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Masaru Takeuchi

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Modern Spherical Functions

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Translated by
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American Mathematical Society
Providence, Rhode Island

現代の球関数

GENDAI NO KYŪKANSŪ
(Modern Spherical Functions)

by Masaru Takeuchi

Translated from the Japanese by Toshinobu Nagura

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ABSTRACT. This book is an exposition of spherical functions on compact symmetric spaces from the viewpoint of Cartan-Selberg. In Chapter I we present spherical functions by making use of representation theory, invariant differential operators, and invariant integral operators. Chapter II treats compact symmetric pairs, the spherical representations for compact symmetric pairs, the fundamental groups of compact symmetric spaces, and the radial parts of invariant differential operators. Chapter III produces classical results for spheres and complex projective spaces by applying the arguments developed in the previous chapters and treats the relation between spherical functions and harmonic polynomials.

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Preface to the English Edition

This work was originally published in Japanese by Iwanami Shoten, Publishers in 1975. It was based on the lectures which I gave at Osaka University in 1970–1971 on spherical functions of symmetric spaces. The original text included several references in Japanese which I have replaced by English ones for this English translation.

The translator T. Nagura kindly pointed out errors in the proofs of some theorems, Theorems 8.1 and 11.2, in particular, and made corrections. Furthermore, he corrected a mistake in the formulation of Theorem 8.3 which I had made in the text, and he gave the proof of the revised theorem. I also owe thanks to the translator for further improvements in the exposition.

I would like to express my deep gratitude to T. Nagura for his valuable contributions.

Masaru Takeuchi
Osaka, January 1993

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Preface

Since the time of Legendre and Laplace in the 18th century, the theory of spherical functions has been studied by many mathematicians. In the 20th century E. Cartan dealt with spherical functions from the point of view of representation theory of Lie groups and obtained very beautiful theorems about spherical functions on compact symmetric spaces. Moreover, in the 1950's Selberg introduced the notion of weakly symmetric spaces which include symmetric spaces and developed the theory of spherical functions on these spaces together with applications to various fields.

In this book I have mainly expounded the spherical functions on compact symmetric spaces from the viewpoint of Cartan-Selberg. These lecture notes are based on the lectures that I gave at Osaka University.

The reader is assumed to have a general knowledge of manifolds and Lie groups. Furthermore, a general knowledge of some other fields is sometimes assumed, for example, representation theory of compact groups and semisimple Lie algebras, Riemannian geometry, and so on. To help the reader who is not familiar with these fields, I collected brief explanations of the prerequisites in the Appendix and included some textbooks in the References.

In concluding the preface, I would like to express my deep gratitude to Professor N. Iwahori and Professor S. Murakami for inviting me to write this book. I am very much indebted to Professor H. Ozeki for his critical reading of the manuscript and for his valuable suggestions for improvements. I would also like to thank Mr. H. Arai of Iwanami Shoten Publishers for his kind cooperation in all phases of the publication of this book.

Masaru Takeuchi
Osaka, October 1974

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Appendix

1. For basic facts on manifolds and Lie groups see Matsushima [19], Ise and Takeuchi [14], and Pontrjagin [21].

2. Actions of groups. Let G be a group, let X be a set, and let

$$\psi : G \times X \longrightarrow X, \qquad \psi(g, x) = gx$$

be a mapping of $G \times X$ to X. The mapping ψ is said to be a (*left*) *action* of G on X if the following conditions are satisfied:

(1) $ex = x, \qquad x \in X$;

(2) $(gh)x = g(hx), \qquad g, h \in G, \ x \in X$,

where e denotes the unit element of G. We say that G *acts* on X (*on the left*) if an action of G on X is given.

Suppose G acts on X. For $g \in G$ the mapping $\hat{g}$ of X to itself defined by the correspondence $x \mapsto gx$ is bijective. We say that G acts *trivially* on X if every $\hat{g}$, $g \in G$, is the identity mapping. The action is called *effective* if $\hat{g}$ is the identity mapping only for $g = e$. The action is called *transitive* if for any $x, y \in X$ there exists $g \in G$ such that $gx = y$. For example, if K is a subgroup of G, then G acts transitively on the set G/K of right cosets by the mapping

$$\psi : G \times G/K \longrightarrow G/K, \qquad \psi(g, hK) = (gh)K.$$

In general, for $x \in X$ we call the subset $Gx = \{gx : g \in G\}$ of X the *G-orbit through* x.

If G and X have certain structures, we often assume that the action preserves those structures. For example, if G is a topological group and X is a topological space, we assume that the action is continuous, and such an action is called a *continuous action.* If G is a Lie group and X is a manifold, we assume that the action is a C^∞ mapping, and such an action is called a C^∞ *action.* Moreover, if this action is effective, G is called a *Lie transformation group* of X. For example, given a Lie group G and its closed subgroup K, there exists a unique manifold structure on the set G/K with the property that the set of C^∞ functions on G/K is identified in a canonical manner with the set of all C^∞ functions f on G which satisfy $f(gk) = f(g)$ for any $g \in G$ and $k \in K$ (refer to Matsushima [19]). Relative to this structure the action ψ mentioned above is a C^∞ action.

