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Volume 123

# Élie Cartan (1869–1951)

M. A. Akivis
B. A. Rosenfeld

American Mathematical Society

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# (1869–1951)


**ÉLIE CARTAN**
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Providence, Rhode Island

ЭЛИ КАРТАН (1869–1951)

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Б. А. Розенфельд

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ABSTRACT. The scientific biography of one of the greatest mathematicians of the 20th century, Élie Cartan (1869–1951), is presented, as well as the development of Cartan's ideas by mathematicians of the following generations.

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# Contents

# Preface

The year 1989 marked the 120th birthday of Élie Cartan (1869–1951), one of the greatest mathematicians of the 20th century, and 1991 marked the 40th anniversary of his death. The publication of this book is timed to these two dates. The book is written by two geometers working in two different branches of geometry whose foundations were created by Cartan. The mathematical heritage of Cartan is very wide, and there is no possibility of describing all mathematical discoveries made by him, at least not in a book of relatively modest size. Because of this, the authors pose for themselves a much more modest problem—to describe and evaluate only the most important of these discoveries. Of course, the authors are only able to describe in detail Cartan's results connected with those branches of geometry in which the authors are experts.

The book consists of seven chapters. In Chapter 1 the outline of É. Cartan's life is given, and in Chapters 2–7 his main achievements are described, namely, in the theory of Lie groups and algebras; in applications of these theories to geometry; in the theory of Lie pseudogroups; in the theory of Pfaffian differential equations and its application to geometry by means of Cartan's method of moving frames; in the geometry of Riemannian manifolds; and, in particular, in the theory of symmetric spaces created by Cartan; in the theory of spaces of affine connection and other generalized spaces. In the same chapters the main routes of the development of Cartan's ideas by mathematicians of the following generations are given. At the end of the book a chronology of the main events of É. Cartan's life and a list of his works are presented. The references to Cartan's works are given by numerals without Cartan's name, and the other references by first letters of the names of the authors, with numerals added for multiple references. The appendices contain H. Poincaré's reference on Cartan's work (1912); Cartan's paper *On a degeneracy of Euclidean geometry*, which was omitted in his *Œuvre Complètes*; his speech at the meeting in the Sorbonne on the occasion of his 70th birthday (1939); and his lecture, *The influence of France in the development of Mathematics* (1940). Chapters 1–3 and 6 were written by B. A. Rosenfeld, Chapters 5 and 7 were written by M. A. Akivis, and Chapter 4 was written by both authors.

The authors express their cordial gratitude to Henri Cartan, a son of É. Cartan, who himself is one of the greatest mathematicians of this century, for providing numerous facts for a biography of his father and for pictures furnished by him.

*Moscow, Russia* *M. A. Akivis*

*University Park, PA, U.S.A.* *B. A. Rosenfeld*

APPENDIX A

# Rapport sur les Travaux de M. Cartan

fait à la Faculté des Sciences de l'Université de Pans

PAR

## H. POINCARÉ[1]

..... Le rôle prépondérant de la théorie des groupes en mathématiques a été longtemps insoupçonné; il y a quatre-vingts ans, le nom même de groupe était ignoré. C'est GALOIS qui, le premier, en a eu une notion claire, mais c'est seulement depuis les travaux de KLEIN et surtout de Lie que l'on a commencé à voir qu'il n'y a presque aucune théorie mathématique où cette notion ne tienue une place importante.

On avait cependant remarqué comment se font presque toujours les progrès des mathématiques; c'est par généralisation sans doute, mais cette généralisation ne s'exerce pas dans un sens quelconque. On a pu dire que la mathématique est l'art de donner le même nom à des choses différentes. Le jour où on a donné le nom d'addition géométrique à la composition des vecteurs, on a fait un progrès sérieux, si bien que la théorie des vecteurs se trouvait à moitié faite; on en a fait un autre du même genre quand on a donné le nom de multiplication à une certaine opération portant sur les quaternions. Il est inutile de multiplier les exemples, car toutes les mathématiques y passeraient. Par cette similitude de nom, en effet, on met en évidence une similitude de fait, une sorte de parallélisme qui aurait pu échapper à l'attention. On n'a plus ensuite qu'à calquer, pour ainsi dire, la théorie nouvelle sur une théorie ancienne déjà connue.

Il faut s'entendre, toutefois: il faut donner le même nom à des choses différentes, mais à la condition que ces choses soient différentes quant à la matière, mais non quant à la forme. A quoi tient ce phénomène mathématique si souvent constaté? Et d'autre part en quoi consiste cette communauté de forme qui subsiste sous la diversité de la matière? Elle tient à ce que toute théorie mathématique est, en dernière analyse, l'étude des propriétés d'un groupe d'opérations, c'est-à-dire d'un système formé par certaines opérations

[1] *Acta Mathematica* **38** (1914), 137–145.

fondamentales et par toutes les combinaisons qu'on en peut faire. Si, dans une autre théorie, on étudie d'autres opérations qui se combinent d'après les mêmes lois, on verra naturellement se dérouler une suite de théorèmes correspondant un à un à ceux de la première théorie, et les deux théories pourront se développer avec un parallélisme parfait; il suffira d'un artifice de langage, comme ceux dont nous parlions tout à l'heure, pour que ce parallélisme devienne manifeste et donne presque l'impression d'une identité complète. On dit alors que les deux groupes d'opérations sont isomorphes ou bien qu'ils ont même structure.

Si alors on dépouille la théorie mathématique de se qui n'y apparaît que comme un accident, c'est-à-dire de sa matière, il ne restera que l'essentiel, c'est-à-dire la forme; et cette forme, qui constitue pour ainsi dire le squelette solide de la théorie, ce sera la structure du groupe.

On distinguera parmi les groupes possibles quatre catégories principales, sans compter certains groupes étranges ou composites qui ne rentrent dans aucune catégorie, ou qui participent des caractères de deux ou plusieurs d'entre elles. Ce sont:

**I.** Les groupes discontinus et finis, ou groupes de Galois; ce sont ceux qui président à la résolution des équations algébriques, à la théorie des permutations, etc.....

**II.** Les groupes discontinus et infinis; ce sont ceux que l'on rencontre dans la théorie des fonctions elliptiques, des fonctions fuchsiennes etc.....

**III.** Les groupes continus et finis ou groupes de LIE proprement dits; ce sont ceux auxquels se rattachent les principales théories géométriques, telles que la géométrie euclidienne, la géométrie non-euclidienne, la géométrie projective, etc.....

**IV.** Les groupes continus et infinis, beaucoup plus complexes, beaucoup plus rebelles aux efforts du géométrie. Ils sont en connexion naturelle avec la théorie des équations aux dérivées partielles.

M. CARTAN a fait faire des progrès importants à nos connaissances sur trois de ces catégories, la $1^{\text{ére}}$, la $3^{\text{é}}$, et la $4^{\text{é}}$. Il s'est principalement placé au point de vue le plus abstrait de la structure, de la forme pure, indépendamment de la matière, c'est-à-dire, dans l'espèce, du nombre et du choix des variables indépendantes.

## Groupes continus et finis

Je commencerai par les groupes continus et finis, qui ont été introduits par LIE dans la science; le savant norvégien a fait commaître les principes fondamentaux de la théorie, et il a montré en particulier que la structure de ces groupes dépend d'un certain nombre de constantes qu'il désigne par la lettre $c$ affectée d'un triple indice et entre lesquelles il doit y avoir certaines relations. Il a enseigné également comment on pouvait construire le groupe quand on connaissait ces constantes. Mais il restait à discuter les diverses manières de satisfaire aux relations qui doivent avoir lieu entre les constantes $c$ ; on pouvait supposer que les divers types de structure seraient extrêmement nombreux et extrêmement variés, de sorte que l'énumération en serait à peu près impossible. Il ne semble pas eu être tout à fait ainsi, au moins en ce qui concerne les groupes simples.

La distinction entre les groupes simples et les groupes composés est due à GALOIS et elle est essentielle, puisque les groupes composés peuvent toujours être construits en partant des groupes simples. Il est clair que le premier problème à résoudre est la construction des groupes simples.

Vers 1890, KILLING a annoncé que tous les groupes simples continus et finis rentrent: soit dans quatre grands types généraux déjà signalés par LIE, soit dans cinq types particuliers dont les ordres sont respectivement 14, 52, 78, 133, et 248. C'était là un résultat d'une très haute importance; malheureusement toutes les démonstrations étaient fausses; il ne restait que des aperçus dénués de toute force probante.

Il était réservé à M. CARTAN de transformer ces aperçus en démonstrations rigoureuses; il suffit d'avoir lu le mémoire de KILLING pour comprendre combien cette tâche était difficile. La méthode repose sur la considération de l'équation caractéristique, et en particulier de la forme quadratique $\psi_r(e)$ qui est le coëfficient de $\omega^{r-2}$ dans cette équation; cette considération permet de reconnaître si le groupe intégrable, ou de trouver son plus grand sous groupe invariant intégrable, ou enfin de reconnaître si le groupe est simple ou semisimple.

M. CARTAN a donné une manière de former, dans chaque type, les groupes linéaires simples dont le nombre des variables est aussi petit que possible.

Une des plus importantes applications des groupes de LIE est l'intégration des équations différentielles ordinaires ou partielles qui sont inaltérées par les transformations d'un groupe. M. CARTAN a appliqué cette méthode au cas des systèmes d'équations aux dérivées partielles don't l'intégrale générale ne dépend que de constantes arbitraires. Les opérations à faire sont toutes de nature rationnelle ou algébrique.

## Groupes discontinus et finis

M. CARTAN a fait faire aussi un progrés important à la théorie des groupes de GALOIS, en les rattachant à celle des nombres complexes. On sait qu'on désigne par nombres complexes des expressions algébriques susceptibles de subir des opérations qui peuvent être regardées comme des généralisations de l'addition et de la multiplication, et auxquelles on peut appliquer les règles ordinaires du calcul avec cette différence que la multiplication, quoique associative, n'est pas commutative. La plus connu des systèmes de nombres complexes a reçu le nom de quaternions et on en a fait des applications nombreuses en Mécanique et en Physique Mathématique.

Ces nombres complexes ont un lien intime avec les groupes de Lie et en particulier avec les groupes linéaïres simplement transitifs; il y a, à ce sujet, un théorème de M. POINCARÉ dont M. CARTAN a donné une nouvelle démonstration. La théorie des nombres complexes a été poussée plus loin par M. M. SCHEFFERS et MOLLIEN qui en ont entrepris la classification et ont les premiers mis en évidence l'importance de la distinction entre les systèmes à quaternions et les systèmes sans quaternions.

M. CARTAN est arrivé à résoudre complètement le problème, par une heureuse adaptation des méthodes qui lui avaient réussi dans l'étude des groupes de Lie. Il a pris comme point de départ une équation caractéristique qui n'est pas tout à fait la même que celle qu'on envisage à propos des groupes de Lie, mais qui se prête à une discussion analogue. M. CARTAN a montré comment on peut construire un système quelconque par la combinaison d'un système pseudonul et de systèmes simples et comment les systèmes simples se réduisent aux quaternions généralisés; comment enfin les systèmes dits de la $2^e$ classe se déduisent facilement de ceux de la $1^{ére}$ classe. Il a étudié aussi le cas où les coëfficients sont des nombres réels.

