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V. V. Prasolov

American Mathematical Society

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Problems and Theorems in Linear Algebra

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10.1090/mmono/134

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ЗАДАЧИ И ТЕОРЕМЫ
ЛИНЕЙНОЙ АЛГЕБРЫ

Translated by D. A. Leites from an original Russian manuscript
Translation edited by Simeon Ivanov

2000 *Mathematics Subject Classification.* Primary 15–01.

ABSTRACT. This book contains the basics of linear algebra with an emphasis on nonstandard and neat proofs of known theorems. Many of the theorems of linear algebra obtained mainly during the past thirty years are usually ignored in textbooks but are quite accessible for students majoring or minoring in mathematics. These theorems are given with complete proofs. There are about 230 problems with solutions.

Library of Congress Cataloging-in-Publication Data

Prasolov, V. V. (Viktor Vasil′evich)
[Zadachi i teoremy lineĭnoĭ algebry. English]
Problems and theorems in linear algebra/V. V. Prasolov; [translated by D. A. Leites from an original Russian manuscript; translation edited by Simeon Ivanov]
p. cm. — (Translations of mathematical monographs, ISSN 0065-9282; v. 134)
Includes bibliographical references.
ISBN 0-8218-0236-4
1. Algebras, Linear. I. Ivanov, Simeon. II. Title. III. Series.

QA184.P7313 1994 94-13332
512′.5—dc20 CIP

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10 9 8 7 6 5 4 13 12 11 10 09 08

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PREFACE

There are very many books on linear algebra, among them many really wonderful ones (see the list of recommended literature). One might think that no more books on this subject are necessary. Choosing the words more carefully, it is possible to deduce that these books contain all that one needs and in the best possible form, and therefore any new book will, at best, only repeat the old ones.

This opinion is manifestly wrong, but nevertheless almost ubiquitous.

New results in linear algebra appear constantly and so do new, simpler and neater proofs of the known theorems. Besides, more than a few interesting old results are ignored by textbooks.

In this book I tried to collect the most attractive problems and theorems of linear algebra still accessible to students majoring in mathematics.

The computational aspects of linear algebra were left somewhat aside. The major part of the book contains results known from journal publications only. I believe that those results will be of interest to many readers.

I assume that the reader is acquainted with the main notions of linear algebra: linear space, basis, linear map, and the determinant of a matrix. Apart from this, all the essential theorems of the standard course of linear algebra are given here with complete proofs, and some definitions from the above list of prerequisites are recollected. I placed the prime emphasis on nonstandard neat proofs of known theorems.

In this book I only consider finite dimensional linear spaces.

The exposition is mostly performed over the fields of real or complex numbers. The peculiarity of the fields of finite characteristics is mentioned when needed.

Cross-references inside the book are natural: 36.2 means subsection 2 of §36; Problem 36.2 is Problem 2 from §36; Theorem 36.2.2 stands for Theorem 2 from 36.2.

ACKNOWLEDGMENTS. This book is based on a course I read at the Independent University of Moscow, 1991/92. I am thankful to the participants for comments and to D. V. Beklemishev, D. B. Fuchs, A. I. Kostrikin, V. S. Retakh, A. N. Rudakov, and A. P. Veselov for fruitful discussions of the manuscript.

For the second printing (1996), the author added the proof of the Kronecker theorem for pairs of linear maps (Sec. 12.6) and corrected several small errors.

