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Masahisa Adachi

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Volume 124

# Embeddings and Immersions 

Masahisa Adachi
Translated by Kiki Hudson

# 埋め込みとはめ込み 

# UMEKOMI TO HAMEKOMI（Embeddings and Immersions） <br> by Masahisa Adachi 

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## Translated from the Japanese by Kiki Hudson

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## Preface to the English Edition

For the convenience of readers of this English edition I have replaced the original Japanese references with the appropriate references in English or French. I have also replaced some other references that are hard to obtain with those that are more readily available.

I wish to sincerely thank Professors Kobayashi and Nomizu for their advice. I am very grateful to Dr. Kiki Hudson, who has provided an excellent translation and pointed out misprints in the original edition.

Masahisa Adachi
October 28, 1992

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## Preface

Among closed surfaces the torus $T^{2}=S^{1} \times S^{1}$ can be thought of as sitting in three-dimensional Euclidean space $\mathbf{R}^{3}$, but the Klein bottle $K^{2}$ cannot be realized there. This observation naturally leads us to the question 'can a general $n$-dimensional manifold $M^{n}$ be smoothly embedded in Euclidean space $\mathbf{R}^{p}$ ?'.

Further, it is possible to embed the circle $S^{1}$ in three-dimensional Euclidean space $\mathbf{R}^{3}$, but there is more than one way to do so. For example, we cannot move one of the two embeddings below to the other via an isotopy; that is, we cannot undo the knot.


This is generalized to the problem 'are two given embeddings $f, g: M^{n} \rightarrow$ $\mathbf{R}^{p}$ isotopic?' Since the concept of topology was first established, these problems have been in its mainstream, and major contributions to solutions have come from H. Whitney and A. Haefliger. Still further research and development can be expected in this field.

In particular, the problem of classifying embeddings of the circle $S^{1}$ in three-dimensional Euclidean space $\mathbf{R}^{3}$ or the three-dimensional sphere $S^{3}$ through isotopies-a bit different from the isotopies mentioned in the previous paragraph -forms a field in topology called the theory of knots, which even today generates many research activities.

The problem of classifying immersions by regular homotopies is slightly easier than that of classifying embeddings by isotopies. Here is an example. In three-dimensional Euclidean space $\mathbf{R}^{3}$, is it possible to turn the sphere $S^{2}$ inside out smoothly allowing self-intersections? Think about it for a minute. It hardly seems likely, but a classification theorem for immersions shows that it can be done.

This classification theorem, the so-called Smale-Hirsch theorem, has been generalized step by step by A. Phillips, M. Gromov, A. Haefliger, and so on to the present stage where it now offers us a tool for finding solutions
(or their candidates) to partial differential inequalities or partial differential equations of certain types. It also provides us with a method for eliminating singularites of certain $C^{\infty}$ maps. There are further applications of these methods as well.

The aim of this book is to give an introduction to this theory in modern topology and its applications. In accordance with the principle of this series we have tried to make the first three chapters easy enough to understand at the level of lower-division mathematics.

In this book, unless otherwise stated, embeddings and immersions will be viewed in the $C^{\infty}$ category. We first explain in detail the classification of regular closed curves in the plane by regular homotopies; this will serve as an intuitive preparation for the contents of the book.

In Chapter I, we give a summary of basic concepts about $C^{r}$ manifolds and $C^{r}$ maps which will be used in Chapter II and beyond.

The discussions in Chapter II evolve around Whitney's theorems. This chapter also serves as a prelude to Chapter VII. We develop Chapter III around the Smale-Hirsch theorem which is generalized to Gromov's theorem.

In Chapter IV we examine the convex integration theory due to Gromov which is another application of the Smale-Hirsch theorem.

In Chapter V we discuss an application of Gromov's theorem, namely, a classification theorem for foliations of open manifolds. In Chapter VI we study complex structures on open manifolds as an application of Gromov's theorem and Gromov's convex integration theory.

We study Haefliger's embedding theorem in Chapter VII, which is a continuation of Chapter II.