Ces résultats ne constituent pas, comme on pourrait être tenté de la croire, une simple curiosité mathématique. Ils sont au contraire susceptibles d'applications nombreuses. En particulier, ils se rattachent à la théorie des groupes de GALOIS; il est clair que les lois de la composition des substitutions d'un groupe de GALOIS sont associatives, sans être commutatives; elles peuvent donc être regardées comme les règles de la multiplication d'un système d'unités complexes; et par conséquent elles définissent un système de nombres complexes. Or si on applique à ce système le théorème de M. CARTAN, on retrouve, de la façon la plus simple et pour ainsi dire d'un trait de plume, les résultats que M. FROBENIUS avait obtenus par une tout autre voie et qui avaient été regardés à juste titre comme le plus grand progrès que la théorie des groupes de GALOIS eût fait depuis longtemps.

On peut, par cette voie, reconnaître quels sont les groupes linéaires les plus simples qui sont isomorthes à un groupe de GALOIS donné, ce qui nous conduit au problème de l'intégration algébrique des équations différentielles linéaires. M. POINCARÉ a eu l'occasion d'appliquer les principes de M. CARTAN à l'intégration algébrique d'une équation linéaire.

## Groupes continus et infinis

La dètermination des groupes continus infinis présente beaucoup plus de difficultés que celle des groupes finis et c'est là que M. CARTAN a déployé le plus d'originalité et d'ingéniosité. Il s'est restreint d'ailleurs à une certaine classe de groupes infinis, la plus importante au point de vue des applications, et celle sur laquelle l'attention de Lie avait surtout été attirée, je veux parler des groupes dont les transformations finies dépendent de fonctions arbitraires d'un ou de pluseurs paramètres, ou, plus généralement, de ceux où les variables transformées, considérées comme fonctions des variables primitives, constituent l'intégral général d'un système d'équations aux dérivées partielles.

M. CARTAN s'est d'ailleurs servi, dans cette étude, de résultats importants qu'il avait obtenus dans des travaux antérieurs relatifs aux équations aux dérivées partielles et aux équations de PFAFF, travaux dont nous parlerons plus loin.

La théorie de la structure, telle que LIE l'expose dans l'étude des groupes finis, n'est pas susceptible d'être immédiatement généralisée et étendue aux groupes infinis. M. CARTAN lui substitue done une autre théorie de la structure, équivalente à la première en ce qui concerne les groupes finis, mais susceptible de généralisation. Si $f$ est une fonction quelconque des variables $x$, et si les $X_i f$ représentent les symboles de Lie, on aura identiquement:

$$df + \sum X_i f \omega_i = 0$$

les $\omega_i$ étant des expressions de Pfaff dépendant des paramètres du groupe et de leurs différentielles.

Au lieu de faire jouer le rôle essentiel aux symboles $X_i f$, comme le faisait LIE, M. CARTAN l'attribue aux expressions de PFAFF $\omega$ qui sont invariantes par les substitutions du groupe des paramètres. Les relations qui définissent la structure se présentent alors sous une autre forme. Au lieu de relations linéaires entre les $X_i f$ et leurs crochets, nous aurons des relations linéaires entre les covariants bilinéaires des $\omega$ et des combinaisons bilinéaires de ces même expressons. Le coëfficients de ces relations sont les mêmes dans les deux cas, quoique dans un autre ordre; ce sont les constantes $c$ de LIE.

Sans sortir encore du domaine des groupes finis, M. CARTAN a illustré cette théorie nouvelle en l'appliquant à des exemples concrets, et en particulier au groupe des déplacements de l'espace; il a montré comment elle se rattachait à la théorie classique du trièdre mobile de M. DARBOUX et comment elle permettait l'étude des invariants différentiels des surfaces et en particulier de ceux de certaines surfaces imaginaires remarquables.

Voyons maintenant comment ces notions peuvent être étendues aux groupes infinis. La notion d'isomorphisme holoédrique peut être facilement définie en ce qui concerne les groupes finis, parce que l'on n'a qu'à faire correspondre une à une les transformations infinitésimales des deux groupes à comparer. Nous ne pouvons plus employer ce procédé lorsque les transformations infinitésimales sont en nombre infini; M. CARTAN donne donc une définition différente, quoique équivalente à la première dans le cas où celle-ci a un sens. Un groupe est le prolongement d'un autre quand il transforme les mêmes variables que cet autre et de la même manière et qu'il transforme en même temps d'autres variables auxiliaires. Par exemple, le groupe des déplacements des points de l'espace aura pour prolongement le groupe des déplacements des droites ou celui des cercles de l'espace. Deux groupes sont alors isomorphes quand deux de leurs prolongements sont semblables.

La théorème fondamental de LIE peut alors être étendu aux groupes infinis; on montre que tout groupe infini est isomorphe au groupe qui laisse invariantes à la fois certaines fonctions $U$ et certaines expressions de PFAFF $\omega$ et $\tilde{\omega}$. Les différentielles totales des $U$ s'experiment linéairement en fonctions des $\omega$, les covariants bilinéaires des $\omega$ (mais non ceux des $\tilde{\omega}$) s'expriment bilinéairement en fonctions des $\omega$ et $\tilde{\omega}$. Les coëfficients de ces relations linéaires ou bilinéaires jouent le rôle des constantes $c$ de LIE. Ce sont des fonctions des invariants $U$. Ce qui caractérise les groupes transitifs, c'est qu'il n'y a pas d'invariants et par conséquent que les coëfficients se réduisent à des constantes. Ce qui caractérise les groupes finis, c'est que les expressions n'existent pas.

Les coëfficients en question peuvent-ils être choisis arbitrairement? Non, ils sont assujettis à certaines conditions que M. CARTAN détermine et que peuvent être regardées comme la généralization des conditions de structure de LIE.

Les trois théorèmes fondamentaux de LIE se trouvent donc étendus aux groupes infinis, de sorte que M. CARTAN a fait pour ces groupes ce que LIE avait fait pour les groupes finis.

Cette analyse a mis en évidence des résultats tout á fait surprenants. Un groupe fini est toujours isomorphe à un groupe transitif, par exemple à celui qu'on appelle son groupe paramétrique, et on aurait pu être tenté de croire qu'il en était de même pour les groupes infinis, puisqu'au premier abord la démonstration ne semblait mettre en œuvre que la notion générale de groupe. Au contraire, M. CARTAN a montré qu'il existe les groupes infinis qui ne sont isomorphes à aucun groupe transitif.

Ce n'est pas tout: un groupe infini peut être mériédriquement isomorphe à lui-même, un groupe infini peut n'admettre aucun sous groupe invariant maximum, etc., . . . . La notion du prolongement normal permet ensuite à M. CARTAN de déterminer tous les groupes isomorphes à un groupe infini donné. Citons un résultat particulier. Les groupes qui ne dépendent que de fonctions arbitraires d'un argument, s'ils sont transitifs, sont isomorphes au groupe général d'une variable.

Etant donné un groupe défini par ses équations de structure, M. CARTAN montre qu'on peut déterminer les équations de structure de tous ses-groupes par des procédé purement algébriques et applique cette méthode à des cas particuliers tels que celles du groupe général de deux variables où il retrouve, par une voie nouvelle, quelques sous groupes déjà connus et importants par leurs applications.

Si l'on se donne deux systèmes différentiels et un groupe, on peut se demander s'il y a des transformations du groupe qui transforment un des systèmes dans l'autre et quelles elles sont; on peut se demander également s'il y a dans le groupe des transformations qui n'altéreront pas l'un de ces systèmes différentiels et qui naturellement formeront un sous-groupe. L'étude de ce sous-groupe a fait également l'objet d'un mémoire de M. CARTAN .

Enfin M. CARTAN s'est proposé en ce qui concerne les groupes infinis, le même problème qu'il avait résolu pour les groupes finis, la formation de tous les groupes simples. Il a montré qu'ici aussi, les groupes simples peuvent se ramener à un nombre restreint de types; ceux qui sont primitifs et d'où l'on peut déduire tous les groupes transitifs simples se répartissent en six grandes classes; quant aux groupes simples qui ne sont isomorphes à aucun groupe transitif, ils peuvent être déduits des précédents par des procédés des procédés que M. CARTAN nous fait connaître.

Le problème proposé se trouve donc entièrement résolu.

## Equations aux dérivées partielles

Le problème de l'intégration d'un système d'équations aux dérivées partielles a fait l'objet de travaux nombreux. M. CARTAN s'est placé pour l'étudier à un point de vue particulier; il remplace le système d'équations aux dérivées partielles par le système correspondant d'équations de PFAFF, c'est-à-dire d'équations aux differentielles totales.

Dans la théorie des expressions de PFAFF, il y a une notion, introduite par M. M. FROBENIUS et DARBOUX, qui joue un rôle extrêmement important, c'est celle du covariant bilinéaire; nous avons déjà vu apparaître ce covariant à propos de la théorie des groupes infinis. M. CARTAN en a donné une interprétation nouvelle à l'aide du calcul de GRASSMANN, et cette interprétation l'a conduit à une généralisation. De chaque expression de PFAFF, il déduit une série d'expressions différentielles qu'il appelle ses dérivées; la dérivée première est la covariant bilinéaire; la dérivée $n^e$ est $n+1$ fois linéaire. C'est en cherchant quelle est la première de ces dérivées qui s'annule identiquement que l'on reconnaîtra si, et jusqu'à quel point, il est possible de réduire le nombre des variables indépendantes sur lesquelles porte l'expression.

Cette considération a permis à M. CARTAN de retrouver sous une forme extrêmement simple tous les résultats connus relatifs au problème de PFAFF et un assez grand nombre de résultats entièrement nouveaux.

Comment maintenant cela peut-il servir à la résolution d'un système d'équations de PFAFF, et surtout à reconnaître quel est le degré d'arbitraire que comporte l'intégrale générale d'un pareil système? C'est en se servant de la notion d'involution que M. CARTAN a résolu cette question. Un système est dit en involution si, jusqu'à une certaine valeur de $m$, par toute multiplicité intégrale à $m$ dimensions passe une multiplicité intégrale à $m+1$ dimensions. M. CARTAN donne une manière de reconnaître si un système est en involution pour les valeurs de $m$ inférieures à un nombre donné, et, par là, de savoir combien la solution générale contient de fonctions arbitraires de 1, de 2, ..., de $n$ variables.

On retrouve ainsi sous une forme nouvelle la théorie des caractéristiques de CAUCHY, celle des caractéristiques de MONGE, celle des solutions singulières, etc., ... ; on retrouve également sous une forme plus simple tous les résultats de M. RIQUIER.

M. CARTAN a appliqué sa méthode à un certain nombre de cas particuliers où l'intégration peut se faire par des équations differentielles ordinaires. Il l'a également complétée en s'aidant de la théorie des groupes qui lui était si familière; il a ainsi reconnu des cas où l'on peut déterminer les invariants d'un système de PFAFF, sans en déterminer les caractéristiques, c'est-à-dire d'une façon rationnelle, et d'autres où les caractéristiques s'obtiennent sans intégration.

## Conclusions

On voit que les problèmes traités par M. CARTAN sont parmi les plus importants, les plus abstraits et les plus généraux don't s'occupent les Mathématiques; ainsi que nous l'avons dit, la théorie des groupes est, pour ainsi diew, la Mathématique entière, dépouillée de sa matière et réduite à une forme pure. Cet extrême degré d'abstraction a sans doute rendu mon exposé un peu aride; pour faire apprécier chacun des résultats, il m'aurait fallu pour ainsi dire lui restituer la matière dont il avait été dépouillé; mais cette restitution peut se faire de mille façons différentes; et c'est cette forme unque que l'on retrouve ainsi sous une foule de vêtements divers, que constitue le lien commun entre des théories mathématiques qu'on s'étonne souvent de trouver si voisines.

