{N}$ for all $1 \le k \le N$. Then the constant α belongs to the weak* closure of Y in $L^\infty(X, \nu)$.*

Proof. Let $\{y_N\}_{N=1}^{\infty}$ be as above. Then we have

$$\lim_{N \to \infty} \int_X (x, y_N)(x, \Psi_k) d\nu(x) = \int_X \alpha(x, \Psi_k) d\nu(x). \tag{1}$$

We get

$$\lim_{N\to\infty}\int_X (x\,,\,y_N)(x\,,\,y)d\nu(x) = \int_X \alpha(x\,,\,y_N)d\nu(x)$$

for all $y \in Y$ by (1), since the set $\{\Psi_k\}_{k=1}^{\infty}$ is dense in Y. Hence we have

$$\lim_{N\to\infty}\int_X (x\,,\,y_N)l(x)d\nu(x) = \int_X \alpha(x)d\nu(x)$$

for any $l(x) \in L^1(X\,,\,\nu)$, since the linear subspace of functions on X generated by $\{(x\,,\,y) : y \in Y\}$ is dense in $L^1(X\,,\,\nu)$ (cf. [HeRa2, §31]). In other words, the sequence $\{(x\,,\,y_N)\}_{N=1}$ converges to α in the weak* topology of $L^\infty(X\,,\,\nu)$. □

A.4.4. Lemma. *Let* $(a\,,\,b\,,\,c) \in L$, $|z| = 1$ *and* $\alpha = a+bz+c\bar{z}$ *be given. Then the set*

$$E(\alpha\,,\,N) = \bigcap_{y\in Y}\bigcup_{k=1}^{N}\left\{\tilde{x} \in U : |\alpha\hat{\nu}_{\tilde{x}}(\Psi_k) - \hat{\nu}_{\tilde{x}}(y + \Psi_k)| \geq \frac{1}{N}\right\}, \qquad N = 1\,,\,2\,,\,\ldots$$

is closed in U *and* $E(\alpha\,,\,N)$ *has no interior point.*

Proof. By Lemma A.4.2, $\hat{\nu}_{\tilde{x}}(y)$ is a continuous function of $\tilde{x} \in U$ for each $y \in Y$. Therefore $E(\alpha\,,\,N)$ is closed in U.

Now suppose that X is an I-group. To force a contradiction, assume that $E(\alpha\,,\,N)$ has nonempty interior. Then there exist finitely many nonempty sets $V_n \subset U_n$, $1 \leq n \leq M-1$, such that

$$V_1 \times V_2 \times \cdots \times V_{M-1} \times \prod_{n=M}^{\infty} U_n \subset E(\alpha\,,\,N).$$

We may assume that M satisfies the following conditions

$$\max\{|a - a_M|\,,\,|b - b_M|\,,\ |c - c_M|\} < \frac{1}{8N}. \tag{1}$$

$$\sup_{x\in U_M} |(x\,,\,\Psi_k) - 1| < \frac{1}{8N}\,, \qquad 1 \leq k \leq N. \tag{2}$$

Choose any points $x_n \in V_n$, $1 \leq n \leq M-1$. Since X is an I-group, we can find $x_M \in U_M$ and $y \in Y$ such that

$$|(x_n\,,\,y) - 1| < \frac{1}{8MN}\,, \qquad 1 \leq n \leq M-1\,, \tag{3}$$

$$|(x_M\,,y) - \bar{z}| < \frac{1}{8N}. \tag{4}$$

Setting $\tilde{x} = (x_1\,,\,x_2\,,\,\ldots\,,\,x_M\,,\,0\,,\,0\,,\,\ldots) \in E(\alpha\,,\,N)$ and $\nu_{\tilde{x}} = *_{n=1}^{M}\ \nu_n$, we have

$$\begin{aligned}|\alpha\hat{\nu}_M(\Psi_k) - \hat{\nu}_M(y+\Psi_k)| &\le |\alpha\{\hat{\nu}_M(\Psi_k) - 1\}| \\ &\quad + |a + bz + c\bar{z} - a_M - b_M(x_n, y) - c_M\overline{(x_M, y)}| \\ &\quad + |\hat{\nu}_M(y) - \hat{\nu}_M(y+\Psi_k)| \\ &< \frac{1}{8N} + \frac{5}{8N} + \frac{1}{8N} = \frac{7}{8N}, \qquad 1 \le k \le N\end{aligned}$$

by (2), (1) and (4), since $\alpha = a + bz + c\bar{z}$ and $\hat{\nu}_M(y) = a_M + b_M(x_M, y) + c_M\overline{(x_M, y)}$. It follows from (3) that,

$$\begin{aligned}|\alpha\hat{\nu}_{\tilde{x}}(\Psi_k) - \hat{\nu}_{\tilde{x}}(y+\Psi_k)| &\le \left|\alpha\prod_{n=1}^{M}\hat{\nu}(\Psi_n) - \prod_{n=1}^{M}\nu_n(y+\Psi_k)\right| \\ \le \sum_{n=1}^{M-1}|\hat{\nu}_n(\Psi_k) - \hat{\nu}_n(y+\Psi_k)| &\le |\alpha\{\hat{\nu}_M(\Psi_k) - \hat{\nu}_M(y+\Psi_k)| \\ < \frac{M-1}{8MN} + \frac{7}{8N} < \frac{1}{N}\end{aligned}$$