Let us give another example. Suppose G acts on X. If we denote by $F(X)$ the set of $\mathbf{C}$-valued functions on X, then G acts on $F(X)$ by

$$(gf)(x) = f(g^{-1}x), \qquad g \in G, \ f \in F(X), \ x \in X.$$

When G acts on X, we usually consider that G acts on a space of functions on X in this manner.

Suppose G acts on X and Y. A mapping $f : X \to Y$ is said to be *G-equivariant* if $f(gx) = gf(x)$ for any $g \in G$ and $x \in X$.

A *right action* $X \times G \to X$ is a mapping which satisfies $xe = x$ and $x(gh) = (xg)h$. Further, for right actions similar notions are defined. For example, in a principal fibre bundle the structure group is usually considered to act on the right.

3. Invariant measures. Let X be a locally compact Hausdorff space, and let $\mathfrak{B}$ be the minimum completely additive class of subsets of X which contains all compact subsets of X. Let $\mathfrak{M}(X)$ be the set of all regular measures on $\mathfrak{B}$ which satisfy $\mu(C) < \infty$ for any compact subset C and $\mu(U) > 0$ for any nonempty open subset $U \in \mathfrak{B}$. Suppose a group G acts on X so that every $\hat{g}$, $g \in G$, is a homeomorphism of X. For $g \in G$ and $\mu \in \mathfrak{M}(X)$ define

$$(g\mu)(A) = \mu(g^{-1}A), \qquad A \in \mathfrak{B}.$$

Then $g\mu \in \mathfrak{M}(X)$ and G acts on $\mathfrak{M}(X)$ in this way. We say that $\mu \in \mathfrak{M}(X)$ is *G-invariant* if $g\mu = \mu$ for any $g \in G$. In particular, if G is a locally compact group, $X = G$, and the action of G on X is the left translation $(g, h) \mapsto gh$, then a G-invariant measure $\mu \in \mathfrak{M}(G)$ is called a *left-invariant Haar measure* on G. Similarly, a *right-invariant Haar measure* on G is defined by making use of the right translation. On a locally compact group there exists a unique left- (resp., right-) invariant Haar measure up to a positive constant factor.

A locally compact group G is said to be *unimodular* if a left-invariant Haar measure is also right-invariant. A compact group is unimodular. In general, for a homeomorphism φ of X and $\mu \in \mathfrak{M}(X)$ we can define $\varphi\mu \in \mathfrak{M}(X)$ as above. For a unimodular group G a Haar measure is also invariant under the homeomorphism φ defined by the correspondence $g \mapsto g^{-1}$. If G is a Lie group, then a left- (resp., right-) invariant Haar measure is the positive C^∞ measure (see §2) determined by a left- (resp., right-) invariant differential form of the highest degree, as is seen from the uniqueness of Haar measure. Therefore, a Lie group G is unimodular if and only if $|\det(\operatorname{Ad} g)| = 1$ for any $g \in G$. Hence, a connected Lie group G is unimodular if and only if $\operatorname{tr}(\operatorname{ad} X) = 0$ for any $X \in \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G. From this we see that if a connected Lie group is semisimple or nilpotent, it is unimodular (see **7**).

For invariant measures refer to Halmos [7].

4. G**-modules.** With the notation in **2**, if X is a vector space V over a field k and if every $\hat{g}$, $g \in G$, is a linear automorphism of V, then we say that G *acts linearly* on V and V is a *G-module.*

Let V_1 and V_2 be G-modules. A linear mapping φ of V_1 to V_2 is called a *G-linear mapping* or *G-homomorphism* if it is G-equivariant, that is, if $g\varphi(x) = \varphi(gx)$ for any $g \in G$ and $x \in V_1$. In particular, if φ is a linear isomorphism, then φ is called a *G-isomorphism.* We say that V_1 is *G-isomorphic* to V_2 if there exists a G-isomorphism between V_1 and V_2. A subspace U of a G-module V is called a *G-submodule* if U is invariant under the action of G. Then U itself is also a G-module. A G-module V is called *irreducible* if the only G-submodules of V are $\{0\}$ and V. If a G-module V is a direct sum of G-submodules $\{V_\lambda\}_{\lambda \in \Lambda}$, then V is called a *direct sum of* $\{V_\lambda\}_{\lambda \in \Lambda}$ as G-module.