M. CARTAN en a donné récemment un exemple curieux. On connait l'importance en Physiqué Mathématiques de ce qu'on a appelé le groupe de LORENTZ; c'est sur ce groupe que reposent nos idées nouvelles sur le principe de relativité .et sur Dynamique de l'Electron. D'un autre côté , LAGUERRE a autrefois introduit en géométrie un groupe de transformations qui changent les sphères en sphères. Ces des groupes sont isomorphes, de sorte que mathématiquement ces deux théories, l'une physique, l'autre géométrique, ne présentent pas de différence essentielle.

Les rapprochements de ce genre se présenteront en foule à ceux qui étudieront avec soin les travaux de LIE et de M. CARTAN. M. CARTAN n'en a pourtant signalé qu'un petit nombre, parce que, courant au plus pressé, il s'est attaché à la forme seulement et ne s'est préoccupé que rarement des diverses matières dont on la pouvait revêtir.

Les résultats les plus importants énoncés par M. CARTAN lui appartiennent bien en propre. En ce qui concerne les groupes de LIE, on n'avait que des énoncés et pas de démonstration; en ce qui concerne les groupes de GALOIS, on avait les théorèmes de FROBENIUS qui avaient été rigoureusement démontrés, mais par une méthode entièrement différente; enfin en ce qui concerne les groupes infinis on n'avait rien: pour ces groupes infinis, l'œuvre de M. CARTAN correspond à ce qu'a été pour les groupes finis l'œuvre de LIE, celle de KILLING, et celle de CARTAN lui-même.

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APPENDIX B

# Sur une dégénérescence de la géométrie euclidienne

PAR

M. ELIE CARTAN

Professeur à la Faculté des Sciences de Paris[1]

La géométrie dans un plan isotrope diffère profondément de la géométrie plane classique; les lignes qui jouent dans un plan nonisotrope le rôle des circonférences sont, dans un plan isotrope, des paraboles toutes tangentes en un même point à la droute de l'infini. Si l'on prend pour axe des $y$ une parallèle à la direction isotrope unique du plan, le groupe de la géométrie euclidienne du plan isotrope est la forme:

$$\begin{cases} x' = x + a, \\ y' = cx + hy + b, \end{cases} \tag{1}$$

l'arc élémentaire $ds$ d'une courbe étant réduit à $dx$. La notion ordinaire de courbure disparait, mais il s'y substitue une *pseudocourbure* égale à $\frac{f'''(x)}{f''(x)}$, lorsque la courbe est définie par $y = f(x)$.

Le groupe (1) est un sous-groupe du plus grand groupe affine qui laisse invariant le point à l'infini dans la direction $Oy$, à savoir:

$$\begin{cases} x' = kx \qquad +a, \\ y' = cx + hy \quad +b; \end{cases} \tag{2}$$

un autre sous-groupe invariant de ce dernier, à savoir le groupe

$$\begin{cases} x' = kx \qquad +a, \\ y' = cx + y \quad +b, \end{cases} \tag{3}$$

peut être pris comme base d'une géométrie plane à direction isotrope privilégiée. Dans cette géométrie, qui est en un certain sens une dégénérescence de la géométrie euclidienne, on peut définir la longueur d'un vecteur parallèle

[1] *Assoc. Franç. Avanc. des Sciences,* 59[e] session, Nantes, 1935, 128–130.

à la direction isotrope comme étant la différence des ordonnées $y'$ et $y$ de son extrémité et de son origine, mais la notion de longueur disparait pour les vecteurs nonisotropes.

La géométrie fondée sur le groupe (3) est intéressante; on voit tout de suite qu'étant donnée une ligne plane autre qu'une droite, on peut définir d'une manière intrinsèque un élément d'arc $ds$ par la formule

$$(4) \qquad ds^2 = \frac{dxd^2y - dyd^2x}{dx} = f''(x)dx^2.$$

Le second membre est en effet le rapport de deux aires, l'aire du parallélogramme construit sur les deux vecteurs $(dx, dy)$ et $(d^2x, d^2y)$, et l'aire du parallélogramme construit sur les deux vecteurs $(dx, dy)$ et $(0, 1)$. Cet élément d'arc est identiquement nul quand la ligne considérée est une droite. Si l'on attache à chaque point de la ligne deux vecteurs $\vec{T}$ et $\vec{N}$, le premier tangent à la ligne et de composantes $\frac{dx}{ds}$, $\frac{dy}{ds}$, le second parallèle à $Oy$ et de longueur 1, on a les formules de Frenêt généralisées:

$$(5) \qquad \frac{dM}{ds} = \vec{T}, \qquad \frac{d\vec{T}}{ds} = k\vec{T} + \vec{N}, \qquad \frac{d\vec{N}}{ds} = 0.$$

Le coefficient $k = \frac{d^2x}{ds^2} / \frac{dx}{ds} = -\frac{1}{2}\frac{f'''(x)}{f''(x)}$ est la *courbure*. Les courbes de courbure nulle sont les paraboles tangentes à la droite de l'infini au point à l'infini sur $Oy$. La courbure est du reste un invariant pour le groupe général (2).

Ce quit donne un certain intérêt à la géométrie précédente, c'est qu'elle se présente d'elle-même quand on veut chercher des propriétés géométriques intrinsèquement attachées à une intégrale $\int F(x, y, y', y'')\, dx$, où $F$ est une fonction donnée de $x, y, y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}$; une propriété est dite intrinsèque si elle ne dépend pas du choix des coordonnées $x, y$. Si la fonction $F$ se réduit à $\sqrt{y''}$, le plus grand groupe qui laisse invariante l'intégrale est précisément le groupe (3). Si $F$ est de la forme $\sqrt{\frac{y''+p(x,y,y')}{By'-A}}$, où $A$ et $B$ sont des fonctions de $x, y$, on a une géométrie que joue par rapport à la géométrie de groupe (3) le même rôle que la géométrie riemannienne par rapport à la géométrie euclidienne, avec cette différence que l'espace doit être regardé comme engendré non par des points $(x, y)$ mais par des éléments linéaires $(x, y, y')$; l'espace est un *espace d'éléments linéaires à connexion affine*, assimilable au voisinage de chaque élément linéaire à un plan euclidien isotrope de groupe (3).

Un autre cas particulier intéressant est celui de l'intégrale $\int \sqrt{y''}\, dx$ qui est liée à la géométrie affine unimodulaire.

APPENDIX C

# Allocution de M. Élie Cartan

A la fin de cette émouvante cérémonie, après tous les éloges dont vous m'avez comblé et que j'ai conscience de n'avoir qu'imparfaitement mérités, permettez que ma pensée se reporte vers ceux qui ne sont plus et qui auraient été si fiers de les entendre. Je pense à mon père et à ma mère, humbles paysans qui pendant leur longue vie ont donné à leurs enfants l'exemple du travail joyeusement accompli et des charges vaillamment acceptées. C'est au bruit de l'enclume résonnant chaque matin dès l'aube que mon enfance a été bercée, et je vois encore ma mère actionnant le métier du canut, aux instants que lui laissaient libres les soins de ses enfants et les soucis du ménage.

En même temps qu'à mes parents je pense à mes premiers maîtres, les instituteurs de l'École primaire de mon village de Dolomieu, M. Collomb, et surtout M. Dupuis; ils donnaient à plus de deux cents garçons un enseignement précis dont j'appréciai plus tard la valeur. Je suis obligé d'avouer—et je n'en ai pas honte—que j'étais un excellent élève; j'étais capable d'énumérer sans hésitation les sous-préfectures de n'importe quel département, et aucune subtilité des règles du participe passé ne m'échappait. Un jour un délégué cantonal qui s'appelait Antonin Dubost et qui devait plus tard devenir un des plus hauts personnages de l'État vint inspecter l'école; cette visite orienta toute ma vie. Il fut décidé que je me présenterais au concours des bourses des lycées; M. Dupuis dirigea ma préparation avec un dévouement affectueux que je n'oublierai jamais. Tout cela me valut un beau voyage à Grenoble, où je subis sans trop d'émoi des épreuves pas trop redoutables. Je fus reçu brillamment, ce qui remplit M. Dupuis de fierté et grâce à l'appui de M. Dubost, qui s'intéressa pendant toute sa vie avec une affection toute paternelle à ma carrière et à mes succès, je fus gratifié d'une bourse complète au Collège de Vienne.

A l'âge de dix ans je quittai donc joyeux le foyer paternel, sans me douter que bien peu de jours me suffiraient pour regretter ce que je perdais. Il fallut m'adapter à la vie d'internat que je devais mener pendant plus de dix ans. Après cinq ans de collège pendant lesquels je dus mettre les bouchées doubles, ma bourse fut transférée au Lycée de Grenoble où j'achevai mes études classiques par la rhétorique et la philosophie, puis au Lycée Janson-de-Sailly, qui était dans toute la fraîcheur de sa première jeunesse, rayonnant du succès que venait d'obtenir Le Dantec reçu premier à l'École Normale. J'eus à Janson des professeurs remarquables, Salomon Bloch en mathématiques élémentaires A, et en mathématiques spéciales Émile Lacour dont tu as su, mon cher Tresse, sans l'avoir connu comme professeur, dépeindre la noblesse de caractère. C'est dans cette classe que j'eus comme camarade, avec Eugène Perreau qui devait entrer avec moi à l'École Normale, Jean Perrin, plus jeune que nous, et qui devait devenir une des plus grandes gloires de la science française.

C'est avec émotion, mon cher Tresse, que je t'ai entendu évoquer nos années d'École Normale. Je ne suis pas sûr que le recul du temps n'ait pas embelli le souvenir que tu as gardé de moi et du rôle que j'aurais joué auprès de mes camarades. Ce que je me rappelle, c'est en effet une camaraderie fraternelle et une collaboration qui s'est montrée surtout assez étroite dans l'année de préparation à l'agrégation. Je vois encore les séances où le soir, réunis dans une salle quelconque, nous écoutions l'un de nous exposer la leçon qu'il devait faire le lendemain. Là les critiques étaient libres et franches et combien profitables. Je me rappelle particulièrement une leçon sur l'intersection des quadriques qui nous frappa pour la manière élégante et neuve dont la question était conçue; l'auteur de cette leçon était Arthur Tresse.

Tu as parlé tout à l'heure, mon cher ami, de l'admiration que nous produisaient les cours de M. Émile Picard, qui excellait à nous ouvrir de vastes perspectives dans un domaine encore nouveau pour nous. A l'École même c'est Jules Tannery qui exerça sur nous la plus profonde influence; par une sorte de transposition mystérieuse due à l'ensemble de toute sa personne, à son regard peut-être, le respect de la rigueur dont il nous montrait la nécessité en mathématiques devenait une vertu morale, la franchise, la loyauté le respect de soi-même. Comme on l'a dit déjà, Tannery était notre conscience: c'est pourquoi nous l'aimions, c'est pourquoi nous avons voué à sa mémoire un culte fidèle.

Nous admirions aussi l'élégance de certaines conférences de Kœnigs, la clarté de l'enseignement de Goursat. A la Sorbonne c'était la limpidité des cours de Mécanique rationnelle d'Appell, l'élégance incomparable des cours de Darboux. Les leçons qui nous produisaient l'impression la plus profonde peut-être étaient celles d'Hermite, dont le visage et les yeux d'une beauté admirable s'illuminaient comme s'il contemplait au sein de la Divinité ce monde éternel des nombres et des formes dont nous parlait tout à l'heureu M. Picard.