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MAIN NOTATIONS AND CONVENTIONS

$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$ denotes a matrix of size $m \times n$; we say that a square $n \times n$ matrix is of *order* n;

a_{ij}, sometimes denoted by $a_{i,j}$ for clarity, is the element or the entry from the intersection of the ith row and the jth column;

(a_{ij}) is another notation for the matrix A;

$\left\|a_{ij}\right\|_p^n$ still another notation for the matrix (a_{ij}), where $p \leq i, j \leq n$;

$\det(A)$, $|A|$ and $\det(a_{ij})$ all denote the *determinant* of the matrix A;

$|a_{ij}|_p^n$ is the *determinant* of the matrix $\left\|a_{ij}\right\|_p^n$;

E_{ij} — the (i, j)th *matrix unit* — the matrix whose only nonzero element is equal to 1 and occupies the (i, j)th position;

AB — the product of a matrix A of size $p \times n$ by a matrix B of size $n \times q$ — is the matrix (c_{ij}) of size $p \times q$, where $c_{ik} = \sum\limits_{j=1}^{n} a_{ij} b_{jk}$, is the scalar product of the ith row of the matrix A by the kth column of the matrix B;

$\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the *diagonal matrix* of size $n \times n$ with elements $a_{ii} = \lambda_i$ and zero offdiagonal elements;

$I = \operatorname{diag}(1, \dots, 1)$ is the *unit matrix*; when its size, $n \times n$, is needed explicitly we denote the matrix by I_n;

the matrix aI, where a is a number, is called a *scalar* matrix;

A^T is the transposed of A, $A^T = (a'_{ij})$, where $a'_{ij} = a_{ji}$;

$\overline{A} = (a'_{ij})$, where $a'_{ij} = \overline{a_{ij}}$;

$A^* = \overline{A}^T$;

$\sigma = \binom{1 \dots n}{k_1 \dots k_n}$ is a permutation: $\sigma(i) = k_i$; the permutation $\binom{1 \dots n}{k_1 \dots k_n}$ is often abbreviated to $(k_1 \dots k_n)$;

$$\operatorname{sign} \sigma = (-1)^\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd;} \end{cases}$$

$\operatorname{Span}(e_1, \dots, e_n)$ is the linear space spanned by the vectors $e_1, \dots, e_n$.

Given bases $e_1, \dots, e_n$ and $\varepsilon_1, \dots, \varepsilon_m$ in spaces V^n and W^m, respectively, we assign to a matrix A the operator $A : V^n \longrightarrow W^m$ which sends the vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ into the vector $\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Since $y_i = \sum_{j=1}^{n} a_{ij} x_j$, we have

$$A\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} x_j \varepsilon_i;$$

in particular, $Ae_j = \sum_i a_{ij}\varepsilon_i$;

everywhere except for §37 the notations $A > 0$, $A \geq 0$, $A < 0$, or $A \leq 0$ mean that a real symmetric or Hermitian matrix A is positive definite, nonnegative definite, negative definite, or nonpositive definite, respectively; $A > B$ means that $A - B > 0$; *whereas in* §37 *they mean that* $a_{ij} > 0$ *for all* i, j, *etc.*

Card M is the cardinality of the set M, i.e, the number of elements of M;

$A|_W$ denotes the restriction of the operator $A : V \longrightarrow V$ onto the subspace $W \subset V$;

sup is the least upper bound (supremum);

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ denote, as usual, the sets of all integer, rational, real, complex, quaternion and octonion numbers, respectively;

$\mathbb{N}$ denotes the set of all positive integers (without 0);

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

CHAPTER I

DETERMINANTS

The notion of a determinant appeared at the end of the 17th century in works of **Leibniz** (1646–1716) and a Japanese mathematician, **Seki Kova**, also known as **Takakazu** (1642–1708). Leibniz did not publish the results of his studies related to determinants. The best known is his letter to **l'Hôspital** (1693) in which Leibniz writes down the determinant condition of compatibility for a system of three linear equations in two unknowns. Leibniz particularly emphasized the usefulness of two indices when expressing the coefficients of the equations. In modern terms he actually wrote about the indices i, j in the expression $x_i = \sum_j a_{ij} y_j$.

Seki arrived at the notion of a determinant while solving the problem of finding common roots of algebraic equations.

In Europe, the search for common roots of algebraic equations soon also became the main trend associated with determinants. Newton, Bezout, and Euler studied this problem.