If a group G acts linearly on a vector space V, G acts linearly on the dual space V^* of V by

$$(g\xi)(x) = \xi(g^{-1}x), \qquad g \in G, \ \xi \in V^*, \ x \in V.$$

This action is called the *contragradient action* of the former one. For G-modules V_1 and V_2 we can define a linear action of G on the tensor product $V_1 \otimes V_2$ by

$$g(x_1 \otimes x_2) = gx_1 \otimes gx_2, \qquad g \in G, \ x_1 \in V_1, \ x_2 \in V_2.$$

This G-module $V_1 \otimes V_2$ is called the *tensor product* of V_1 and V_2. Let G_1 and G_2 be groups, let V_1 be a G_1-module, and let V_2 be a G_2-module. Then we can define an action of the direct product $G_1 \times G_2$ on $V_1 \otimes V_2$ by

$$(g_1, g_2)(x_1 \otimes x_2) = g_1x_1 \otimes g_2x_2, \qquad g_i \in G_i, \ x_i \in V_i \quad (i = 1, 2).$$

This $G_1 \times G_2$-module $V_1 \otimes V_2$ is called the *exterior tensor product* of V_1 and V_2.

5. Representations of topological groups. Let G be a topological group. If V is a finite-dimensional complex vector space and a homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

is continuous, then ρ is called a *representation of* G, V is called the *representation space of* ρ, and the dimension of V is called the *degree of* ρ. Given a representation ρ we define a continuous linear action of G on V by

$$gx = \rho(g)x, \qquad g \in G, \ x \in V,$$

where V is endowed with the natural topology. Therefore, V becomes a G-module. Conversely, from a continuous linear action of G on V we obtain a representation $\rho : G \to \mathrm{GL}(V)$ by the relation above.

A representation ρ of G is called *trivial* if G acts trivially on the representation space. Two representations ρ_1 and ρ_2 of G are called *equivalent* if the representation spaces V_1 and V_2 are G-isomorphic. Given a representation $\rho : G \to \mathrm{GL}(V)$ and a G-submodule U of V, the induced representation $\rho : G \to \mathrm{GL}(U)$ is called a *subrepresentation* of ρ. A representation ρ of G is called *irreducible* if the representation space is an irreducible G-module. For irreducible representations $\rho_i : G \to \mathrm{GL}(V_i)\,(i = 1, 2)$ the vector space consisting of G-homomorphisms of V_1 to V_2 is of dimension 1 if ρ_1 and ρ_2 are equivalent, or of dimension 0 if not (Schur's lemma). If the representation space of a representation ρ of G is a direct sum of G-submodules $V_i\,(1 \leqq i \leqq n)$, then ρ is called a *direct sum* of the subrepresentations $\rho_i\,(1 \leqq i \leqq n)$ determined by V_i, which is denoted by $\rho = \rho_1 \oplus \cdots \oplus \rho_n$. Similarly, the notions of *contragradient representation*, *tensor product*, and *exterior tensor product* are defined, and for these we use the notation ρ^*, $\rho_1 \otimes \rho_2$, and $\rho_1 \boxtimes \rho_2$, respectively. For a representation ρ of G the *character* χ_ρ of ρ is the continuous function defined by

$$\chi_\rho(g) = \mathrm{tr}\,\rho(g), \qquad g \in G.$$

The function χ_ρ is a class function on G, that is, a function which is constant on every conjugate class of G. For characters we have the following:

(1) if ρ_1 and ρ_2 are equivalent, we have

$$\chi_{\rho_1} = \chi_{\rho_2};$$

(2) $\chi_{\rho^*}(g) = \chi_\rho(g^{-1})$, $g \in G$;

(3) $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$;

(4) $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\chi_{\rho_2}$;

(5) for a representation ρ_1 of G_1 and a representation ρ_2 of G_2 we have

$$\chi_{\rho_1 \boxtimes \rho_2}(g_1, g_2) = \chi_{\rho_1}(g_1)\chi_{\rho_2}(g_2), \qquad g_1 \in G_1,\ g_2 \in G_2.$$

In what follows, a topological group G is assumed to be compact. Let dg be the normalized bi-invariant Haar measure on G, that is, a bi-invariant Haar measure satisfying

$$\int_G dg = 1.$$

For any representation ρ of G there exists a *G-invariant inner product* $(\ ,\ )$ on the representation space V, that is, an inner product satisfying

$$(\rho(g)x, \rho(g)y) = (x, y), \qquad g \in G,\ x, y \in V.$$

Indeed, take an inner product $(\ ,\ )_0$ on V, and set

$$(x, y) = \int_G (\rho(g)x, \rho(g)y)_0\, dg, \qquad x, y \in V.$$

Then this inner product $(\ ,\ )$ is G-invariant. If ρ is irreducible, such an inner product is unique up to a positive constant factor. From the existence of invariant inner product we see that if G is compact, then property (2) of a character is also written as

(2′) $\chi_{\rho^*}(g) = \overline{\chi_\rho(g)}$, $g \in G$.

From the same fact we see that any representation ρ of G is *completely reducible*, that is, for any G-submodule U of the representation space V there exists a G-submodule W such that $V = U + W$ (direct sum). Indeed, if we take as W the orthogonal complement of U with respect to a G-invariant inner product, this satisfies the condition above.