Tannery, Goursat, Appell, Darboux, Picard, Hermite, que de grands noms s'offraient à l'admiration de notre jeunesse. Je n'ai pas parlé du géant des Mathématiques, Henri Poincaré, dont les leçons passaient bien au-dessus de nos têtes; il n'est aucune branche des mathématiques modernes qui n'ait subi son empreinte, et vous comprendrez que je garde à sa mémoire une particulière reconnaissance puisque le dernier travail de sa vie si brusquement interrompue a été un rapport sur mon œuvre scientifique. De cette illustre pléiade de grands mathématiciens, vous seul, mon cher Maître, nous restez; nous admirons toujours votre jeunesse et je me félicite que mon âge me donne encore le privilège d'entendre retracer ma carrière scientifique par le maître admiré qui, il y a un demi-siècle, m'initiait à l'Analyse mathématique, présentait mes premières notes à l'Académie et était le rapporteur de mon jury de thèse.

Après ma thèse dont le sujet, tu l'as peut-être oublié, mon cher Tresse, me fut signalé par toi à ton retour de Leipzig où tu avais été l'élève de Sophus Lie, je fus nommé maître de conférences à Montpellier. Je garde le meilleur souvenir des quinze ans que j'ai passés en province, à Montpellier d'abord, à Lyon, et à Nancy ensuite. Ce furent des années de méditation dans le calme, et tout ce que j'ai fait plus tard est contenu en germe dans mes travaux mûrement médités de cette période. C'est à Nancy que je commençai à me familiariser avec les vastes auditoires. J'avais à y enseigner les éléments de l'Analyse aux élèves de l'Institut électrotechnique et de Mécanique appliquée. Institut encore jeune, mais déjà prospère sous la direction de l'homme au dévouement admirable qu'était Vogt. Cet enseignement m'intéressait beaucoup et j'eus la satisfaction de sentir tout de suite le contact s'établir avec les élèves. Je me trouvai ainsi préparé à l'enseignement des mathématiques générales qui devait m'être confié un peu plus tard à la Sorbonne.

C'est un enseignement analogue que je donne à l'École de Physique et de Chimie depuis vingt-neuf ans. Dans la mesure où je mérite les éloges affectueux que votre amitié m'a prodigués, mon cher Langevin, je suis très heureux d'avoir pu vous aider à réaliser le dessein qui vous tient à cœur, celui de faire de l'École technique que vous dirigez un véritable établissement d'enseignement supérieur en assurant aux élèves une culture théorique fortement organisée. La tâche, là encore, m'a étè rendue facile par le courant de sympathie qui n'a cessé d'unir le mâtre et les élèves, toujours attentifs et désireux d'acquérir les connaissances dont ils reconnaissent eux-mêmes l'utilité pour leur carrière future. Ce n'est pas sans un vif regret que je quitterai bientôt, cette École à laquelle me rattachent tant de liens; mon départ ne pourra affaiblir les sentiments d'admiration que j'éprouve pour le savant et l'homme qui la dirige.

Tu as retracé tout à l'heure, mon cher Maurain, en termes qui m'ont particulièrement touché, venant de l'ami, du doyen affectueusement vénéré de tous ses collègues, ma carrière de professeur à la Sorbonne. Cela a toujours été pour moi une grande joie que d'enseigner; je me suis toujours intéressé à ce que j'enseignais: c'est une condition nécessaire et peut-être suffisante pour intéresser ceux qui vous écoutent. Si ma prochaine mise à la retraite ne me vieillit pas prémeturément, il me sera agréable de donner de temps en temps quelques séries de leçons sur des sujets que je n'ai pas encore eu l'occasion d'enseigner.

C'est à l'École Normale que s'est exercée une grande partie de ma carrière de professeur; pendant quelque quatorze ans j'y ai eu tout mon service. Il est vrai que j'y comprends les années de guerre, pendant lesquelles je vous ai accueilli à plusieurs reprises, mon cher Julia, lorsque grand blessé vous veniez vous reposer dans notre vieille École des opérations successives qu'on était obligé de vous faire subir au Val de Grâce. Il est difficile d'imaginer un auditoire plus intéressant que celui l'École Normale; devant lui on peut aborder tous les problèmes et j'en ai abordé un certain nombre. J'ai été heureux d'entendre de vous, mon cher Bruhat, et de vous, mon cher Julia, l'opinion qu'ont bien voulu garder de moi mes élèves. Ce sont maintenant des maîtres; un grand nombre enseignent dans les Facultés. L'un d'eux, celui que ses camarades de Janson envoyaient passer leurs colles chez Cartan, est l'un des plus jeunes membres de l'Académie des Sciences.

Nous, leurs aînés, nous avons la grande joie de voir sortir de l'École Normale des générations successives de brillants mathématiciens; nous sommes assurés ainsi qu'elle n'abdique pas le rôle de pépinière des mathématiques qu'elle joue depuis longtemps et qui inspira autrefois à Sophus Lie l'idée de lui dédier son grand traité sur la théorie des groupes. Et puisque, par une pensée touchante, le fils de Sophus Lie a voulu marquer ce Jubilé par l'envoi du buste de son père, ne serait-il pas naturel que la place de ce buste soit à la bibliothèque des Sciences de l'École Normale? Il rappellerait aux promotions successives à la fois le grand mathématicien norvégien et les normaliens qui ont été ses élèves à Leipzig et ont illustré l'École, les Vessiot, les Tresse, les Drach.

Mon cher Bruhat, vous avez parlé en termes qui me sont allés au cœur de la dynastie normalienne des Cartan. Me permettrez-vous d'adjoindre aux deux noms d'Henri Cartan et d'Hélène Cartan les noms de deux autres normaliens qui m'ont été très chers? Le premier est celui de mon beau-frère Antoine Bianconi, cacique littéraire de la promotion de 1903, dont la mort sur le champ de bataille interrompit l'œuvre philosophique qu'il méditait et qui promettait d'être importante. Le second est celui de ma plus jeune sœur Anna Cartan, dont le succès au concours d'entrée à Sèvres m'avait rempli de joyeuse fierté; élève elle aussi de Jules Tannery, dont elle ne pouvait parler sans émotion, elle a terminé prématurément sa brillante carrière comme professeur au Lycée annexe de Sèvres. Il m'est doux de penser qu'elle est un peu présente ici, en voyant au milieu de nous la compagne de promotion à qui la liait une tendre affection, ma chère amie Madame la Directrice de l'École de Sèvres.

Mon cher Julia, c'est avec empressement que je me suis associé à votre projet de fonder pour les jeunes mathématiciens un cercle d'études, votre séminaire, où ces jeunes gens, travaillant en collaboration, exposeraient chaque année une question importante de Mathématiques. Vous nous avez dit à ce propos que les jeunes sentent; sans peut-être trop se l'avouer, le besoin de s'appuyer sur leurs aînés. En entendant tout à l'heure Dieudonné, nous avons compris combien vous aviez raison.

Mon cher Dieudonné, les paroles que vous m'avez adressées me touchent au delà de toute expression. Elles montrent que vous avez l'enthousiasme de la jeunesse, vertu que je vous souhaite de conserver toute votre vie. Cet enthousiasme ne vous a-t-il pas fait dépasser la mesure? J'aurais certes mauvaise grâce à vous contredire, mais je suis assez âgé pour savoir ne pas tirer de vos éloges un orgueil déplacé, sachant très bien que si j'ai les qualités que vous m'attribuez, il m'en manque un certain nombre d'autres qui m'auraient permis de rendre plus de services à l'enseignement et à la science; elles ne sont sans doute pas dans ma nature, mais je n'ai peut-être pas eu assez de ferme volonté pour les acquérir.

Mon cher Demoulin, nous sommes liés par une vieille amitié et de nombreux souvenirs communs; nous avons écouté ensemble les maîtres dont je rappelais les noms tout à l'heure. Je suis très sensible aux félicitations que vous m'apportez au nom des savants étrangers. Je remercie particulièrement tous ceux d'entre eux, et je les vois ici nombreux, qui ont tenu à assister en personne à cette cérémonie. Leur présence m'est précieuse et l'empressement avec lequel des savants de nombreuses nations étrangères ont bien voulu s'associer à mon Jubilé m'a vivement touché. Dans le monde troublé où nous vivons, il est indispensable que la collaboration internationale, au moins dans le domaine scientifique, soit maintenue malgré tous les obstacles.

En même temps qu'aux délégués étrangers, j'adresse mes remerciements aux amis, aux collègues, aux élèves qui ont bien voulu répondre à l'appel du Comité jubilaire. Je remercie les membres de ce Comité qui ont accepté de donner leur concours à l'organisation de cette fête, et surtout mon collègue et ami Darmois qui, avec l'aide de mon élève Ehresmann, a pris sur lui la part la plus lourde de cette organisation.

Plusieurs des orateurs précédents, et j'en suis particulièrement touché, ont tenu à associer le nom de la compagne de ma vie à cette commémoration de ma carrière scientifique. Depuis plus de trente-six ans elle est la flamme ardente qui anime le foyer familial. Nos enfants nous ont réservé de grandes joies; la douleur ne nous a pas été épargnée. Nous n'oublierons jamais l'empressement avec lequel le Comité a tenu à faire sienne la pieuse pensée de rendre présente ici, grâce au grand artiste qu'est M. Charles Münch, l'âme de l'enfant disparu dont toi, mon cher Tresse, vous, mon cher Julia, et vous, mon cher Dieudonné, avez su évoquer la mémoire en termes si émouvants. La cérémonie de ce matin, où vous avez tenu à ne pas dissocier l'homme du professeur et du savant, nous a donné à ma femme et à moi les plus grandes joies qui puissent encore nous être réservées.

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APPENDIX D

# The Influence of France in the Development of Mathematics[1]

Like any science, mathematics is a common, international possession; it is the commonwealth that belongs to all developed nations, the commonwealth to which every nation contributes according to its abilities. It would be unacceptable if any well-regarded mathematician would decline to pay awed respect to the great foreign minds of the past: Galilei from Italy, Newton from England, Euler from Switzerland, Abel from Norway, Leibniz, Gauss, and Riemann from Germany, to mention but the most significant. They opened new routes in different fields of the science that, without them, would not be what it is today. However, I hope to make you realize that the French mathematicians made one of the most noteworthy contributions to the development of mathematics, and that, when it comes to the number of great mathematical minds, France does not take second place to any other nation. I am honored and pleased to be given this opportunity to talk about this particular subject in front of a friendly audience and in a country tied with my own by many common memories.

In mathematics, as in any other science, there are two kinds of scientists: those who open royal avenues by coming up with new ideas, usually simple ones but nevertheless ones that have not occurred to anyone else; and those who, on the vast land cleared by the first, till their own gardens, often picking tasty fruits, and sometimes collecting magnificent harvests. When it comes to the development of any science, the latter are not simply significant but rather indispensable; however, it is clear that the names of the former are those that are remembered and honored. Those are the people about whom I speak today.