Seki did not have the general notion of the derivative at his disposal, but he actually got an algebraic expression equivalent to the derivative of a polynomial. He searched for multiple roots of a polynomial $f(x)$ as common roots of $f(x)$ and $f'(x)$. To find common roots of polynomials $f(x)$ and $g(x)$ (for f and g of small degrees) Seki got determinant expressions. The main treatise by Seki was published in 1674; there applications of the method are published, rather than the method itself. He kept the main method secret confiding only in his closest pupils.

In Europe, the first publication related to determinants, due to **Cramer**, appeared in 1750. In this work Cramer gave a determinant expression for a solution of the problem of finding the conic through 5 fixed points (this problem reduces to a system of linear equations).

The general theorems on determinants were proved only *ad hoc* when they were needed to solve some other problem. Therefore, the theory of determinants developed slowly, left behind as compared with the general development of mathematics. A systematic presentation of the theory of determinants is mainly associated with the names of **Cauchy** (1789–1857) and **Jacobi** (1804–1851).

1. Basic properties of determinants

The *determinant* of a square matrix $A = \|a_{ij}\|_1^n$ is the alternated sum

$$\sum_{\sigma} (-1)^{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)},$$

where the summation is over all permutations $\sigma \in S_n$. The determinant of the matrix $A = \|a_{ij}\|_1^n$ is denoted by $\det A$ or $|a_{ij}|_1^n$. If $\det A \neq 0$, then A is called *invertible* or *nonsingular*.

over $\mathbb{C}$ the polynomials f and g have a nontrivial common divisor and, therefore, f and g have a nontrivial common divisor, r, over $\mathbb{Q}$ as well. Since f is an irreducible polynomial with the leading coefficient 1, it follows that $r = \pm f$. □

A.2. THEOREM (Eisenstein's criterion). *Let*

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

be a polynomial with integer coefficients and let p be a prime such that the coefficient a_n is not divisible by p whereas $a_0, \dots, a_{n-1}$ are, and a_0 is not divisible by p^2. Then the polynomial f is irreducible over $\mathbb{Z}$.

PROOF. Suppose that $f = gh = (\sum b_k x^k)(\sum c_l x^l)$, where g and h are not constants. The number $b_0 c_0 = a_0$ is divisible by p and, therefore, one of the numbers b_0 or c_0 is divisible by p. Let, for definiteness sake, b_0 be divisible by p. Then c_0 is not divisible by p because $a_0 = b_0 c_0$ is not divisible by p^2. If all numbers b_i are divisible by p, then a_n is divisible by p. Therefore, b_i is not divisible by p for some i with $0 < i \le \deg g < n$.

We may assume that i is the least index for which the number b_i is nondivisible by p. On the one hand, by the assumption, the number a_i is divisible by p. On the other hand, $a_i = b_i c_0 + b_{i-1} c_1 + \cdots + b_0 c_i$ and all numbers $b_{i-1} c_1, \dots, b_0 c_i$ are divisible by p whereas $b_i c_0$ is not divisible by p. Contradiction. □

COROLLARY. *If p is a prime, then the polynomial $f(x) = x^{p-1} + \cdots + x + 1$ is irreducible over $\mathbb{Z}$.*

Indeed, we can apply Eisenstein's criterion to the polynomial

$$f(x+1) = \frac{(x+1)^p - 1}{(x+1) - 1} = x^{p-1} + \binom{p}{1} x^{p-2} + \cdots + \binom{p}{p-1}.$$

A.3. THEOREM. *Suppose the numbers*

$$y_1,\ y_1^{(1)},\ \dots,\ y_1^{(\alpha_1 - 1)},\ \dots,\ y_n,\ y_n^{(1)},\ \dots,\ y_n^{(\alpha_n - 1)}$$

are given at points $x_1, \dots, x_n$ and $m = \alpha_1 + \cdots + \alpha_n - 1$. Then there exists a polynomial $H_m(x)$ of degree not greater than m for which $H_m(x_j) = y_j$ and $H_m^{(i)}(x_j) = y_j^{(i)}$.