Finally, as references we give a list of books and papers we have either used, adapted, or quoted from directly, and also books and papers basic to embeddings and immersions.

The author thanks Kazuhiko Fukui, Shigeo Kawai, and Goo Ishikawa for their valuable help in writing this book.

We are deeply indebted to Professor Itiro Tamura who encouraged us to write this book and gave us valuable advice concerning the first draft.

Last but not least our deepest gratitude goes to Mr. Hideo Arai of Iwanami Shoten Publishers, without whose help this book would never have been realized.

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## Afterword

Chapter 0 featured Whitney [C20] as a means for giving the reader an intuitive preview of the book.

In Chapter I we dealt with the fundamentals of differential topology- $C^{r}$ manifolds, $C^{r}$ maps, fiber bundles, and other related concepts; we limited our discussion both in selection and scope to the parts essential for this book. For more in-depth information on the subject, we recommend Differentiable manifolds by Y. Matsushima. ${ }^{1}$ )
In Chapter II we discussed embeddings of manifolds, our focal point being Whitney's embedding theorems [C19], [C21]. Here we used, among other references, the book of Tamura. $\left({ }^{2}\right)$ Our topics in this chapter included a method for eliminating double points in completely regular immersions. We saved Haefliger's generalization of this method for Chapter VII.

The theme of Chapter III was immersions of $C^{\infty}$ manifolds. Here our discussion centered around the Smale-Hirsch theory-a natural generalization of (the topic in) Chapter 0-and included the theorems of Phillips and Gromov. The theorem of Gromov encompasses the submersion theorem of Phillips and is a generalization of the Smale-Hirsch theorem. Consequently, we presented the proofs of these theorems as corollaries of Gromov's theorem. For the proof of Gromov's theorem we followed Haefliger's presentation of the subject [B3]. We also mentioned that Gromov founded his theorem on Smale's "homotopy covering technique" [C17], "taking an idea out of an old wise man's paper". (*)

In Chapter IV we introduced yet another generalization of the SmaleHirsch theory. Gromov's integration theory [C5], unlike his theorem in Chapter III, does not require openness for the base spaces of jet bundles. This constitutes an essential difference between these two works. Here we used a report paper of Shigeo Kawai. In our opinion this chapter points to a promising future direction for the subject of this book.

We presented in Chapter V an application-Haefliger's classification the-

[^0]orem for foliations of open manifolds-of the theorem of Gromov in Chapter III. We recommend Tamura [A9] for the fundamentals of foliation theory.

We devoted Chapter VI to complex structures on open manifolds as an application of the theorems of Gromov in Chapters III and IV. The integrability of almost complex structures on open manifolds of arbitrary dimensions remains unsolved to date.

We gave an outline of a proof of Haefliger's embedding theorem in Chapter VII. As we mentioned above, we had wished to shed some light on the fact that Haefliger's proof is a natural extension of Whitney's method as used in eliminating double points of immersions; we are somewhat dubious as to what extent we succeeded in doing so. We feel that finding a sufficient condition, independent of connectivity, for the existence of embeddings of manifolds is a major open problem in this field.

We did not mention Haefliger's alternative proof for his embedding theorem [C8]. This is based on the so-called Whitney-Thom theory concerning singular sets of differentiable maps.

We also omitted any solid application of the Smale-Hirsch theory to manifolds; for this we refer the reader to Smale [B9] and James [B4].

From the historical perspective we marvel at the evolution of the simple problem of Chapter 0 as developed throughout our book to its present stage, and we expect further progress in the future.

Addendum. A videotaped version of the main topic of Chapter 0 is available. $\left({ }^{3}\right)$

[^1]
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[^0]:    $\binom{1}{2}$ Y. Matsushima, Differentiable manifolds, Marcel Dekker, New York, 1972.
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    $\left(^{*}\right)$ Editor's note. The phrase in quotes was added in translation by the author.

[^1]:    $\left({ }^{3}\right)$ Regular homotopies in the plane, International Film Bureau Inc., Chicago, Illinois.