$1 \le k \le N$. Hence $x \notin E(\alpha, N)$, which contradicts our choice of x. □

A.4.5. Proposition. *If X is an I-group, then, for quasi-all $\tilde{x} \in U$, $S_{\tilde{x}}$ contains the multiplicative compact semigroup in $\mathscr{D}$ generated by the set $\{a + bz + c\bar{z} : (a, b, c) \in L$ and $|z| = 1\}$. If X is not an I-group, this conclusion fails for some admissible local base $\{U_n\}_{n=1}^{\infty}$ at zero in X and some admissible sequence $\{(a_n, b_n, c_n)\}_{n=1}^{\infty}$.*

Proof. Suppose X is an I-group and take any countable dense subset A of the set

$$\{a + bz + c\bar{z} : (a, b, c) \in L \text{ and } |z| = 1\}. \tag{1}$$

If $\tilde{x} \in U$ does not belong to $\bigcup\{E(\alpha, N) : \alpha \in A$ and $N \ge 1\}$, then we have $A \subset S_{\tilde{x}}$ by Lemma A.4.3. On the other hand, it is easy to see that $S_{\tilde{x}}$ is a compact semigroup in $\mathscr{D}$ for every $\tilde{x} \in U$. Therefore, for each $\tilde{x}$ as above, $S_{\tilde{x}}$ contains the compact semigroup generated by set (1). Thus the first assertion of Proposition A.4.5 follows from Lemma A.4.4.

Now assume that X is not an I-group. Then X contains an open subgroup of the form $\mathbb{R}^n + K$, where $n \ge 0$ and K is a compact group. Since X is not an I-group, then $n = 0$ and the group K is periodic [HeRa1, §25]. So K is a compact open periodic subgroup of X. Let n_0 be a positive integer such that $n_0 x = 0$ for all $x \in K$. Set $a_n = 0$ and $b_n = c_n = \frac{1}{2}$ for all $n \ge 1$; then $L = \{(0, \frac{1}{2}, \frac{1}{2})\}$, and the multiplicative compact semigroup in $\mathscr{D}$ generated by the set $\{a + bz + c\bar{z} : (a, b, c) \in L$ and $|z| = 1\}$ is the segment $[-1, 1]$. If $y \in Y$ and $x \in K$, then we have either $(x, y) = 1$ or $|\operatorname{Re}(x, y)| \le |\cos(\frac{2\pi}{n_0})|$, since $(x, y)^{n_0} = (n_0 x, y) = 1$. Let $\{U_n\}_{n=1}^{\infty}$ be any admissible local base at zero in K. Then, for every $\tilde{x} \in U$ and $y \in Y$, we have $\hat{\nu}_{\tilde{x}}(u) = \prod_{n=1}^{\infty}\operatorname{Re}(x_n, y)$. Therefore either $|\hat{\nu}_{\tilde{x}}(y)| = 1$ (if $n_0 = 2$), or $|\hat{\nu}_{\tilde{x}}(y)| \le |\cos\frac{2\pi}{n_0}|$ (if $n_0 \ge 2$). Hence every $S_{\tilde{x}}$ is disjoint from the open interval $(-1, 1)$ if $n_0 = 2$ and from $(|\cos\frac{2\pi}{n_0}|, 1)$ if $n_0 \geq 3$. This proves Proposition A.4.5. □

A.4.6. PROPOSITION. *Suppose that* X *is not an* I*-group, and define* $q = q(X)$ *as the largest natural number such that every neighborhood of zero in* X *contains an element of order* q. *Then, for quasi-all* $\tilde{x} \in U$, $S_{\tilde{x}}$ *contains the compact semigroup in* $\mathscr{D}$ *generated by all complex numbers of the form* $a+bz+c\bar{z}$, *where* $(a, b, c) \in L$ *and* $z^q = 1$.

The proof of Proposition A.4.6 is almost the same as the proof of the first assertion of Proposition A.4.5, and we omit the details. □

A.4.7. THEOREM. *Let* X *be an arbitrary group. Suppose that the sequence* $\{(a_n, b_n, c_n)\}_{n=1}^{\infty}$ *has a limit point* (a, b, c) *such that* $\max\{a, b, c\} < 1$. *Then*:

(i) *If every neighborhood of zero in* X *contains either an element of infinite order or an element of order* ≥ 4, *then quasi-all* $\tilde{x} \in U$ *have the property that, for any* $t \in X$, $\nu_{\tilde{x}}^{*m} * E_t$ *is singular with respect to* $\nu_{\tilde{x}}^{*n}$ *for* $n \neq m$.

PROOF. Suppose that the hypothesis of Theorem A.4.7 holds. Then, for quasi-all $\tilde{x} \in U$, $S_{\tilde{x}}$ contains a complex number α with $0 < |\alpha| < 1$. By Propositions A.4.5 and A.4.6 there exists $f \in \Delta[\mathscr{M}(X)]$ such that $f_{\nu_{\tilde{x}}}(u) = \alpha(\nu_{\tilde{x}})$-a.e.