PROOF. Let $k = \max(\alpha_1, \dots, \alpha_n)$. For $k = 1$ we can make use of the Lagrange interpolation polynomial

$$L_n(x) = \sum_{j=1}^{n} \frac{(x - x_1) \dots (x - x_{j-1})(x - x_{j+1}) \dots (x - x_n)}{(x_j - x_1) \dots (x_j - x_{j-1})(x_j - x_{j+1}) \dots (x_j - x_n)} y_j.$$

Let $\omega_n(x) = (x - x_1) \dots (x - x_n)$. Take an arbitrary polynomial H_{m-n} of degree not greater than $m - n$ and assign to it the polynomial $H_m(x) = L_n(x) + \omega_n(x) H_{m-n}(x)$. It is clear that $H_m(x_j) = y_j$ for any polynomial H_{m-n}. Besides,

$$H'_m(x) = L'_n(x) + \omega'_n(x) H_{m-n}(x) + \omega_n(x) H'_{m-n}(x),$$

i.e., $H'_m(x_j) = L'_n(x_j) + \omega'_n(x_j)H_{m-n}(x_j)$. Since $\omega'_n(x_j) \neq 0$, then at points where the values of $H'_m(x_j)$ are given, we may determine the corresponding values of $H_{m-n}(x_j)$. Furthermore,

$$H''_m(x_j) = L''_n(x_j) + \omega''_n(x_j)H_{m-n}(x_j) + 2\omega'_n(x_j)H'_{m-n}(x_j).$$

Therefore, at points where the values of $H''_m(x_j)$ are given we can determine the corresponding values of $H'_{m-n}(x_j)$, etc. Thus, our problem reduces to the construction of a polynomial $H_{m-n}(x)$ of degree not greater than $m-n$ for which $H^{(i)}_{m-n}(x_j) = z_j^{(i)}$ for $i = 0, \dots, \alpha_j - 2$ (if $\alpha_j = 1$, then there are no restrictions on the values of H_{m-n} and its derivatives at x_j). It is also clear that $m - n = \sum(\alpha_j - 1) - 1$. After $k-1$ of similar operations it remains to construct Lagrange's interpolation polynomial. □

A.4. Hilbert's Nullstellensatz. We will only need the following special case of *Hilbert's Nullstellensatz.*

THEOREM. *Let $f_1, \dots, f_r$ be polynomials in n indeterminates over $\mathbb{C}$ without common zeros. Then there exist polynomials $g_1, \dots, g_r$ such that $f_1g_1 + \dots + f_rg_r = 1$.*

PROOF. Let $I(f_1, \dots, f_r)$ be the ideal of the polynomial ring $\mathbb{C}[x_1, \dots, x_n] = K$ generated by $f_1, \dots, f_r$. Suppose that there are no polynomials $g_1, \dots, g_r$ such that $f_1g_1 + \dots + f_rg_r = 1$. Then $I(f_1, \dots, f_r) \neq K$. Let I be a nontrivial maximal ideal containing $I(f_1, \dots, f_r)$. As is easy to verify, K/I is a field. Indeed, if $f \notin I$, then $I + Kf$ is the ideal strictly containing I and, therefore, this ideal coincides with K. It follows that there exist polynomials $g \in K$ and $h \in I$ such that $1 = h + fg$. Then the class $\overline{g} \in K/I$ is the inverse of $\overline{f} \in K/I$.

Now, let us prove that the field $A = K/I$ coincides with $\mathbb{C}$.