REMARK. We have assumed up to this point that V is a complex vector space. However, even if V is a real vector space, similar notions are defined as in the complex case. For example, a continuous homomorphism ρ of G to $\mathrm{GL}(V)$ is called a *real representation*, and so on. Moreover, if G is compact, then the following holds as in the complex case: V has an invariant inner product; such an inner product is unique up to a positive constant factor if ρ is irreducible; ρ is completely reducible.

For continuous functions f_1 and f_2 on G set

$$\langle f_1, f_2 \rangle = \int_G f_1(g)\overline{f_2(g)}\, dg.$$

Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation. Take a G-invariant inner product $(\ ,\ )$ on V, and let $\{x_1, \dots, x_d\}$ be an orthonormal basis of V with respect to this innner product. Define $\mathbf{C}$-valued continuous functions $\rho_j{}^i$ $(1 \leqq i, j \leqq d)$ on G by

$$\rho_j{}^i(g) = \big(\rho(g)x_j, x_i\big), \qquad g \in G.$$

These functions $\rho_j{}^i$ are called the *matrix elements* of ρ. Let ρ' be another irreducible representation of G which is not equivalent to ρ, and let $\rho'{}_l{}^k$ $(1 \leqq k, l \leqq d')$ be its matrix elements. Then we have the following, which is called the *orthogonality relations of matrix elements*:

(1) $\langle \rho_j{}^i, \rho_l{}^k \rangle = \dfrac{1}{d}\delta_{ik}\delta_{jl}$, $\qquad 1 \leqq i, j, k, l \leqq d$,

(2) $\langle \rho_j{}^i, \rho'{}_l{}^k \rangle = 0$, $\qquad 1 \leqq i, j \leqq d$, $1 \leqq k, l \leqq d'$.

These follow from Schur's lemma. From (1) and (2) we have the *orthogonality relations of characters*:

(1′) $\langle \chi_\rho, \chi_\rho \rangle = 1$,

(2′) $\langle \chi_\rho, \chi_{\rho'} \rangle = 0$.

From the complete reducibility of a representation and the orthogonality relations of characters it follows that two representations ρ_1 and ρ_2 are equivalent if and only if $\chi_{\rho_1} = \chi_{\rho_2}$.

For these refer to Chevalley [3].

6. Algebras. Let A be a vector space over a field k, and assume that in A a product $(a, b) \mapsto ab$ is defined. Then A is called an *algebra* over k if the following is satisfied:

$$a(b+c) = ab + ac, \quad (b+c)a = ba + ca, \qquad a, b, c \in A,$$
$$a(bc) = (ab)c, \qquad a, b, c \in A,$$
$$\lambda(ab) = (\lambda a)b = a(\lambda b), \qquad \lambda \in k, \ a, b \in A.$$

If there exists an element $e \in A$ such that $ea = ae = a$ for any $a \in A$, this element e is called the *unit element* of A and we shall denote it by 1. If $ab = ba$ for any $a, b \in A$, the algebra A is called *commutative*. For example, let k be a field, and let $k[X_1, \dots, X_n]$ be the set of polynomials in n variables with coefficients in k. Then $k[X_1, \dots, X_n]$ is a commutative algebra with unit element.

Let A be an algebra over k. For subsets B and C of A, BC will denote the subspace spanned over k by $\{bc : b \in B, \ c \in C\}$. A subspace B of A with $BB \subset B$ is called a *subalgebra* of A. A subalgebra B itself is an algebra. A subalgebra B is called a *left ideal* (resp., *right ideal*) if $AB \subset B$ (resp., $BA \subset B$). A subalgebra is called a *two-sided ideal* if it is both a left and a right ideal. For a subset S of A, the smallest subalgebra which contains S, say B, is called the *subalgebra generated by* S. Here when A contains the unit element 1, we assume that B also contains 1. If a subalgebra B is identical with the subalgebra generated by S, we say that B is *generated by* S and S is a *system of generators*. A subalgebra B is said to be *finitely generated* if B is generated by some finite set. Also, for ideals similar notions are defined. Any ideal of the polynomial ring $k[X_1, \dots, X_n]$ is finitely generated (Hilbert's basis theorem; refer to Van der Waerden [29]). For a two-sided ideal B of A the quotient vector space A/B has a natural algebra structure over k called a *quotient algebra*.

A linear mapping φ of an algebra A_1 to an algebra A_2 is called an (*algebra*) *homomorphism* if $\varphi(ab) = \varphi(a)\varphi(b)$ for any $a, b \in A_1$. Here we assume that $\varphi(1) = 1$ when both A_1 and A_2 have the unit elements 1. Moreover, if φ is a linear isomorphism, then φ is called an (*algebra*) *isomorphism*. A homomorphism (resp., isomorphism) of an algebra A to itself is called a *endomorphism* (resp., *automorphism*). A linear endomorphism D of an algebra A is called a *derivation* if $D(ab) = (Da)b + a(Db)$ for any $a, b \in A$.