Suppose that there exist different positive integers m and n, an element $t \in X$ and a distribution $\mu \in \mathscr{M}^1(X)$ absolutely continuous with respect to both $\nu_{\tilde{x}}^{*m} * E_t$ and $\nu_{\tilde{x}}^{*n}$. We have

$$f_{\nu_{\tilde{x}}^{*m}} * E_t(t) = f_{\mu}(u) = f_{\nu_{\tilde{x}}^{*n}}(u) \qquad (\mu\text{-a.e.}) \tag{1}$$

by A.4.1(i). On the other hand

$$f_{\nu_{\tilde{x}}^{*m} * E_t}(u) = \alpha^m f(E_t) \qquad (\nu_{\tilde{x}}^{*m} * E_t\text{-a.e.}) \tag{2}$$

and

$$f_{\nu_{\tilde{x}}^{*n}}(u) = \alpha^n \qquad (\nu_{\tilde{x}}^{*n}\text{-a.e.}) \tag{3}$$

Since $|f(E_t)| = 1$ for all $t \in X$ and since $|\alpha^m| \neq |\alpha^n|$ unless $m = n$, relations (1)–(3) prove Theorem A.4.7. □

A.4.8. Assume now that $X = \mathbb{R}$. We give some results about Bernoulli convolutions with mutually singular powers. Let A be the class of symmetric Bernoulli convolutions, i.e., distributions $\mu \in \mathscr{M}^1(\mathbb{R})$ of the form

$$\mu = \mathop{\ast}_{n=1}^{\infty} \tfrac{1}{2}[E_{-x_n} + E_{x_n}], \qquad x_n \in \mathbb{R}, \tag{1}$$

where $\sum_{n=1}^{\infty} x_n^2 < \infty$, and let B be the class of antisymmetric Bernoulli convolutions

$$\mu = \mathop{\ast}_{n=1}^{\infty} \tfrac{1}{2}[E_0 + E_{x_n}], \qquad x_n \in \mathbb{R},\ x_n > 0, \tag{2}$$

where $\sum_{n=1}^{\infty} x_n < \infty$. Of course there is a close connection between these two classes. In fact, if $\sum_{n=1}^{\infty} x_n = x$, then (1) is the shift of $*_{n=1}^{\infty} \frac{1}{2}[E_0 + E_{2x_n}]$ by x. It is easy to verify that the distributions in A and B are continuous unless $x_n = 0$ for all sufficiently large n. We shall consider only continuous Bernoulli convolutions, so that A and B consist solely of continuous distributions.

A.4.9. Theorem. *Let μ belong to A or to B. Then either $\mu^{*n} \in L^1(\mathbb{R})$ for some integer n or $\mu^{*m} * E_x$ is singular with respect to μ^{*n} for all $x \in \mathbb{R}$, unless $m = n$.*

In order to make this result workable, we require necessary conditions for $L^1(\mathbb{R})$ to contain a power of μ. One such necessary condition is that the support $\sigma(\mu)$ of μ contains a basis for $\mathbb{R}$. This follows from the equality

$$\sigma(\mu^{*n}) = (n)\sigma(\mu)$$

and from the fact that every set of positive Lebesgue measure contains basis for $\mathbb{R}$. Šreider [Sr] gave necessary conditions for certain perfect subsets of $\mathbb{R}$ to contain a basis. In our case when μ belongs either to A or to B, support $\sigma(\mu)$ does not contain a basis if $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 0$. Using this fact, we obtain the following result.

A.4.10. Corollary. *Let μ be as in* A.4.8(1) *or* A.4.8(2), *and suppose that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 0$. Then, for any $x \in \mathbb{R}$, $\mu^{*m} * E_x$ is singular with respect to μ^{*n}, unless $n = m$.*

Another necessary condition for some power of μ to belong to $L^1(\mathbb{R})$ is that the Fourier-Stieltjes transform of μ vanishes at infinity. This gives us a result for distributions of the form (A.4.8.1), where $x_n = \alpha^{-n}$ and α is a Pisot number (i.e., α is an algebraic integer all of whose conjugates lie inside the unit circle) and $\alpha \neq 2$. Salem has shown that the Fourier-Stieltjes transforms of such distributions do not vanish at infinity [Sal, Chapter 4]. From these remarks we obtain the following result.

A.4.11. Corollary. *Let μ be the measure*

$$\mu = \mathop{*}_{n=1}^{\infty} [E_{-\alpha^{-n}} + E_{\alpha^{-n}}],$$

where α is a Pisot number and $\alpha \neq 1, 2$. Then, for any $x \in \mathbb{R}$, $\mu^{*m} * E_x$ is singular with respect to μ^{*n}, unless $n = m$.

As a special case of this result we see that the Lebesgue- Stieltjes distribution of the Cantor set

$$\mu_0 = \mathop{*}_{n=1}^{\infty} \tfrac{1}{2}[E_0 + E_{(\frac{2}{3})^n}]$$

must have the property that every shift of μ_0^{*m} is singular to μ_0^{*n} unless $m = n$. This follows since μ_0 is the shift by $\frac{1}{2}$ of

$$\mathop{\ast}_{n=1}^{\infty} \tfrac{1}{2}[E_{-(\frac{1}{3})^n} + E_{(\frac{1}{3})^n}].$$

Unsolved problems

1. Prove or disprove that any Gaussian distribution on the group $\mathbb{T}^\infty$ is either absolutely continuous or singular (with respect to $m_{\mathbb{T}^\infty}$). (See §§5.13–5.15.)