Joseph Bertrand tells us that, at a Fontainebleau reception for the Dutch ambassador, King Henri IV took pleasure in recalling great Frenchmen who, by their achievements in literature and art, exceeded their foreign rivals. "Those I myself admire," said the Dutchman, by training a mathematician

[1]This talk was presented by Élie Cartan in the French Institute in Belgrade, Yugoslavia, on February 27, 1940. The talk was translated from French into Serbian by Milorad B. Protić, published in 1940 in the Yugoslavian journal *Saturn* and in 1941 as a separate book with the introduction written by Mihailo Petrović (see [190]). For this Appendix the lecture was translated from Serbian into English by Dr. Jelena B. Gill, who also wrote all footnotes.

whose field was geometry, "but I must notice that, so far, France failed to produce any mathematicians." "Romanus se trompe!" cried Henri IV and, having at once turned to one of the servants, asked that M. de la Bigottiere be brought in. The first great French mathematician, M. de la Bigottiere—whose real name was François Viète (1540–1603)—was the founder of modern algebra. He was the first to realize that the procedure for solving special numeric equations would be simplified if the operational symbolism—whose beginnings can be traced back to the ancient times—was applied to letters as well; also, he deserves most of the credit for the systematic development of that idea, and he predicted its unbounded expansion. At the end of the sixteenth century, when Galilei and an advanced geometry school brought fame to Italy, it was François Viète who secured for France a distinguished place in the process of founding modern mathematics. I should tell you that, for quite some time, Viète was in contact with one of your first mathematicians, Marin Getaldić (1566–1626), who was born in Dubrovnik and who, in Paris, in the year 1600, published one of Viète's last works.

For France, the seventeenth century was particularly glorious. In the history of mathematics, mechanics, and physics, three names from this period especially stand out: Descartes, Pascal, and Fermat.

A philosopher, mathematician, and physicist, René Descartes (1596–1650) is frequently considered the originator of a new era in the history of the human mind. As a physicist, he witnessed a defeat of his attempts to explain the world; however, his idea that all physical phenomena can be expressed in terms of space and motion has retained its attractiveness until the present day, because the founder of the general theory of relativity himself believed that it may be possible to interpret physics by using geometric terms (it was nothing but the past development of mathematics that enabled Einstein to carry his ideas further than Descartes could have). Even if we deny him credit for the creation of analytical geometry (1637), we must not undermine his role in mathematics. It is known that Greek geometers freely used numbers and computations in their thinking, but for them the numbers had not yet completely lost the geometric character they had in hellenistic science; as the words "square" and "cube" stand for both the numbers and the geometric forms it is clear that the common speech of today still shows traces of this double use. Descartes was the first to use abstract numbers systematically to represent geometric forms and to convert geometric reasoning into computations. In that way he created an extraordinarily powerful tool. To him we must ascribe the growth of geometry that stemmed primarily from analytical and differential geometry; he enriched the latter with a general method for finding tangents of algebraically defined curves. Thanks to analytical geometry, mathematicians not only succeeded in understanding a space of any number of dimensions but also learned to think geometrically in such a space. It is possible to say that it is in fact analytical geometry that taught mathematicians to feel comfortable in, for example, a spheric three-dimensional space, i.e., the one that, only recently, physicists started using to explain physical phenomena. All of this represents, although remote, nevertheless unquestionable consequences of Descartes's ideas and results. In algebra, it is to him that we owe the rule about the signs. In pure geometry, he should be credited with a theorem that, having been independently discovered by Euler, now bears Euler's name. A result of *analysis situs*[2], a science unknown at the time, this theorem establishes the relationships between the number of vertices, edges, and sides of a convex polyhedron. Finally, in mechanics, Descartes's principle of conservation of linear momentum provides an illustration of the intuition that required nothing more than a proper refinement to bring about one of the basic principles of classic mechanics.

Even in his early youth, Blaise Pascal (1623–1662), a somewhat strange but extraordinary genius, exhibited an unusual talent for geometry by writing, at the age of sixteen, *Traité sur les sections coniques*, a treatise about curves that are most frequently studied as flat conic section and play an important role in Kepler's planetary laws. Pascal used the results of his contemporary Gerald Desargues, who was one of the most significant French geometers and who, alongside Pascal, was a forefather of projective geometry. By taking, in a way similar to Desargues's, the perspective as a starting point, Pascal succeeded in reducing all properties of conic sections to a property that he called "*L'hexagramme mystique*": if a hexagon is inscribed into a cone, the three points at which pairs of opposite sides cross each other always lie on a straight line. Even by this result Pascal demonstrated the creative power of an eminent geometer.

As soon as Pascal the forefather of projective geometry established himself, Pascal the founder of mathematical probability took the stage. When his friend Chevalier de Méré asked him a couple of questions concerning a game of chance, Pascal answered them by reducing all possible outcomes to those most basic. Pierre de Fermat, on the other hand, came up with the same answer but in a completely different way. The evolution of the principles of mathematical probability is well illustrated in the letters exchanged between Pascal and Fermat. The scope of this new research did not escape Pascal: "By connecting the exactness of a mathematical approach with the uncertainty of chance," he was known to say, "the new science can rightly be given an astounding name—Geometry of Chance." From the famous betting proof, it is known to what extent his research and thinking were influenced by his interest in this new geometry. It is also known that this geometry played an instrumental role in the development of modern science, in which entire portions of physics are nothing but chapters of mathematical probability, and many of the laws of physics are nothing but laws of chance.

Pierre de Fermat (1601–1665), whom we mentioned earlier, is one of the greatest mathematical geniuses. He became a counselor of the parliament at

[2]The old name for topology.

the age of thirty and held that position until his death. Although his vocation did not predestine him for mathematical fame, he made sure to devote enough time to his favorite avocation. Fermat is especially famous for his research in arithmetic and number theory. On the margins of a copy of Diophantus's work about undefined equations (which was published in 1612 by Bache de Meriziac, the author of *Problèmes plaisants et délectables*) he wrote a number of important theorems without proofs; it is a matter of common belief that he was in possession of their proofs. The most famous among those theorems is the one frequently called Fermat's Last Theorem—according to which the sum of the $n$th degrees of two integers cannot equal the $n$th degree of a third integer for any integer $n$ that is greater than two. This theorem inspired a wealth of results whose authors, in spite of having at their disposal modern algebraic results that had been unknown to Fermat, have never been able either to prove or disprove it. It has been believed for a long time that, even if the theorem is wrong in general, it might in fact be wrong only for some values for $n$; however, it is by no means known if the number of the values for which it is wrong is finite or infinite. Through the research prompted by this single theorem—conducted in nearly all mathematically developed theories—Fermat influenced the growth of number theory. His contemporaries readily recognized his extraordinary skills in that field. In one of his letters, Pascal wrote that his own results in number theory were surpassed by Fermat's and that his was but to admire them.

The first half of the seventeenth century was an era of strong advancement of integral and differential calculus. With respect to integral calculus (determining areas and volumes, finding centers of gravity), it is enough to mention Cavalieri[3] and de Roberval[4]. As Fermat's own research, however, went quite far in this field, we are indebted to him for the classical integration procedures. On the other hand, once while trying to fight a tremendous toothache by solving roulette problems, Pascal accidentally discovered a procedure for obtaining integrals of higher powers of trigonometric functions. The names of those whom we have been talking about are found in differential calculus as well (the tangent problem). By his method "*de maximis et minimis*", Fermat introduced the notion of an infinitesimally small number. Lagrange and Laplace considered Fermat to be the actual founder of infinitesimal calculus, while Emile Picard[5] believed Pascal's works about roulette to represent the beginnings of integral calculus. Originally, Leibniz scribbled his formulae of infinitesimal calculus on a copy of one of Pascal's manuscripts, which, as he himself put it, had suddenly showed him the way.

It would be unfair to conclude the account of these great minds without mentioning that, at the age of twenty-eight, Pascal constructed the first

[3]Bonaventura Cavalieri (1598–1647).
[4]Gilles Personne Roberval (1602–1675).
[5]Charles Emile Picard (1856–1941).

arithmetic machine, capable of adding and subtracting. Due to his work *Traité de l'équilibre des liqueurs,* Pascal can be considered—together with Archimedes—one of the founders of hydrostatics; this is why it comes as no surprise that the barrel he used to check what is today known as Pascal's Principle is displayed next to his death mask in the little chapel erected in the churchyard of Port Royal. Finally let me mention the experiments concerning atmospheric pressure, which, it is suspected, he conducted under the influence of Mersenne[6], the soul of a small group of philosophers, mathematicians, and physicists that, before the creation of the Academy of Sciences in 1666, represented the first small but lively academy.

Those were fortunate times when one and the same man could be accomplished in philosophy, mathematics, and physics, and when a philosopher such as Malebranche[7] could have the extraordinary feeling that colors might be related to the number of vibrations of which light is composed!

## II

The second half of the seventeenth and the beginning of the eighteenth century were dominated by Christian Huygens (1629–1695) from the Netherlands, Isaac Newton (1642–1727) from England, and Gottfried Wilhelm von Leibniz (1646–1716) from Germany. It should be enough to mention that the last two are credited with the discovery or, rather, the systematization of infinitesimal calculus, while the first is famous for his works in differential geometry, rational and applied mechanics, and especially his works concerning the theory of light (in which he originated and developed an undulatory theory as opposed to Newton's particle theory). In this period, a remarkable scientific revolution was triggered by Newton's proof that stars and objects on Earth move according to the same laws of mechanics, namely, that one and the same law, the law of gravitation, explains the motion of planets, the moon, and comets as well as the existence of Earth's gravity, high and low tide, and so on. It was Newton's genius that created an entirely new science—celestial mechanics. But even if the earliest beginnings of this science did take place in England, it was France that provided a particularly fertile soil for its future development. To realize this, it is enough to recall the names of those whose works contributed the most to its growth: Clairaut, d'Alembert, Euler, Lagrange, Laplace, Gauss, Cauchy, Poisson, Le Verrier, Tisserand, and finally and especially—Henri Poincaré.

I pause for a moment on the first of them, Clairaut. The second in a family of twenty-one children, with a father who was a teacher of mathematics, Alexis Claude Clairaut (1713–1765) demonstrated talents similar to those of Pascal; however, unlike Pascal, his first works in no way revealed the significance of those that followed. He sent his first announcement to the Academy of Sciences before reaching the age of thirteen, and addressed an

[6]Marin Mersenne (1588–1648).

[7]Nicolas de Malebranche (1638–1715).

article about lines with double curvatures at the age of sixteen. He was eighteen when, against the existing rules, the king named him a member of the Academy of Sciences, the Division of Mechanics. I shall refrain from telling you about his research in the field of pure mathematics in general and about the part connected with solving differential equations in particular—the latter of which should be well known to all who studied differential equations—and focus instead on those results that made him famous. Newton and Huygens came up not only with a theoretical proof that, instead of being a perfect sphere, the earth is a sphere flattened at the poles, but also with a way to calculate the measure of flatness. However, when in 1701, at the Pyrénées, Cassini[8] determined the degree of arc of the Paris meridian, their conclusions came to be questioned. After debates that were occasionally confusing but always lively, in 1736 the Academy of Sciences decided to launch, under the guidance of de Maupertuis[9], an expedition that would travel to Lapland to determine the degree of the Lapland meridian arc. Working under very hard conditions, which were further complicated by snow and polar night, the team—which included Clairaut as well—came up with a numerical value that was remarkably larger than the one Cassini had obtained in France, hence proving beyond any doubt that the earth is indeed flattened at the poles. Understandably, de Maupertuis won laurels for the success of the expedition: with his head wrapped in a bear skin, his hand pressing against a globe, he posed for a portrait. But Clairaut continued to think about a possible cause of the earth's polar flatness and tried theoretically to determine the shape that a fluid planet would assume under the influence of Newton's attraction. The results of his research were published in 1743 in *La Théorie de la Figure de la Terre*, the book that d'Alembert characterized as a classical account of everything that had been done by that time, the account that marked an important date in the history of celestial mechanics. In addition, Clairaut explained the motion of the moon and in so doing contributed to Newton's lunar theory. He summarized his results from this field in *Théorie de la Lune*, a book published in 1732, to which, two years later, he added numerical tables, which, as Fontaine had put it, made it possible to find out "every step that the moon makes in the sky". A few years later, by predicting the next return of Halley's comet, Clairaut reached popular recognition and fame. After explaining that the perturbations caused by Saturn would delay the return of Halley's comet for about one hundred days and the influence of Jupiter would delay it for an additional five hundred and eighteen days, he predicted that its next passage through the perihelion would occur around April 13, 1759, but cautioned that, due to numerous other factors that he had to neglect, this date might be off by up to one month—indeed, Halley's comet passed through the perihelion on March 13, 1759. Almost one century

[8]Jacques Cassini (1677–1756).