Next we give some examples.

(i) Let X be a topological space, and let $C(X)$ be the set of $\mathbf{C}$-valued continuous functions on X. Define

$$(f+g)(x) = f(x) + g(x), \qquad f, g \in C(X), \ x \in X,$$
$$(\lambda f)(x) = \lambda f(x), \qquad \lambda \in \mathbf{C}, \ f \in C(X), \ x \in X,$$
$$(fg)(x) = f(x)g(x), \qquad f, g \in C(X), \ x \in X.$$

Then $C(X)$ is a commutative algebra over $\mathbf{C}$.

(ii) Let V be an n-dimensional vector space over $\mathbf{R}$. A mapping f of V to $\mathbf{R}$ (resp., $\mathbf{C}$) is called a *polynomial function* or simply a *polynomial* on V with values in $\mathbf{R}$ (resp., $\mathbf{C}$) if there exist a basis $\{\xi^1, \dots, \xi^n\}$ of the dual space of V and a polynomial F in n variables with coefficients in $\mathbf{R}$ (resp., $\mathbf{C}$) such that

$$f(x) = F(\xi^1(x), \dots, \xi^n(x)), \qquad x \in V.$$

Polynomial functions are continuous with respect to the natural topology of V. The set of $\mathbf{C}$-valued polynomial functions on V is a subalgebra of the algebra $C(V)$ of continuous functions in (i) and is isomorphic to the polynomial ring $\mathbf{C}[X_1, \dots, X_n]$.

A *graded vector space* A is a vector space endowed with a direct sum decomposition into subspaces

$$A = \sum_k A_k, \qquad k \in \mathbf{Z}.$$

Then we say that A is *graded by* the subspaces A_k. Given two graded vector spaces

$$A = \sum_k A_k, \qquad A' = \sum_k A'_k,$$

a linear mapping φ of A to A' is said to be *gradation-preserving* if $\varphi A_k \subset A'_k$ for any $k \in \mathbf{Z}$. An algebra A is said to be a *graded algebra* if it is graded by subspaces A_k, $k \in \mathbf{Z}$, and its gradation satisfies the condition that $A_k A_l \subset A_{k+l}$ for any $k, l \in \mathbf{Z}$.

A *filtered vector space* A is a vector space endowed with subspaces A_k, $k \in \mathbf{Z}$, such that $A_k \subset A_l$ for $k \leqq l$ and

$$A = \bigcup_k A_k.$$

Then we say that A is *filtered by* the subspaces A_k. Given two filtered vector spaces

$$A = \bigcup_k A_k, \qquad A' = \bigcup_k A'_k,$$

a linear mapping (resp., linear isomorphism) φ of A to A' is called a *filtration-preserving linear mapping* (resp., *linear isomorphism*) if $\varphi A_k \subset A'_k$ (resp., $\varphi A_k = A'_k$) for any $k \in \mathbf{Z}$. An algebra A is said to be a *filtered algebra* if it is filtered by subspaces A_k, $k \in \mathbf{Z}$, and its filtration satisfies the condition that $A_k A_l \subset A_{k+l}$ for any $k, l \in \mathbf{Z}$. An algebra homomorphism (resp., algebra isomorphism) between two filtered algebras is called a *filtration-preserving algebra homomorphism* (resp., *algebra isomorphism*) if it is a filtration-preserving linear mapping (resp., linear isomorphism).

7. Lie groups and Lie algebras. Let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra (consisting of left-invariant vector fields). If

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

is a representation of G, that is, a continuous homomorphism of G to $\mathrm{GL}(V)$, then ρ is actually differentiable of class C^∞. Therefore, by taking the differential of ρ we obtain a representation of $\mathfrak{g}$

$$d\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V),$$

which is called the *differential* of ρ (as mentioned in the Introduction, a representation of a real Lie algebra means a complex representation of finite dimension). For example, the differential of the adjoint representation Ad of G is the adjoint representation ad of $\mathfrak{g}$. In general, a representation of $\mathfrak{g}$ is not necessarily obtained by taking the differential of a representation of G. However, if G is simply connected, any representation of $\mathfrak{g}$ is obtained by this procedure, and thus, the representations of G correspond bijectively to those of $\mathfrak{g}$. This follows from the following general fact: let G be a simply connected Lie group, and let H be a Lie group; let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively, then for any Lie algebra homomorphism $\Phi : \mathfrak{g} \to \mathfrak{h}$ there exists a unique C^∞ homomorphism $\varphi : G \to H$ with $d\varphi = \Phi$. From this it is seen that for a simply connected Lie group G with Lie algebra $\mathfrak{g}$ the automorphisms of $\mathfrak{g}$ correspond bijectively to those of G. For these refer to Matsushima [19] or Hochschild [12].