2. Prove or disprove that any two Gaussian distributions on the group X are either mutually absolutely continuous or mutually singular. (V. V. Sazonov and V. N. Tutubalin [ST]. (See Propositions 5.16 and 5.19.)

3. Prove or disprove that any symmetric Gaussian distribution of the class I_0 on a connected infinite-dimension group X is a continuous monomorphic image of a Gaussian distribution, concentrated on some linear subspace of $\mathbb{R}^\infty$. (See Proposition 5.29.)

4. Let $\gamma \in \Gamma^S(X)$ and $\sigma(\gamma)$ be a connected infinite dimension subgroup of X and let π be a Poisson distribution on X. If $\gamma, \pi \in I_0$, then $\gamma * \pi \in I_0$. (See Proposition 7.5.)

5. Let the group X be such that $\dim C_X = \infty$ and let γ be an arbitrary symmetric Gaussian distribution of the class I_0 on X. Prove or disprove that there exists a continuous measure $\phi \in M_+(X)$ such that $\gamma * e(\phi) \in I_0$. (See Theorem 7.13.)

6. Let $X = \mathbb{R} + D$, where D is a discrete group, $\gamma \in \Gamma(X)$, $\phi \in M_+(X)$ and $\gamma * e(\phi) \in I_0$. Must the measure ϕ be discrete? (See the Linnik theorem 7.6.)

7. For a given group X describe these sets of integers $\{a_j\}_1^m$ and $\{b_j\}_1^m$ admissible for X that have the following property: if ξ_j are X-valued independent random variables with distributions μ_j such that the linear forms $L_1 = a_1\xi_1 + \cdots + a_m\xi_m$ and $L_2 = b_1\xi_1 + \cdots + b_m\xi_m$ are independent, then $\mu_j \in I(X) * \Gamma(X)$, $j = 1, \dots, m$. (See Theorem 10.5, 10.9.)

8. For a given group X describe those sets of integers $\{a_j\}_1^m$ and $\{b_j\}_1^m$ admissible for X that possess the following property: if ξ_j are X-valued independent random variables with distributions μ_j, satisfying

$$\prod_{j=1}^{m} \hat{\mu}_j(y) \neq 0$$

for any $y \in Y$, and if the linear forms $L_1 = a_1\xi_1 + \cdots + a_s\xi_s$ and $L_2 = b_1\xi_1 + \cdots + b_s\xi_s$ are independent, then $\mu_j \in \Gamma(X)$. (See Theorem 10.13 and 10.15–10.17.)

9. Describe those groups X for which the relation

$$\{\mu \in \Gamma_A(X) : \hat{\mu}(y) \neq 0 \text{ for any } y \in Y\} \subset \Gamma(X) \tag{1}$$

holds for any $A \in \mathscr{A}(X)$.

10. For a given group X describe those sets $A \in \mathscr{A}(X)$ for which equality 11.4(3) holds. (See Theorem 11.9.)

11. For a given group X describe those sets $A \in \mathscr{A}(X)$ for which relation (1), Problem 9, holds.

12. Extend the main results of §11 to the case of two linear forms having the same distributions instead of a monomial and a linear form with the same distributions.

13. Develop a technique for obtaining estimates of distribution decomposition stability for groups. (See Theorem A.2.3.) In particular, let γ be a Gaussian distribution of the class I_0. Estimate the value $\beta_{d_1 d_2}(\gamma, \varepsilon)$ for pairs of metrics d_1, d_2 on $\mathscr{M}^1(X)$.

14. Investigate problems of stability for characterization problems on groups. (See Chapter III.)

15. Investigate the structure of an infinitely divisible distribution μ, whose characteristic function has representation (x_0, ϕ, φ) (see §2.22), subject to the properties of the Lévy measure ϕ and the function φ. In particular, determine the structure of the distribution $\Pi(\phi)$ in the case $V = \infty$, $\phi \in \mathscr{L}_{1/n}$, $n \geq 2$, $\phi \in \mathscr{L}_0$ (see Appendix 3) (V. M. Zolotarev, V. M. Kruglov [ZK]).

Comments

Section 4

Theorem 4.1 and the group analogs of the Khinchin theorems (Theorems 4.3 and 4.4) were proved by Parthasarathy, Rao, and Varadhan in [PRV1]. Our exposition follows this article.

There are many sufficient conditions for a distribution on the groups $X = \mathbb{R}$ and $X = \mathbb{R}^n$ to be indecomposable [LinO, Chapter III, §3]. For a survey of results obtained up to 1974 see [LivOCh].

The main results on indecomposable distributions on topological groups satisfying the second axiom of countability were obtained by Parthasarathy, Rao, and Varadhan in [PRV2]. We present here the most important ones:

THEOREM C1. *Let X be an infinite complete group satisfying the second axiom of countability, perhaps nonabelian. Then the set of all indecomposable distributions is a dense G_δ set in $\mathscr{M}^1(X)$.*

Under the additional assumption that X is nondiscrete, the set of all nondiscrete indecomposable distributions taking positive values on open sets is also a dense G_δ set in $\mathscr{M}^1(X)$.