[9]Pierre Louis Moreau de Maupertuis (1698–1759).

later, by determining the position of an until-then-unknown planet that had been the main cause of the disturbance of Uranus, French astronomer Le Verrier[10] attained nearly the same glory.

III

The second half of the eighteenth century was dominated by Euler and Lagrange and, in a somewhat lesser degree, by d'Alembert.

Leonhard Euler (1707–1783), "the prince of mathematicians", was born in Basel and spent part of his life in St. Petersburg and Berlin. His genius glowed in all areas of mathematics, and his work has had significant and lasting influence. I will always remember the delight I experienced while reading his *Introduction to the infinitesimal analysis*, the book that was given to me as an award at the end of my final year of gymnasium: it opened a whole new world in front of me, preparing me to understand better the lectures I would attend at the Sorbonne and in l'École Normale.

Jean Le Rond D'Alembert (1717–1783) left his trace in many different areas of mathematics. A well-known algebraic theorem that bears his name asserts that the total number of solutions (real and complex) of a rational equation equals the highest degree of the variable. Although d'Alembert's proof of this result was wrong, it should be mentioned that Euler's proof, based on completely different principles, was not without flaws. Only when the famous mathematician Gauss entered the mathematical scene was a correct proof found, and only with Cauchy's appearance was a real and very simple justification of this theorem established. In analysis I shall mention only the first correct formulation—which came from d'Alembert—of a partial differential equation describing vibrations of strings. And finally, it is well worth mentioning that, in mechanics, d'Alembert came up with a principle—nowadays known as d'Alembert's principle—which paved the way for Lagrange's analytical mechanics.

Joseph Louis Lagrange (1736–1813) was born in Torino, in a French family; although, like Euler, he spent a few years in Berlin, in 1787 he made his permanent home in Paris, entitling France to consider him one of her very own most celebrated minds. He is truly one of the most significant mathematicians of all times. He worked in all fields of mathematics. In the theory of numbers he proved Fermat's theorem for the power four. In algebra, through developing a unique method for solving a polynomial equation by reducing it to an equation of a lower degree, he cleared a path for Abel, Gauss, and Galois; in addition, he demonstrated that polynomial equations of the fifth degree cannot be solved in the way used for solving those of the third and fourth degree. In analysis, he gave the method for solving partial differential equations of the first order and came up with the notion of a singular solution. In function theory, he attempted but did not quite succeed in establishing a rigorous foundation for infinitesimal calculus, the area whose

[10]Urbain Jean Joseph Le Verrier (1811–1877).

principles had not yet been developed with desired exactness but whose consequences were nevertheless trusted. However, in spite of this lack of full success, his method of considering functions in an abstract way, independent of their geometric or mechanical meaning, had remarkable influence in preparing the terrain for the modern theory of functions. Lagrange's talent for generalizing became truly obvious in his works concerning the calculus of variations.

The calculus of variations was developed during the eighteenth century, through the works of Bernoulli and Euler, both from Switzerland. Its roots are in some problems of geometry and mechanics, the simplest of which might be the problem of determining the shortest path between two points on the same surface; here, the unknown quantity is not a number but, much more complexly, a line consisting of infinitely many points. De Maupertuis was the one who, by his Principle of Least Action, reduced the problem of determining a trajectory of a particle in a given force field to a problem of maxima and minima, giving special importance to this kind of calculus. It should not be forgotten, however, that by that time Fermat had already reduced the laws of optics to a similar principle, according to which the path chosen by light is the shortest in terms of time. By applying the infinitesimal variation on an unknown line and by showing how that variation can be calculated, Lagrange introduced a general method into a theory in which nearly every problem required a special procedure in order to be solved.

I shall omit Lagrange's work in celestial mechanics and, instead, devote more time to his most significant work, *Mécanique Analytique* (1788). Galilei, Descartes, Huygens, Leibniz, Newton, and d'Alembert gradually developed all of the grand principles of modern mechanics. But the problem of determining the trajectory of a system governed by given forces was frequently complicated by the necessity to take into account unknown relations between the forces. With ingenious intuition, in the case without friction Lagrange completely removed the difficulty and gave a general procedure for determining equations that would give the trajectory in question: to achieve this it is enough to determine the active force of that system as well as the work of that force for an infinitely small movement of the system. Aside from practical importance, this wonderful creation has remarkable philosophical importance because it completely illuminates everything that is, from the point of view of mechanical properties, important in a system of particles. In this respect, Lagrange's genius is equal to that of Descartes, the creator of analytical geometry.

The so-called Lagrange's equations in *Mechanique Analytique* represented an analytical model for various mechanical explanations of certain physical theories. From that point of view this work has great philosophical significance; but, although it is the most important work of the nineteenth century, it created the impression that everything can be explained by the principles of mechanics—an impression as erroneous as Descartes's belief that everything can be explained in terms of geometry—which is the reason that, today, it is completely abandoned. Nevertheless, it illustrates the ability of mathematics to provide physicists with the tools they require to carry out their theories.

Some of the extraordinary minds were inclined to see danger in the manufacturing of structures (similar to the one created by Lagrange) that offered insights into infinite arrays of phenomena; they feared that such structures might cause a loss of connection with reality. For instance, the great geometer Poncelet, known for his works in mechanics, avoided using Lagrange's method and, instead, preferred following to the last detail the influences and interactions of various forces in order to determine, step by step, their actual works. The same type of skepticism prevented Poncelet from using analytical geometry and prompted him, instead, to examine directly relations between various geometric figures by applying principles of classic geometry. With respect to accepting the latest results, there are indeed two kinds of minds, both equally important for the development of science and both found among great French mathematicians.

## IV

Visible as early as the end of the eighteenth century, the French superiority in mathematics became especially clear during the French Revolution and at the beginning of the nineteenth century. Among the great names of that era one must include Monge, Laplace, and Legendre.

Pierre Simon de Laplace (1749–1827) owed his reputation to his research in celestial mechanics, summarized in his charming treatise *Exposition du Système du Monde.* The peculiar result stating that even the finest details of almost all celestial phenomena can be explained evolved into scientific determinism, according to which, in order to be able to determine positions and velocities of cosmic particles at a given time, it is enough to know their positions and velocities at any other time, provided it is known, in addition, which principles regulate the forces—modeled after the forces of Newton's gravitation—that the particles are governed by. For a long time mathematical physics developed according to this result; only recently, electromagnetism and atomic physics succeeded in proving it to be wrong. Still, this result had strong influence on the development of science. A very significant treatise, *Théorie Analytique des Probabilities* (1812), is another one for which we are grateful to Laplace; the most important part of this work deals with the application of the notion of probability in the theory of least squares, the possibility of which had been indicated by Legendre. While studying the inclination of an ellipsoid, Laplace introduced spherical functions by means of which one can express any function dependent on a point on a sphere. We should not forget Laplace's famous equation which is satisfied by Newton's potential function; this equation is of extraordinary importance in many problems of analysis, geometry, mechanics, and physics.

Adrien Marie Legendre (1752–1833) is responsible for the rejuvenation of number theory, previously successfully treated by Euler. Although Euler was the first one to publish the reciprocity law in arithmetic, Legendre explained it clearly and partially proved it; the law is named after Legendre. Gauss was the third mathematician discover this same law, but the first one to construct a correct and complete proof. Legendre's significant work of several years, *Sur les Intégrales Elliptique*, a tract in two volumes, was published in 1825 and 1826. There he presented a complete study of integrals involving square roots of fourth-degree polynomials and developed different forms that can be given to them. Although with this work Legendre became a forefather of the marvelous theory of elliptical functions, he let Jacobi and Abel take credit for its founding. Finally, let us mention his *Éléments de Géométrie* (1794), a work which had numerous editions and which, in schools of the Anglo-Saxon countries, soon replaced Euclid's theory; in the history of the non-Euclidean geometries, this work had definitive importance.

Gaspard Monge (1746–1818) was one of the best French geometers. There are two reasons why. First, by founding modern projective geometry, he joined the long process of development of perspective, the theory whose principles had been known to Italian renaissance painters, which Desargues and Pascal applied to the theory of conic sections, and which, following the previous two, the French geometer de la Hire[11] expanded to the theory of poles and polars of a circle. Monge systematized projective geometry and enriched it with constructions on surfaces that are not flat. On the other hand, by his treatise *Applications de l'Analyse à la Géométrie* he gave a substantial boost to differential geometry, the field that was separated from Descartes's analytical geometry by Euler's and Meusnier's significant works concerning the properties of surfaces; it is Monge to whom we are indebted for the notion of measure of curvature, as well as for its application in stereometry; it was his idea to characterize a vast family of surfaces by obtaining them as a solution set of a single partial differential equation. He managed to integrate the equation of minimal surfaces, surfaces which have been and still are an object of important research, and which had been obtained first in Plateau's experiments. Monge presented his theories during his lectures at l'École Normale—the school founded in 1795 as a convent—as well as at l'École Polytechnique (at which Lagrange and Laplace taught as well). I am pleased to have a chance to mention Dupin[12], for he was one of the numerous students with whom Monge worked; Dupin is known for his work *Développement de Géométrie*, in which he introduced the notions of conjugated tangents and indicatrix at a point of a surface; also, Dupin can be considered a creator of a new branch of geometry.

[11] Phillipe de la Hire (1640–1718).

[12] Francois Pierre Charles Dupin (1784–1873).

## V

The most remarkable names in France during the first half of the nineteenth century were those of Fourier, Cauchy, Poncelet, and Galois. Although quite different from each other, they all cleared new paths in science.

Jean Baptiste Joseph Fourier (1768–1830) can be considered the founder of mathematical physics. I shall neglect his important results in algebra and instead tell you about *Théorie Mathématique de la Chaleur*, the work he did not publish until 1822 but which must have been in his thoughts since at least 1807. With this work Fourier opened up a new field in mathematical analysis. "Unknown to the ancient geometers, and for the first time used by Descartes for researching curved lines and surfaces," Fourier says, "analytical equations are by no means limited to these general phenomena. Since mathematical analysis determines the most diverse relations and measures time, space, forces and temperature, it is safe to say that it is as wide and rich as Nature itself. It always follows the same paths and gives the same interpretations, in that way certifying about the unity, simplicity and stability of the Universe." It should not be forgotten that, according to Fourier, the richest source of all mathematical discoveries lies in the study of nature. As, for instance, the mathematical theory of heat had a significant influence on the development of pure mathematics, we may say that Fourier's viewpoint was correct. Created by Fourier to help him integrate frequently encountered partial differential equations, the theory of trigonometric series prompted incredibly many articles, all of which were trying to establish a rigorous foundation for this theory as well as to complete and further develop it. The basic problem that needed to be solved was determining which functions can be represented in the form of a Fourier series. As even many of Fourier's own examples were peculiar, it did not take much to make the mathematicians truly puzzled, in a way in which a musician would be puzzled upon discovering that, by combining finite or infinite numbers of pure sounds and their various multiples (harmonics), it is possible to create any disconnected sequence of sounds. These unusual results forced mathematicians to check once more and specify the notion of a function and to start thinking, bit by bit, about the foundations of their own science. This is what brought about unbelievable consequences which have not yet fully presented themselves. Group theory—a field which so frequently failed mathematicians and which caused many paradoxes that, I am afraid, have not yet been successfully resolved—was one of the branches of mathematics that eventually evolved from these efforts; another branch that had its origins in the same efforts is the theory of functions of one real variable, a creation of French mathematics from the end of the nineteenth and the beginning of the twentieth century.