Let $\mathfrak{g}$ be a Lie algebra of finite dimension over **R** or **C**. For subsets $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{g}$ we shall denote by $[\mathfrak{a}, \mathfrak{b}]$ the subspace of $\mathfrak{g}$ spanned by $\{[X, Y] : X \in \mathfrak{a},\ Y \in \mathfrak{b}\}$. A subspace $\mathfrak{a}$ with $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$ (resp., $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$) is called a *subalgebra* (resp., an *ideal*). The subspace $[\mathfrak{g}, \mathfrak{g}]$ is an ideal and called the *commutator subalgebra* of $\mathfrak{g}$. If $[\mathfrak{g}, \mathfrak{g}] = \{0\}$, $\mathfrak{g}$ is called *commutative.* We call

$$\mathfrak{c} = \{X \in \mathfrak{g} : [X, \mathfrak{g}] = \{0\}\}$$

the *center* of $\mathfrak{g}$. A Lie algebra is called *semisimple* if it does not contain any commutative ideal except for $\{0\}$. If $\mathfrak{g}$ is semisimple, its commutator subalgebra is identical with $\mathfrak{g}$ itself. Therefore, since

$$\mathrm{tr}\Big(\mathrm{ad}([X, Y])\Big) = 0, \qquad X, Y \in \mathfrak{g},$$

it follows that if $\mathfrak{g}$ is semisimple, $\mathrm{tr}(\mathrm{ad}\, X) = 0$ for any $X \in \mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called *simple* if it is not commutative and if it does not contain any ideal except for $\{0\}$ and $\mathfrak{g}$ itself. A direct sum of simple Lie algebras is semisimple. A Lie algebra $\mathfrak{g}$ is called *nilpotent* if $\mathrm{ad}\, X$ is a nilpotent linear endomorphism of $\mathfrak{g}$ for any $X \in \mathfrak{g}$. It is easy to see that if $\mathfrak{g}$ is nilpotent, we have $\mathrm{tr}(\mathrm{ad}\, X) = 0$ for any $X \in \mathfrak{g}$. A Lie group is called *semisimple*, *simple*, or *nilpotent* according to whether its Lie algebra is semisimple, simple, or nilpotent. For these refer to Humphreys [13].

A semisimple Lie algebra $\mathfrak{g}$ over $\mathbf{R}$ is called *compact semisimple* if it admits a $\mathfrak{g}$-invariant inner product $(\ ,\ )$, that is, admits an inner product satisfying

$$([X, Y], Z) + (Y, [X, Z]) = 0, \qquad X, Y, Z \in \mathfrak{g}.$$

A connected Lie group is compact if its Lie algebra is compact semisimple. There are several proofs of this fact. For example, see Kobayashi and Nomizu [17].

8. Banach spaces and Hilbert spaces. Let X be a vector space over $\mathbf{R}$ or $\mathbf{C}$. A *norm* on X is an $\mathbf{R}$-valued function $x \mapsto \|x\|$ on X with the following properties:

(1) $\|x\| \geqq 0$ and $\|x\| = 0$ if and only if $x = 0$;
(2) $\|\lambda x\| = |\lambda|\,\|x\|$, $\lambda \in \mathbf{R}$ (or $\mathbf{C}$);
(3) $\|x + y\| \leqq \|x\| + \|y\|$.

A vector space with norm is called a *normed linear space.* If we set $d(x, y) = \|x-y\|$ for $x, y \in X$, then d satisfies the axioms of distance. A normed linear space X is called a *Banach space* if the metric space (X, d) is complete. A subspace of a Banach space X is called a *closed subspace* if it is closed with respect to the topology defined by the distance d. A Banach space X is called a *Banach algebra* if it is an algebra and satisfies the condition that $\|xy\| \leqq \|x\|\,\|y\|$ for any $x, y \in X$. Let us cite an example. For a compact topological space E, the algebra $C(E)$ of $\mathbf{C}$-valued continuous functions on E is a Banach algebra over $\mathbf{C}$ with norm

$$\|f\|_\infty = \max_{x \in E} |f(x)|, \qquad f \in C(E).$$

A subalgebra of a Banach algebra is called a *closed subalgebra* if it is a closed subspace.

A linear endomorphism of a Banach space X is called a *compact operator* if it maps every bounded subset (with respect to the distance d) to a relatively compact set (with respect to the topology defined by the distance d). For example, let $C(E)$ be the Banach space above, and let K be an integral operator with kernel k which is a $\mathbf{C}$-valued continuous function on $E \times E$, that is,

$$(Kf)(x) = \int_E k(x, y) f(y)\, d\mu(y), \qquad x \in E,\ f \in C(E),$$

where μ is a bounded measure on E. Then K is a compact operator on $C(E)$. If φ is a compact operator on a Banach space X over $\mathbf{C}$, then for any $\lambda \in \mathbf{C}$ with $\lambda \neq 0$ the eigenspace of φ belonging to the eigenvalue λ is finite dimensional.