Let α_i be the image of x_i under the natural projection

$$p : \mathbb{C}[x_1, \dots, x_n] \longrightarrow \mathbb{C}[x_1, \dots, x_n]/I = A.$$

Then

$$A = \left\{\sum z_{i_1\dots i_n}\alpha_1^{i_1}\dots\alpha_n^{i_n} \mid z_{i_1\dots i_n} \in \mathbb{C}\right\} = \mathbb{C}[\alpha_1, \dots, \alpha_n].$$

Further, let $A_0 = \mathbb{C}$ and $A_s = \mathbb{C}[\alpha_1, \dots, \alpha_s]$. Then $A_{s+1} = \{\sum a_i\alpha^i_{s+1} | a_i \in A_s\} = A_s[\alpha_{s+1}]$. Let us prove by induction on s that there exists a ring homomorphism $f : A_s \longrightarrow \mathbb{C}$ (which sends 1 to 1). For $s = 0$ the statement is obvious. Now, let us show how to construct a homomorphism $g : A_{s+1} \longrightarrow \mathbb{C}$ from the homomorphism $f : A_s \longrightarrow \mathbb{C}$. For this let us consider two cases.

a) The element $x = \alpha_{s+1}$ is transcendental over A_s. Then for any $\xi \in \mathbb{C}$ we can define a homomorphism g such that $g(a_nx^n + \dots + a_0) = f(a_n)\xi^n + \dots + f(a_0)$. Setting $\xi = 0$ we get a homomorphism g such that $g(1) = 1$.

b) The element $x = \alpha_{s+1}$ is algebraic over A_s, i.e., $b_mx^m + b_{m-1}x^{m-1} + \dots + b_0 = 0$ for certain $b_i \in A_s$. Then for all $\xi \in \mathbb{C}$ such that $f(b_m)\xi^m + \dots + f(b_0) = 0$ there is determined a homomorphism $g(\sum a_kx^k) = \sum f(a_k)\xi^k$ which sends 1 to 1.

As a result we get a homomorphism $h : A \longrightarrow \mathbb{C}$ such that $h(1) = 1$. It is also clear that $h^{-1}(0)$ is an ideal and there are no nontrivial ideals in the field A. Hence, h is a monomorphism. Since $A_0 = \mathbb{C} \subset A$ and the restriction of h to A_0 is the identity map h is an isomorphism.

Thus, we may assume that $\alpha_i \in \mathbb{C}$. The projection p maps the polynomial $f_i(x_1, \dots, x_n) \in K$ to $f_i(\alpha_1, \dots, \alpha_n) \in \mathbb{C}$. Since $f_1, \dots, f_r \in I$, we have $p(f_i) = 0 \in \mathbb{C}$. Therefore, $f_i(\alpha_1, \dots, \alpha_n) = 0$. Contradiction. □

A.5. THEOREM. *Let the polynomials* $f_i(x_1, \dots, x_n) = x_i^{m_i} + P_i(x_1, \dots, x_n)$, *where* $i = 1, \dots, n$, *be such that* $\deg P_i < m_i$ *and let* $I(f_1, \dots, f_n)$ *be the ideal generated by* $f_1, \dots, f_n$.

a) *Let* $P(x_1, \dots, x_n)$ *be a nonzero polynomial of the form* $\sum a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$, *where* $i_k < m_k$ *for all* $k = 1, \dots, n$. *Then* $P \notin I(f_1, \dots, f_n)$.

b) *The system of equations* $x_i^{m_i} + P_i(x_1, \dots, x_n) = 0 \quad (i = 1, \dots, n)$ *is always solvable over* $\mathbb{C}$ *and the number of solutions is finite.*