Augustin Cauchy (1789–1857), an extraordinarily fruitful theorist, was successful in all areas of mathematics: number theory, geometry, analysis and celestial mechanics. Unlike Euler, he did not explore series without first finding out whether they made sense, that is, whether they were convergent; in that way, we may say, he opened up the era of exactness. Cauchy discovered the general rule—later found independently by J. Hadamard[13] —which explains how to determine those values of the variable for which a power series is convergent. Creation of a theory of functions with a complex (or imaginary) variable is another of Cauchy's great accomplishments. For more than three centuries, imaginary quantities were a scandal in mathematics. They were encountered for the first time in the sixteenth century, by Italian algebraists, in the formula for the roots of a third-degree equation in the paradoxical case when all of the roots are real. But, once researchers got adjusted to these new quantities and learned how to use them, it was easy to determine important results concerning real numbers, some of which could not have been obtained in any other way. Sometime toward the end of the eighteenth century, the Swiss mathematician Argand explained the secret of imaginary quantities by finding their importance in the possibility of expressing a vector in a plane whenever one needed to give not only the length of the vector but its orientation as well. When Cauchy started representing a point in the plane by just one imaginary (or, better, complex) quantity instead of two real coordinates, he got the idea of a function with a complex variable, a function which would assign one point in the plane to another point in the plane. In this way Cauchy created a whole new world. The elements of that world are perfectly organized: just as Cuvier[14] was able to reconstruct a creature from the antediluvial era from just one piece of its skeleton, a mathematician became able to reconstruct one of Cauchy's functions, provided he knew its values at every point of the arc, no matter how small the arc might be. The perfect order in this world, its marvelous harmony, and—with the exclusion of number theory—a long sequence of theorems determining properties of functions and their numerous applications, all leave the most magnificent impression.

As Cauchy created the right conditions for more discoveries than he could have possibly anticipated, the significance of his opus should be measured by the length of the sequence of works concerning functions of a complex variable. A single theorem from this sequence, whose beauty is in its simplicity, was nearly enough to immortalize the name of Liouville[15]. Another theorem on the same subject—named after Emile Picard, perhaps the greatest among the living mathematicians—opened vast and until-then hidden horizons, and created a stream of articles that has not yet ceased.

By using a viewpoint different from Cauchy's, the German mathematician Weierstrass also developed a theory of functions of a complex variable. For a long time it had been believed that the viewpoint one chose was irrelevant,

[13]Jacques Salomon Hadamard (1865–1963).

[14]Georges Cuvier (1769–1832), a French naturalist.

[15]Joseph Liouville (1809–1882).

but Borel[16] demonstrated—in one of his most charming results—that this was not true, and that Cauchy's viewpoint penetrates deeper into the heart of the matter. Borel indeed took out of the plane so much that no circle, regardless of how small, was left intact, and yet inside of what remained he managed to construct a function that, although satisfying all of Cauchy's requirements, did not satisfy Weierstrass's definition, that conditions the existence of a function of a complex variable by the existence of an intact portion of the plane. By starting his celebrated collection of monographs about the theory of functions—the collection whose past and present contributors include mathematicians from all countries—Borel himself contributed a lot to the development of functions of a complex variable.

With Jean Victor Poncelet (1788–1867) we enter the era of pure geometry. Poncelet is considered the founder of projective geometry, the field whose subject is studying those properties of objects that do not change in projections. He is the one who discovered the new and very useful notion of transformations by means of reciprocal polars, the transformations which make it possible to derive one flat figure from another, with a provision that, peculiarly, the sides of the new figure correspond to the vertices of the old one, and vice versa. Frequently, a transformation of this type makes it possible to explore the properties of some figure by reducing them to the easier-to-explore properties of another. Somewhat later, Gergonne[17] used this to derive the duality principle, a principle very important in projective geometry. Finally, Poncelet was the one who discovered the continuity principle, according to which if a figure had a certain property, it will retain the same property even after being deformed, provided that the ratios between its various elements were taken into account. By many simple examples Cauchy proved that this principle, as formulated by Poncelet, was wrong; however, if formulated in a slightly different and much more precise way, this principle is in fact correct. Being very helpful, this principle is frequently used. In geometry, Poncelet's influence was remarkable: in Germany, Steiner and Staudt owe the existence of their works to Poncelet; in France, Chasles[18], the first member of the department of higher geometry at the Sorbonne, was the most outstanding representative of modern pure geometry. To Chasles we are indebted for the important historical monument *L'Aperçu historique sur le Développement de la Géométrie*, which led to the correction of a certain number of wrong opinions.

Before ending our discussion of Poncelet, I note that he played an important role in developing applied mechanics, which he taught for a long time, first in Metz and then at the Sorbonne.

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[16] Emile Borel (1871–1956).

[17] Joseph Dιez Gergonne (1771–1859).

[18] Michel Chasles (1793–1880).

Evariste Galois (1811–1832) is one of the most unusual figures in the history of science. Having twice failed the entrance exam at l'École Polytechnique, in 1831 he was accepted to l'École Normale, only to leave it a year later. Taking an active part in politics earned him several months in prison; not quite twenty-one years old, he was killed in a duel triggered by an insignificant quarrel. He had presented his mathematical discoveries in equation theory to the Academy of Sciences in two different announcements, but both of them were later lost; fortunately, he had also published them in several small articles in *Bulletin de Ferussac* in 1830 and also talked about them to his friend Chevalier in a letter written shortly before his death. Some other results, discovered among his papers, were published in 1846, in Liouville's magazine.

The significance of his work can be explained quickly. Tartaglia, Cardano, and Ferrari, Italian algebraists of the sixteenth century, used the second and third roots to solve equations of the third and fourth degree; however, all efforts to solve equations of higher degrees in the same way were in vain. By showing that some classes of equations can indeed be solved in that same way, Lagrange, Abel, and Gauss contributed a great deal to this problem. Abel first showed, in 1826, that a general equation of the fifth degree cannot be solved by means of radicals. In that way it became clear that the problem, with which mathematicians had wrestled since the sixteenth century, had not been well formulated. The glory for solving it belongs to Galois, for he showed that each equation determines a certain number of permutations of its roots, the permutations forming a so-called group; although applied to the roots, these permutations do not disrupt their rational interactions (the meaning of the term "rational interactions" needs an additional explanation). The nature of that group determines the basic properties of the equation, whether it is possible to find its roots or not, and, in a general case, the nature of auxiliary equations whose solving would result in solving the original equation. By starting from his own idea, Galois easily found the results of his predecessors and successfully incorporated them into his own result.

The theory of substitution groups, i.e., groups of permutations of a certain number of objects, which was founded by Cauchy, demonstrated its full value through Galois's works. Galois improved its important aspects and demonstrated how basic was the role of ordinary groups. Moreover, he enriched number theory by introducing new classes of imaginary quantities (Galois's imaginary numbers), each of which was tied to a power of a prime number; Galois's name is frequently encountered not only in the theory of equations but also in modern algebra. The letters he sent to his friend Chevalier make it clear that in analysis he had as many important results as in algebra and that his works on Abel integrals were twenty-five years ahead of those of the famous German mathematician Riemann. Although it makes me sad to think how much science lost by Galois's early death, I must also say that, as Emile Picard once put it, "When confronted with such a short and turbulent life, one's respect for the extraordinary mind which left so deep trace in science gets even greater."

It was Galois's theory that made it possible to explain the miracle which allowed imaginary quantities to appear in the formula for solving a third-degree equation with real roots; indeed, it became possible to show that, if an equation has all roots real and if it can be solved by means of radicals, then it can be solved by means of square roots only. By using the same theory, it can also be shown that some of the ancient problems—such as the problem of doubling a cube or the problem of trisecting an angle—cannot be solved with a ruler and a compass. By his significant work *Traité des Substitutions*, Jordan[19] erected a monument in honor of Galois.

Being both simple and profound, Galois's main idea permitted applications in areas other than algebraic equations. Emile Picard and Ernest Vessiot, for example, considered it highly important in integration of linear differential equations. It is noteworthy that Drach and Vessiot attempted to extend Galois's theory to solving the most general differential equations but encountered difficulties that could be overcome only if the original theory were altered or if, at least, some of its magnificent simplicity were sacrificed.

The development of science after Galois demonstrated the growth of the importance of groups in the most diverse branches of mathematics and physics. Norwegian mathematician Sophus Lie, the founder of the theory of groups of transformations, introduced them into analysis and geometry. A great admirer of Galois, he dedicated his momentous opus about groups of transformations (in 1889) to l'École Normale Supérieure. Indeed, the most significant results concerning developing, refining, extending, and finding new applications of Galois's theory were made in France. Poincaré claimed that the notion of group had already existed in the spirit of geometry; the axiom that two geometric figures are equal to each other if each of them is equal to a third is in fact identical to the statement that there is a group that regulates geometry, more precisely a family of procedures by which one figure turns into another that is equal to the first. It is extraordinarily important that group theory is capable of giving us all concrete, connected meanings that can be given to the expression "equal figures"; as it was shown in 1872 by the great German mathematician Felix Klein, exactly this implies the existence of infinitely many geometries, each ruled by a special group, as well as by the fact that each geometry can be investigated independently, without resorting to elementary geometry. This framework encompasses projective geometry, the field in which two figures are considered equal if one of them can be obtained from the other by a sequence of projections.

## VI

Since Galois's death one century has passed. During that period mathematics has developed remarkably; innumerable volumes have been written,

[19] Camille Jordan (1838–1922).

some of which, I must say, take undeserved space in libraries. Some of the theories, just formulated at the time of Galois, have since been profoundly explored, and some of them have penetrated other areas of mathematics; in a word, as mathematics, like other sciences, has been constantly and dramatically changing, it became difficult for a mathematician, no matter who he might be, to have true insight into its current state. There are fewer and fewer minds capable of making significant discoveries in either pure or applied mathematics. It is rare to encounter a genius similar to that of the Frenchman André Ampère (1775–1836), who was also a physicist, the founder of electrodynamics, and a remarkable mathematician (he and Monge share the credit for creating the theory of partial differential equations of the second order). The Frenchman Gabriel Lamé (1795–1870) was an analyst, geometer, and the founder of elasticity theory, while the Frenchman Siméon Poisson (1781–1840) is famous for his works in analysis and mathematical physics; Augustin Fresnel (1788–1827), the creator of physical optics—whose works had finally ensured, at least until the appearance of quantum physics, a triumph of the modular theory of light—can be considered a mathematician as well.

Instead of giving you a long, and likely tedious, list of names, let us focus on just a few of the greatest contemporary French mathematicians, those who were my professors and to whom I am honored and happy to have a chance to pay respect.