Let X be a vector space over $\mathbf{R}$ or $\mathbf{C}$ with inner product $(\ ,\ )$. For $x \in X$ set $\|x\| = \sqrt{(x, x)}$. Then the function $x \mapsto \|x\|$ is a norm on X. We call X a *Hilbert space* if X is a Banach space with respect to this norm. A subspace of a Hilbert space X is called a *closed subspace* if it is a closed subspace as Banach space. A linear isomorphism $\varphi : X_1 \to X_2$ between Hilbert spaces X_1 and X_2 is called an *isomorphism* of Hilbert spaces if $(\varphi(x), \varphi(y)) = (x, y)$ for any $x, y \in X_1$. In particular, an isomorphism of a Hilbert space X to itself is called a *unitary operator.* Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a countable family of closed subspaces of a Hilbert space X which are orthogonal to each other. The Hilbert space X is called a *direct sum* of $\{X_\lambda\}_{\lambda \in \Lambda}$ if for any $x \in X$ and $\lambda \in \Lambda$ there exists a unique element $x_\lambda \in X_\lambda$ such that

$$\sum_\lambda \|x_\lambda\|^2 < \infty, \qquad \sum_\lambda x_\lambda = x,$$

where the second equation means that the left-hand side is convergent to x with respect to the norm $\| \ \|$. Then we shall express this by

$$X = \sum_{\lambda \in \Lambda} \bigoplus X_\lambda.$$

A subset S of a Hilbert space X is called an *orthogonal system* of X if $(x, y) = 0$ for any two different elements $x, y \in S$. Furthermore, if $(x, x) = 1$ for every $x \in S$, then it is called an *orthonormal system.* A *complete orthogonal* (resp., *orthonormal*) *system* of X is a maximal orthogonal (resp., orthonormal) system of X.

As examples of Hilbert spaces, *L_2-spaces* which we discuss below are well known. Let $(X, \mathfrak{B}, \mu)$ be a measure space, that is, $\mathfrak{B}$ is a completely additive class of subsets of X, and μ is a measure on $\mathfrak{B}$. A **C**-valued function f on X is called *square integrable* if it is a $\mathfrak{B}$-measurable function with

$$\|f\|_2 = \left(\int_X |f(x)|^2 \, d\mu(x)\right)^{\frac{1}{2}} < \infty.$$

We shall denote by $L_2(X, \mathfrak{B}, \mu)$ or simply by $L_2(X)$ the vector space consisting of **C**-valued square integrable functions, where we consider that $f = g$ in $L_2(X)$ if $\|f - g\|_2 = 0$. Then $L_2(X)$ is a Hilbert space over **C** with inner product

$$\langle f, g \rangle = \int_X f(x)\overline{g(x)}\, d\mu(x), \qquad f, g \in L_2(X).$$

If $L_2(X)$ has a countable complete orthonormal system $\{\varphi_\lambda\}_{\lambda \in \Lambda}$, every $f \in L_2(X)$ can be expanded in the form

$$f = \sum_\lambda \langle f, \varphi_\lambda \rangle \varphi_\lambda$$

which means that the right-hand side is convergent to f with respect to the norm $\| \ \|_2$. We sometimes call this the *Fourier expansion* of f and the $\langle f, \varphi_\lambda \rangle$'s the *Fourier coefficients.* For example, for a locally compact group G, the L_2-space $L_2(G)$ with respect to a Haar measure always has a countable complete orthonormal system. For these refer to Yosida [31].

9. Riemannian geometry. Let M be a manifold of dimension n. For $x \in M$ a linear isomorphism of $\mathbf{R}^n$ onto the tangent space $T_x(M)$ is called a *frame* of M at x. The set $L(M)$ of frames of M has a principal fibre bundle structure with structure group $\mathrm{GL}(\mathbf{R}^n)$. For $u \in L(M)$ let G_u be the tangent space to the fibre of $L(M)$ through u at u. An n-dimensional C^∞ distribution $\Gamma : u \mapsto \Gamma_u$ on $L(M)$ is called a *linear connection* on M if it is invariant under the action of $\mathrm{GL}(\mathbf{R}^n)$ on $L(M)$ and satisfies the condition that $\Gamma_u \cap G_u = \{0\}$ for any $u \in L(M)$. There always exists a linear connection on M.