PROOF. Substituting the polynomial $(f_i - P_i)^{t_i} x^{q_i}$ instead of $x_i^{m_i t_i + q_i}$, where $0 \le t_i$ and $0 \le q_i < m_i$, we see that any polynomial $Q(x_1, \dots, x_n)$ can be represented in the form

$$Q(x_1, \dots, x_n) = Q^*(x_1, \dots, x_n, f_1, \dots, f_n) = \sum a_{js} x_1^{j_1} \dots x_n^{j_n} f_1^{s_1} \dots f_n^{s_n},$$

where $j_1 < m_1, \dots, j_n < m_n$. Let us prove that such a representation Q^* is uniquely determined. It suffices to verify that by substituting $f_i = x_i^{m_i} + P_i(x_1, \dots, x_n)$ in any nonzero polynomial $Q^*(x_1, \dots, x_n, f_1, \dots, f_n)$ we get a nonzero polynomial $\tilde{Q}(x_1, \dots, x_n)$. Among the terms of the polynomial Q^*, let us select the one for which the sum $(s_1 m_1 + j_1) + \dots + (s_n m_n + j_n) = m$ is maximal. Clearly, $\deg \tilde{Q} \le m$. Let us compute the coefficient of the monomial $x_1^{s_1 m_1 + j_1} \dots x_n^{s_n m_n + j_n}$ in $\tilde{Q}$. Since the sum

$$(s_1 m_1 + j_1) + \dots + (s_n m_n + j_n)$$

is maximal, this monomial can only come from the monomial $x_1^{j_1} \dots x_n^{j_n} f_1^{s_1} \dots f_n^{s_n}$. Therefore, the coefficients of these two monomials are equal and $\deg \tilde{Q} = m$.

Clearly, $Q(x_1, \dots, x_n) \in I(f_1, \dots, f_n)$ if and only if $Q^*(x_1, \dots, x_n, f_1, \dots, f_n)$ is the sum of monomials for which $s_1 + \dots + s_n \ge 1$. Besides, if $P(x_1, \dots, x_n) = \sum a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$, where $i_k < m_k$, then

$$P^*(x_1, \dots, x_n, f_1, \dots, f_n) = P(x_1, \dots, x_n).$$

Hence, $P \notin I(f_1, \dots, f_n)$.

b) If $f_1, \dots, f_n$ have no common zero, then by Hilbert's Nullstellensatz the ideal $I(f_1, \dots, f_n)$ coincides with the entire polynomial ring and, therefore, $P \in I(f_1, \dots, f_n)$; this contradicts part a). Therefore, the given system of equations is solvable. Let $\xi = (\xi_1, \dots, \xi_n)$ be a solution of this system. Then $\xi_i^{m_i} = -P_i(\xi_1, \dots, \xi_n)$, where $\deg P_i < m_i$, and, therefore, any polynomial $Q(\xi_1, \dots \xi_n)$ can be represented in the form $Q(\xi_1, \dots, \xi_n) = \sum a_{i_1 \dots i_n} \xi_1^{i_1} \dots \xi_n^{i_n}$, where $i_k < m_k$ and the coefficient $a_{i_1 \dots i_n}$ is the same for all solutions. Let $m = m_1 \dots m_n$. The polynomials $1, \xi_i, \dots, \xi_i^m$ can be linearly expressed in terms of the basic monomials $\xi_1^{i_1} \dots \xi_n^{i_n}$, where $i_k < m_k$. Therefore, they are linearly dependent, i.e., $b_0 + b_1 \xi_i + \dots + b_m \xi_i^m = 0$, not all numbers $b_0, \dots, b_m$ are zero and these numbers are the same for all solutions (do not depend on i). The equation $b_0 + b_1 x + \dots + b_m x^m = 0$ has, clearly, finitely many solutions. □

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SUBJECT INDEX

There are a number of very good books available on linear algebra. However, new results in linear algebra appear constantly, as do new, simpler, and better proofs of old results. Many of these results and proofs obtained in the past thirty years are accessible to undergraduate mathematics majors, but are usually ignored by textbooks. In addition, more than a few interesting old results are not covered in many books. In this book, the author provides the basics of linear algebra, with an emphasis on new results and on nonstandard and interesting proofs. The book features about 230 problems with complete solutions. It can serve as a supplementary text for an undergraduate or graduate algebra course.

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