THEOREM C2. *Let X be a locally compact noncompact abelian group satisfying the second axiom of countability. Then the set of all absolutely continuous (with respect to m_X) indecomposable distributions is a dense G_δ set in the set of all absolutely continuous distributions on X endowed with the $L^1(X, m_X)$ topology.*

In [UU1], [UU2] V. G. Ushakov and N. G. Ushakov proved that for a wide class of groups X the set of all indecomposable distributions is dense in $\mathscr{M}^1(X)$ with respect to the variation distance, i.e., with respect to the norm $\|\cdot\|$ (see 2.26).

THEOREM C3 ([UU1], [UU2]). *Let X be an uncountable metric group, satisfying at least one of the following conditions:*

(I) *X is noncompact and any distribution is dense on X.*
(II) *X is nonbounded.*
(III) *X is separable.*

Then the set of all indecomposable distributions on X is dense in $\mathscr{M}^1(X)$ with respect to the variation distance.

The article [UU2] also contains conditions that ensure a mixture of discrete and continuous distributions to be indecomposable. Let X be an arbitrary metric group. Following [UU2] denote by $W(X)$ the class of all functions φ on $\mathscr{M}^1(X)$ that satisfy the following conditions:

1. Every function $\varphi \in W(X)$ takes only two values 0 and 1.
2. If $\varphi(\mu) = \varphi(\nu) = 0$, then $\varphi(\mu * \nu) = 0$.
3. If $\varphi(\mu) = 1$, then $\varphi(\mu * \nu) = \varphi(\nu * \mu) = 1$ for any $\nu \in \mathscr{M}^1(X)$.
4. If $\varphi(\mu) = 1$, then $\varphi(\alpha\mu + (1-\alpha)\nu) = 1$ for any $\nu \in \mathscr{M}^1(X)$, $0 < \alpha < 1$.

THEOREM C4 [UU2]. *Let ν_1 be a discrete and ν_2 a continuous distribution such that for some $\varphi \in W(X)$ $\varphi(\nu_1) = 1$ and $\varphi(\nu_2) = 0$. For the mixture*

$$\mu = \alpha\nu_1 + (1-\alpha)\nu_2, \qquad 0 < \alpha < 1,$$

to be indecomposable it is necessary and sufficient that ν_1 and ν_2 do not have common right or left nondegenerate divisors.

Among examples of application of Theorem C4 that are considered in [UU2], we note that for the group $X = \mathbb{R}$ the mixture of a Gaussian distribution with any discrete infinitely divisible distribution is indecomposable.

Two theorems by L. S. Kudina related to indecomposable distributions should also be noted. Their group analogs are still unknown.

1. For any closed subset $A \subset \mathbb{R}^n$ that contains at least two points there exists an indecomposable distribution μ such that $\sigma(\mu) = A$ [Kud1].
2. The weak closure of the set of all indecomposable distributions supported on A can differ from the set of all distributions μ such that $\sigma(\mu) = A$ [Kud2].

Proposition 4.6 was proved by G. M. Fel′dman. Decomposition 4.16(ii) was mentioned by Heyer in [He1]. Theorem 4.17 was proved by G. M. Fel′dman in [F2].

Section 5

Parthasarathy, Rao, and Varadhan defined a Gaussian distribution as a distribution satisfying conditions 5.30(i), (ii) and then proved that this class coincides with the class defined in 5.1. This equivalence is an important step in their proof of formula 2.22(i).

It was noted in [PRV1] that Gaussian distributions appear naturally from the point of view of limit theorems. Let $\{\Phi_n\}$ be a sequence of finite measures on a group X satisfying the following conditions:

1. $\lim_{n\to\infty} \Phi_n(X \backslash U) = 0$ for any neighborhood U of zero in X.
2. The distributions $e(\Phi_n)$ converge to a limit after being appropriately shifted.

If the sequence $\{\Phi_n\}$ is unbounded and the limit of $e(\Phi_n)$ is nondegenerate, then this limit is a nontrivial Gaussian distribution.

Consider the class of distributions μ that may be included into a one-parameter distribution semigroup (μ_t), $t \geq 0$, $\mu_0 = E_0$, i.e., there exists such a semigroup for which $\mu_1 = \mu$. Forst [Fo] (see also [BeF]) proved that Gaussian distributions of this class admit the following description.

THEOREM C5. *Suppose a distribution* $\mu \in \mathscr{M}^1(X)$ *may be included in a one-parameter distribution semigroup* (μ_t), $t \geq 0$, $\mu_0 = E_0$. *Then the following statements are equivalent*:

(i) $\mu \in \Gamma(X)$;

(ii) $\lim_{t \to 0}[\mu_t(X \backslash U)/0] = 0$ *for any neighborhood of zero in* X.

Remark 5.2 and Propositions 5.4 and 5.5 are due to Parthasarathy, Rao, and Varadhan [PRV1]. We follow [He2] in the proof of Proposition 5.4 and [ST] in that of Proposition 5.5. Remark 5.3 is due to G. M. Fel′dman.