Soon after being admitted to l'École Polytechnique, Charles Hermite (1822–1901) wrote to the well-known professor Jacobi—who, along with Abel, was one of the founders of the theory of elliptical functions—and sent him an article about classifying Abel's transcendental functions, the functions related to integration of the most general algebraic differentials. Jacobi, who was once, under similar circumstances, kindly received by Legendre, congratulated the young Hermite on his marvelous results. That was only the beginning of regular correspondence between these two great mathematicians. It was Jacobi to whom, at the age of twenty-four, Hermite sent his discoveries in advanced algebra, the discoveries that ultimately secured him a place among the most prominent geometers. Building on the most famous Gauss's results, he confidently approached the algebraic theory of shapes in their most general form and introduced continuous variables into number theory, a field characterized by discontinuity. The fact that he was the one who introduced quadratic forms with indefinite conjugate terms, today known as Hermite's forms, is the reason that his name is one of the most frequently found in works from quantum physics. In 1873, Hermite became famous by discovering the transcendentality of $e$, the base of Neper's logarithm (the existence of transcendentals, the numbers that satisfy no algebraic equation whose coefficients are rational numbers, had first been demonstrated by Joseph Liouville). As Hermite's result made a strong impression, some expected him to prove transcendentality of $\pi$, and thus, consequently, to destroy forever the hope that a circle can be squared with a ruler and a compass; however, having found inspiration in Hermite's method and having devised a way to modify it properly, Ferdinand Lindemann, a German mathematician, came up with a proof instead, securing the honor for himself.

Hermite always left a profound impression on his listeners. "No one will ever forget the sermon-like sound of Hermite's lectures," said the well-known mathematician Painlevé[20], "or the feeling of beauty and revelation that one had to experience while listening to him talk about a marvelous discovery or something that was still waiting to be discovered. His word had the ability to open vast horizons of science; it conveyed affection and respect for high ideals." Every time I had a chance to listen to Hermite, I had before me an image of quiet and pure joy caused by contemplations about mathematics, joy similar to the one that Beethoven must have felt while feeling his music inside of himself.

Gaston Darboux (1847–1917) was an analyst and geometer at the same time. Although he was the initiator of some results in analysis, I shall not talk about that part of his work because it was his work in geometry that brought him recognition. He surely was not one of the geometers who avoided tarnishing the beauty of geometry by flattering analysis, and neither was he one of the analysts inclined to reduce geometry to calculations without any concern for or interest in their geometric meanings. In this respect he followed in Monge's footsteps, connecting fine and well-developed geometric intuition with skilled applications of analysis. All of his methods are extraordinarily elegant and perfectly suited for the subject under investigation. While teaching in the department of higher geometry at the Sorbonne, where he succeeded Michel Chasles, he frequently and with reverence spoke about the theory of triple orthogonal systems, with pleasure stressing the importance of Lamé's works; not less frequently he spoke about the theory of deformations of planes, the theory which originated in Gauss's *Disquisitiones circa Superficies Curvas* and which, even before Darboux, was a subject of significant works of French mathematicians, among whom Ossian Bonnet certainly deserves a mention. Finally, Darboux demonstrated the usefulness of a system of local coordinates, i.e., coordinates connected with the investigated figure rather than independent of it. Thanks to the theory of groups, Élie Cartan further developed this approach and adapted it to the most diverse spaces created as a consequence of general relativity theory. Darboux had tremendous influence on the development of geometry; of his numerous students and followers, I shall mention only the well-known Roumanian geometer Tzitzeica, one of the founders of the Mathematical Reviews of the Balkan Union, a man whose recent death is still mourned in the world of science. Classic in its field, Darboux's work *Théorie des Surfaces* is a splendid monument erected in honor of both analysis and geometry.

[20]Paul Painléve (1863–1933).

A story has it that, when a young German mathematician expressed his puzzlement over Lagrange's refusal to recognize Gauss as the greatest German geometer, Lagrange told him, "No, he cannot be the greatest German geometer for he is the greatest European geometer!" In the same spirit one could say that Henri Poincaré (1854–1912) was not only a great mathematician but mathematics itself. It is impossible to find a branch of mathematics—a branch of physics even—in which he did not leave a trace or which he did not rejuvenate or from which he did not infer a completely new field. After creating Fuchsian functions[21], he used uniform functions with the same parameter to express the coordinates of a point on an algebraic surface, and in that way obtained the result which, before him, was known only for some special classes of surfaces. He solved the uniformization problem in a way that, at the time, was quite brave. He was a forerunner of the theory of functions with several complex variables. Also, he created the theory of differential equations in a real field; due to that theory, he was then able to restore the methods of celestial mechanics, to study periodic solutions of problems of this field, and to investigate stability problems. In *analysis situs*, the part of geometry interested only in those properties of objects that are not affected by continuous transformations, Poincaré authored several treatises that would become the starting point for nearly all later results in that field. At the Sorbonne, by lecturing on all areas of mathematical physics, he influenced the ideas triggered by Michelson's experiment[22]. With his early death, science lost one of its most prominent leaders. Translated to many languages, his scientific-philosophical works *La Science et l'Hypothèse* and *La Valeur de la Science* are well known to the entire world. In some ways—one of which is well illustrated by Poincaré's words, "Thought is only a flash in the middle of a long night, but the flash that means everything"—Poincaré can be compared with Pascal. It will take a long time to develop all of Poincaré's ideas and to explore all of the paths that he had paved by his rich and diverse work.

Finally,I would like to mention Paul Appell and Edouard Goursat—the first of whom is the author of *Traité de Mécanique Rationnelle*, and the second of *Traité de Calcul Différentiel et Intégral*—and also, once again, Emile Picard, the last living from that celebrated generation. Two years ago, together with the great German mathematician David Hilbert, Emile Picard received a gold medal from the Mittag-Leffler Institute, and only several weeks ago, at the celebration of the fifty years since Picard was elected a member of the Academy of Sciences, Emile Borel talked about his scientific opus. I already mentioned the famous theorem named after him, as well as those among his works that developed Galois's theory. His work concerning algebraic functions with two variables represents the foundation of algebraic geometry, a

[21] It was Poincaré himself who named them this way after the German mathematician Lazarus Fuchs; nowadays, these functions are called automorphic.

[22] Also known as the Michelson-Morley experiment.

branch of geometry especially well developed in Italy. It is true that the viewpoint of Italian geometers from the past century was more clearly defined than that of Emile Picard, but, as Emile Borel put it, algebraic geometry would be certainly crippled without Picard's contributions.

## VII

The glory of French mathematics created by the greatest results of Hermite, Darboux, Poincaré, and Picard has not darkened. Indeed, the flame is as strong as it has ever been. As the time is short, to justify this statement I am forced to limit myself to just a few names.

Gabriel Kœnigs was a fine geometer; the elegance of some of his works can be compared with that of Darboux. By creating new transcendentals, Paul Painlevé solved a problem that even to Poincaré seemed unapproachable; Poincaré characterized Painlevé's results in analysis by saying: "Mathematics is a well-ordered continent whose countries are united; the work of Paul Painléve is a magnificent island in an ocean." But this judgment is somewhat incomplete because Painlevé—who, for a long time, taught mechanics at l'École Polytechnique—also remarkably advanced mechanics; besides, his theoretical research prompted development of aviation in such a measure that one may say that, thanks to Painlevé, aviation is an exclusively French creation.

The results of Jacques Hadamard were numerous and significant: in arithmetic, he worked on the Riemann's function related to the complicated problem of distribution of prime numbers; in geometry, he researched geodesic lines with opposite curvatures; in analysis, he published works about partial differential equations in mathematical physics. Also, he gave a strong stimulus to the calculus of variations and functional analysis, the new science founded by the Italian mathematician Volterra. Finally, his seminar at Collège de France, where all foreign mathematicians wished to present their latest results, influenced international collaboration in mathematics. As he is still young, I may say with certainty that his work is far from finished.

The research of functions with complex variables has always been very successful in France. Here I mention Emil Borel; the short-lived analyst Fatou; Paul Montel, famous for his theory concerning families of normal functions; Gaston Julia, known for his works about elevation of rational functions; and so forth.

The theory of functions with real variables is of almost exclusively French origin. Set up by Camille Jordan's *Traité d'Analyse* (which, like Emile Picard's treatise of the same name, had international influence), founded by the works of Emile Borel, Henri Lebesgue (who defined measure of a set), René Baire (who introduced integrals which today bear his name), and Denjoy (the creator of the totalization theory), it introduced unexpected harmony into a field that had been neglected for a long time, testimony to the daring and talent of its creators.

I cannot but mention Maurice Fréchet's theory of abstract spaces, Bouligand's infinitesimal geometry, and Élie Cartan's works in analysis and geometry, the last of which I am not qualified to judge.

Institut Henri Poincaré is born from new French enthusiasm for research in the field of mathematical physics. Emile Borel, the soul of probability theory, started a praise-deserving series of publications in this field, similar to the one in function theory—a series in which Fréchet, Paul Levy, and Georges Darmois presented their excellent results. The Department of Theoretical Physics is headed by Louis de Broglie, the creator of wave mechanics, who restored atomic physics and reconciled the undulatory and corpuscular theory of light. I should not forget to mention the Institut of Mechanics, headed by Henri Villat, known for his results in hydrodynamics, who is also editor of the internationally known collection *Mémorial des Sciences Mathématiques* and editor-in-chief of *Journal de Mathématiques Pures et Appliquées*, a journal which, nearly a century ago, was started by Liouville and which for quite some time was edited by Camille Jordan.

The account of French mathematical activity would be incomplete without a mention of l'École Polytechnique and l'École Normale. For more than a century, great French mathematicians have owed their education to one of the two institutions; in the last half-century that marvelous role belonged almost exclusively to l'École Normale, which, even a good fifty years ago, Sophus Lie considered a nursery of French mathematics. Young talents from many countries have been coming here to get the same education as their French colleagues. That is why it is difficult not to consider Georges Tzitzeica, whom I already mentioned, to be a French mathematician. For the same reason, I am inclined to include among French mathematicians my good friend Mihailo Petrovic, a doyen of Yugoslav mathematics, who is widely recognized for his great originality in inventing the spectral method in arithmetic, algebra, and analysis, and also for creating general phenomenology, the field which systematically examines the problems of existence of analytical molds that could be used to present simultaneously several apparently different physical theories. I hope that you will not object if I credit his results to the accomplishments which mathematics owes to France.

Thanks to l'École Normale, young mathematicians are ready to replace the older ones. One might say that it is too early to mention names, but some of them are nevertheless already well known. I shall mention only Jacques Herbrand, whose works, mercilessly interrupted by his early death, were announcing a great mathematician, perhaps similar to Evariste Galois.

Ladies and gentlemen, it is time for me to finish this talk, for I have already used a great deal of your kind attention. In conclusion, I would like to make just one remark of general nature.

More than any other science, mathematics develops through a sequence of consecutive abstractions. A desire to avoid mistakes forces mathematicians to find and isolate the essence of the problems and entities considered. Carried to an extreme, this procedure justifies the well-known joke according to which a mathematician is a scientist who neither knows what he is talking about or whether whatever he is indeed talking about exists or not. French mathematicians, however, never enjoyed distancing themselves from reality; they do know that, although needed, logic is by no means crucial. In mathematical activity, like in any other type of human activity, one should find a balance of values: there is no doubt that it is important to think correctly, but it is even more important to formulate the right problems. In that respect, one can freely say that French mathematicians not only always knew what they were talking about, but also had the right intuition to select the most fundamental problems, those whose solutions produced the strongest influence on the overall development of science.

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