Propositions 5.6 and 5.9 were proved by G. M. Fel′dman in [F3]. Concerning the linear spaces $\mathbb{R}^\infty$ and $\mathbb{R}_0^\infty$ see [RR]. It should be noted that the group $\mathbb{R}_0^\infty$ is not locally compact and does not satisfy the second axiom of countability. Therefore $\mathbb{R}_0^\infty$ is nonmetrizable. Nevertheless in [Se] it was proved that if s is a limit point of a set $B \subset \mathbb{R}_0^\infty$, then there exists a sequence of elements $\{s^{(k)}\} \subset B$ such that $s^{(k)} \to s$.

Remarks 5.11 and 5.12 belong to Parthasarathy, Rao, and Varadhan and were proved in [PRV1] by a different method.

Proposition 5.14 was proved in [F3]. Earlier Siebert had proved in [Si] that absolutely continuous Gaussian distribution exist only on connected locally connected groups.

Let μ be an infinitely divisible distribution without nondegenerate divisors, and let its characteristic function have representation $(0, \Phi, \varphi)$. V. M. Zolotarev and V. M. Kruglov [ZK] posed the problem of studying the structure of μ subject to properties of Φ and φ and solved this problem in the case $\varphi \equiv 0$ (see Appendix 3). Proposition 5.14 may be viewed as a partial solution of this problem for the case $\Phi = 0$.

Gaussian distributions on a finite-dimensional torus $\mathbb{T}^n$ were studied by Siebert [Si] (see also [He2, §5.5]). The detailed investigation of Gaussian distribution μ on an infinite-dimensional torus $\mathbb{T}^\infty$ corresponding to a diagonal matrix A in Proposition 5.9 was carried out by Berg [Be].

The comments to Chapter 5 in [He2] contain a complete bibliography related to those aspects of Gaussian distributions that are not considered in the present book.

Propositions 5.16 and 5.19 were proved in [F3]. They give a partial answer to the question (posed by V. V. Sazonov and V. V. Tutubalin in [ST]): are two Gaussian distributions on the group X either mutually singular or mutually absolutely continuous? Lemma 5.17 was proved by G. M. Fel′dman in [F1], [F3].

In 1936, answering a question of Lévy, Cramér proved that any divisor of a Gaussian distribution on $\mathbb{R}^n$ is Gaussian. This was the first result on distribution arithmetic. Somewhat later Marcinkiewicz noted [Mark] that any Gaussian distribution on $\mathbb{T}$ has non-Gaussian divisors (Theorem 5.20). This theorem together with the Lévy theorem on decomposition of a Poisson distribution on the group $\mathbb{T}$ (see the comment on §6 below) was the first result concerning distribution arithmetic for groups differing from $\mathbb{R}^n$. The Marcinkiewicz theorem was reproved by Martin-Löf (see [Gr, Remark 4.5.1]) and Carnal [C].

Theorems 5.22 and 5.23 were proved by G. M. Fel′dman in [F1]. It should be noted that the fact that a Gaussian distribution on the group $X = \mathbb{R}+\mathbb{Z}(2)$ belongs to the class I_0 was proved by V. M. Zolotarev [Z1]. Propositions 5.28 and 5.29 were proved by G. M. Fel′dman and A. E. Fryntov [FFr2].

It was Urbanik [Ur] who studied distributions, satisfying conditions 5.31(i), (ii). To be more precise, instead of condition (i) he considered the condition:

(i′) μ can be included into a continuous one-parameter distribution semigroup (μ_t), $t \geq 0$, $\mu_0 = E_0$.

It was mentioned in Proposition 5.31, that the class of distributions, satisfying conditions (i), (ii), is the same as $\Gamma(X)$. At the same time an example was constructed in [Ur] of a non-Gaussian distribution on the group $X = \mathbb{T}^2$ satisfying condition 5.31(i). This work stimulated investigation of distributions Gaussian in the Urbanik sense (see Definition 5.32), that proved useful in the study of characterization problems on groups (see Chapter III). The results of §§5.32–5.37 are due to G. M. Fel′dman.

Section 6

The problem of whether the generalized Poisson distribution $\mu = e(\Phi)$ belongs to the class I_0 is important for distribution arithmetic on the groups $X = \mathbb{R}$ and $X = \mathbb{R}^n$. The first result in this direction was obtained by D. A. Raĭkov (1937) who got the affirmative answer for $X = \mathbb{R}$. For further results on this subject for the groups $X = \mathbb{R}$ and $X = \mathbb{R}^n$ see the monograph [LinO] and also the surveys [O4], [O2] by I. V. Ostrovskiĭ.

The problem of decomposition of a Poisson distribution on a group different from $\mathbb{R}^n$ was first considered by Lévy [Lévy2], who proved that the Poisson distribution $\mu = e(\psi(E_x))$, $\psi > 0$ on the group $X = \mathbb{T}$ belongs to the class I_0 if x is either of infinite order or of order two. Proposition 6.6 that generalizes this result was proved by A. L. Rukhin [Ruh2], [Ruh3]. Theorems 6.5 and 6.6 were proved by G. M. Fel′dman [F2]. It should also be noted that using Proposition 4.18 one may reduce the Rukhin theorem in the case of an element x of infinite order to Raĭkov's theorem. It suffices to consider the monomorphism $p: \mathbb{Z} \to X$ defined by the relation $p(nx) = nx$, $n \in \mathbb{Z}$.

The statement that the generalized Poisson distribution $\mu = e(\Phi)$, where $\Phi = \psi_1 E_{x_1} + \psi_2 E_{x_2}$, $x_i \in \mathbb{R}$, $\psi_i > 0$, $i = 1, 2$, belongs to the class I_0 on the group $\mathbb{R}$ was obtained by D. A. Raĭkov [Ra] in the case when x_1 and x_2 belong to the same semiaxis and are independent, and by Lévy [Lévy2] in the case when x_1 and x_2 belong to the same semiaxis and are dependent. Therefore, Theorem 6.7 in the case of dependent elements x_1, x_2 is a generalization of the Lévy theorem.

If the elements x_1 and x_2 belong to different semiaxes, then the inclusion $e(\Phi) \in I_0$ is a consequence of the Linnik theorem [LinO, Chapter I, §1]. Theorem 6.10 was proved by G. M. Fel′dman and A. E. Fryntov in [FFr1].

Theorem 6.13 was proved by G. M. Fel′dman in [F4], [F6]. The proof in the book belongs to G. P. Chistyakov [Ch2]. This theorem is new even for the case $X = \mathbb{R}$. It implies, for example, that if $\Phi_0 = *_{n=1}^{\infty} \frac{1}{2}(E_0 + E_{(2/3)^n})$ is the Lebesgue-Stieltjes distribution on the standard Cantor set, then $e(\Phi_0) \in I_0$ (the pairwise singularity of powers of the distribution Φ_0 is a consequence of a result of A. M. Vershik [V]; see also [BrM1]) and contains the Ostrovskiĭ-Cuppens theorem [LinO, Chapter VI, §4] stating that $e(\Phi) \in I_0$ if $\Phi \in \mathcal{M}_+(\mathbb{R}^n)$ and that the measure Φ is concentrated on an independent set of points. The Ostrovskiĭ-Cuppens theorem in its turn completed a series of results that involved some additional assumptions (see [LinO, Comment to Chapter VI]). For more details on distribution with mutually singular powers on a group X, see Appendix 4.

Theorem 6.18, Lemma 6.21, and Proposition 6.23 were proved in [F4]. Theorem 6.18 is the group analog of Ostrovskiĭ's theorem that was proved in connection with a problem posed by Yu. V. Linnik on the existence of distributions of the class I_0 on the group $X = \mathbb{R}$ with nondiscrete Lévy measure.

Section 7

The fact that the convolution of a Gaussian and a Poisson distribution on the group $X = \mathbb{R}$ belongs to the class I_0 was proved by Yu. V. Linnik [LinO, Chapter VI, §1]. The corresponding result for the group $X = \mathbb{R}^n$ was obtained by I. V. Ostrovskiĭ and Cuppens [LinO, Chapter VI, §3]. Theorem 7.2 was proved by G. M. Fel′dman and A. E. Fryntov in [FFr1].

Theorem 7.13 as well as Lemmas 7.7–7.12 were proved by G. M. Fel′dman in [F7]. It should be noted that Lemma 7.7 follows from a result by I. V. Ostrovskiĭ [O3] on the Cartesian product of one-dimensional distributions of the class I_0. The construction in Lemma 7.12 is a generalization of the construction used in [LinO, Chapter VI, §6] to prove that, for the case $X = \mathbb{R}^n$, both images and preimages of Borel sets in the isomorphism H of semigroups $M^+(A)$ and $M^+(A')$ are Borel sets.

Section 8

The fact that the class I_0 is dense in the class of all infinitely divisible distributions on the group $X = \mathbb{R}$ was proved by I. V. Ostrovskiĭ and on the group $X = \mathbb{R}^n$ by L. Z. Livshits and I. V. Ostrovskiĭ [LinO, Chapter VI, §4].

Lemma 8.2 is a consequence of Rudin's theorem [Rud] on the existence of a perfect independent set on a group any neighborhood of zero in which contains an element of infinite order. Propositions 8.3 and 8.6 were proved by G. M. Fel′dman in [F2], [F6]. Lemma 8.4 is due to Parthasarathy, Rao, and Varadhan [PRV1] and Lemma 8.5 to Dugué [LinO], [Fr].

The fact that any infinitely divisible distribution on the group $X = \mathbb{R}^n$ may be represented as a finite or infinite convolution of distributions of the class I_0 was proved by I. V. Ostrovskiĭ [O1]. Theorem 8.8 was proved by G. M. Fel′dman [F6].

Section 9

The problem of extending the Bernstein characterization of Gaussian distribution to groups was considered by A. L. Rukhin [Ruh1], [Ruh3] and independently by Heyer and Rall [HR] (see also [He2, Chapter 5, §3]). Proposition 9.5 was proved by these authors. Equality (9.5.2) was obtained by A. L. Rukhin [Ruh1], [Ruh3] under the assumption that Y is a Corwin group and by Heyer and Rall under the assumption that X is a Corwin group and condition 9.11(i) holds.

Theorem 9.10 and Lemmas 9.6–9.9 are due to G. M. Fel′dman [F8] as well as the results of 9.13–9.16. Some sufficient conditions for a group X to satisfy 9.13(i), 9.15(i) and 9.16(i) are mentioned in [HR].

Theorems 9.19 and 9.21 and Proposition 9.18 were proved by G. M. Fel′dman [F8]. It should also be noted that A. L. Rukhin proved [Ruh1], [Ruh3] that the assertion of Theorem 9.19 is true under the assumption that X and Y are Corwin groups.

Let $\tau\colon X^2 \to X^2$ be the mapping defined by the equality $\tau(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$. Corwin [Co1] considered finite complex-valued measures on a group X satisfying the condition

$$\tau(\mu \otimes \mu) = (\mu^{*2}) \otimes (\mu * \overline{\mu}) \tag{1}$$

and studied their properties [Co2]–[Co4]. In the case $\mu \in \mathscr{M}^1(X)$ condition (1) is the same as $\mu \in \Gamma_B(X)$ (see [He2]).

Section 10

The results of this section are due to G. M. Fel′dman. It should be noted that in the proof of Theorem 10.3 the group analog of the following Marcinkiewicz theorem was used: if $\mu \in \mathscr{M}^1(\mathbb{R})$ and the characteristic function $\hat{\mu}(s)$ has the form $\hat{\mu}(s) = \exp\{P(s)\}$ where $P(s)$ is a polynomial, then $\mu \in \Gamma(\mathbb{R})$ [LinO, Chapter II, §5]. The complete description of those groups X for which this analog is valid is given in [F13].

Section 11

The results of this section are due to G. M. Fel′dman [F11]. Theorem 11.9 and the lemmas it uses were proved in [F14]. Theorem 11.16 for the case $m = 2$ was proved in [F10]. This proof is based on the results of 11.18–11.20. The results of 11.25–11.32 were obtained in [F12].

It should be noted that the problem of constructing a theory of equidistribution of forms on algebraic structures was posed by A. M. Kagan, Yu. V. Linnik, and Rao in [KLR].

Appendix 1

The results of Appendix 1 were obtained by G. M. Fel′dman [F13].

Appendix 2

Theorem A.2.2 was proved by A. P. Ushakova [Ush]. The first estimates of the decomposition stability were obtained by N. A. Sapogov (for Gaussian distributions on the group $X = \mathbb{R}$), see [LinO, Chapter VIII]. On the further development of this subject for the groups $X = \mathbb{R}$ and $X = \mathbb{R}^n$ see the survey article [53] by G. P. Chistyakov.

The results of A.2.3–A.2.5 were obtained by G. P. Chistyakov in [Ch2]. This is the only article known to the author that gives estimates of stability for distributions on general locally compact abelian groups.

Appendix 3

The results of this Appendix are due to V. M. Zolotarev and V. M. Kruglov [ZK]. It should be noted that Lemma A.3.3 is a group analog of the well-known result of Jessen and Wintner [JW]. Examples constructed in [ZK] for the group $X = \mathbb{R}$ demonstrate that the classes $\mathscr{L}_{1/n}$ for any $n \geq 2$ and $\mathscr{L}_0$ are nonempty.

Appendix 4

Infinite convolutions of discrete distributions appear naturally in many parts of analysis, number theory, probability, often as a source of various examples and counterexamples. We should single out the paper by Brown and Moran [BrM2] among a number of investigations devoted to this subject. They studied the Bernoulli convolution on the circle group $\mathbb{T}$. Lin and Saeki [LiSa] proved some analogs of the main result in [BrM2] for nondiscrete metric locally compact Abelian group. The results of A.4.2–A.4.7 belong to them. We give an account of their results following [LiSa]. It was Theorem A.4.7 that enabled us to prove the density of the class I_0 in the class of infinitely divisible distributions on a nondiscrete group X (Proposition 8.3).

In [BrM1] Brown and Moran considered more a general class of distributions, which they called ergodic distributions and used to prove that some symmetric Bernoulli convolutions and antisymmetric Bernoulli convolutions have relatively singular powers.

Let D be a countable subgroup of X and let $\lambda \in \mathscr{M}^1(K)$ be quasi-invariant under the action of D on X, i.e., if $N \in \mathscr{B}(X)$ and $\lambda(N) = 0$, then $\lambda(d + N) = 0$ for every $d \in D$. Such a distribution λ is called ergodic with respect to the action of D if whenever N is a D-invariant Borel set of X (i.e., $d + N = N$ for all $d \in D$), then either $\lambda(N) = 0$ or $\lambda(N) = 1$.

A group G is called a refinement of X if G is algebraically isomorphic to X but has a finer locally compact topology.

The main result of the paper [BrM1] is the following.

THEOREM C.6. *Let λ be ergodic with respect to the action of a countable subgroup D of X. Then either*

(i) *λ^{*m} is singular to λ^{*n} unless $m = n$, or*

(ii) *there are a refinement G of X and a positive integer p such that $\lambda \in \mathscr{M}^1(G)$ and $\lambda^{*p} \in L^1(G)$.*

Theorem A.4.9 follows from this theorem. Theorem A.4.9 and Corollary A.4.11 are due to Brown and Moran [BrM1]. Corollary A.4.10 is due to Kaufman [Kau]. We note that the relative singularity of powers of the Lebesgue-Stielties distribution μ_0 on the standard Cantor set follows from the Vershik's results [V].

Detailed references concerning these problems could be found in the papers [BrM1], [LiSa], and [BrM2].

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