Let Γ be a linear connection on M, and let π be the projection of $L(M)$ onto M. Let x_t, $0 \leqq t \leqq a$, be a C^∞ curve in M. Then for any point $u_0 \in L(M)$ with $\pi(u_0) = x_0$, there exists a unique C^∞ curve u_t, $0 \leqq t \leqq a$, in $L(M)$ with initial point u_0 which is everywhere tangent to the distribution Γ and satisfies $\pi(u_t) = x_t$, $0 \leqq t \leqq a$. A linear isomorphism τ_t of $T_{x_0}(M)$ onto $T_{x_t}(M)$ is defined by $\tau_t = u_t u_0^{-1}$. This is determined only by the curve x_t and is independent of the choice of u_0. This linear isomorphism τ_t induces a linear isomorphism

$$\left(\bigotimes^r \tau_t\right) \otimes \left(\bigotimes^s \tau_t^*\right) : \left(\bigotimes^r T_{x_0}(M)\right) \otimes \left(\bigotimes^s T_{x_0}(M)^*\right) \longrightarrow \left(\bigotimes^r T_{x_t}(M)\right) \otimes \left(\bigotimes^s T_{x_t}(M)^*\right),$$

where τ_t^* denotes the inverse of the tranposed mapping of τ_t. We shall also denote this by τ_t which is called the *parallel displacement* (with respect to the linear connection Γ) along the curve x_t. Let X be a vector field on M, and let S be a tensor field of type (r, s) on M. Let $x_0 \in M$ and take a C^∞ curve x_t, $0 \leqq t \leqq a$, with initial point x_0 whose tangent vector $\dot{x}_0$ at x_0 is X_{x_0}. Set

$$(\nabla_X S)_{x_0} = \lim_{t \to 0} \frac{1}{t}(\tau_t^{-1} S_{x_t} - S_{x_0}).$$

Then $(\nabla_X S)_{x_0}$ is determined only by X_{x_0} and is independent of the choice of curve x_t. Therefore, we have a tensor field $\nabla_X S$ of type (r, s) on M which is called the *covariant derivative of S relative to X*. The correspondence $X \mapsto \nabla_X S$ determines a tensor field of type $(r, s+1)$. We shall denote this by ∇S which we call the *covariant differential of S*. We sometimes call this "differential operator" ∇ a linear connection.

Let $(x^1, \dots, x^n)$ be a local coordinate system on an open set U of M. Define C^∞ functions $\Gamma_{jk}{}^i$ on U by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^n \Gamma_{ij}{}^k \frac{\partial}{\partial x^k}.$$

We call this system $\{\Gamma_{jk}{}^i\}$ the *local expression of Γ*, and these functions $\Gamma_{jk}{}^i$ the *components* of Γ with respect to the local coordinate system $(x^1, \dots, x^n)$.

Let $(\bar{x}^1, \dots, \bar{x}^n)$ be another local coordinate system on U, and let $\{\bar{\Gamma}_{\alpha\beta}{}^{\gamma}\}$ be the corresponding local expression of Γ. Then we have

$$\bar{\Gamma}_{\alpha\beta}{}^{\gamma} = \sum_{i=1}^{n} \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} \frac{\partial \bar{x}^\gamma}{\partial x^i} + \sum_{i,j,k=1}^{n} \Gamma_{ij}{}^{k} \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial x^j}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\gamma}{\partial x^k}.$$

For example, given a covariant tensor field S of degree s, its covariant differential ∇S is written locally

$$(\nabla S)_{i_1 \cdots i_s k} = \frac{\partial S_{i_1 \cdots i_s}}{\partial x^k} - \sum_{\mu=1}^{s} \sum_{h=1}^{n} \Gamma_{k i_\mu}{}^{h} S_{i_1 \cdots i_{\mu-1} h i_{\mu+1} \cdots i_s}.$$

Tensor fields T and R of type $(1, 2)$ and of type $(1, 3)$, respectively, are defined by the conditions

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$
$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where X, Y, and Z are any vector fields on M. We call T and R the *torsion tensor field* and the *curvature tensor field*, respectively.

Let x_t, $0 \leqq t \leqq a$, be a C^∞ curve in M. From the definition of covariant derivative we see that given a smooth vector field X_t, $0 \leqq t \leqq a$, along the curve x_t the covariant derivative $\nabla_{\dot{x}_t} X_t$ can be defined. A *geodesic* x_t, $0 \leqq t \leqq a$, is a C^∞ curve such that $\nabla_{\dot{x}_t} \dot{x}_t = 0$ for any $0 \leqq t \leqq a$. Let $(x^1, \dots, x^n)$ be a local coordinate system around $x_0 \in M$. It is called a *normal coordinate system around* x_0 if $x^i(x_0) = 0$ for $1 \leqq i \leqq n$ and the curve x_t defined by $x_t{}^i = a_i t$, $1 \leqq i \leqq n$, is a geodesic for any $a_1, \dots, a_n \in \mathbf{R}$. There always exists a normal coordinate system.

A tensor field g on M of type $(0, 2)$ is called a *pseudo-Riemannian metric* if every g_x, $x \in M$, is a nondegenerate symmetric bilinear form on the tangent space $T_x(M)$. In what follows, we assume that M has a pseudo-Riemannian metric g. We then call M a *pseudo-Riemannian manifold.* Then there exists a unique linear connection Γ on M with

$$\nabla g = 0, \qquad T = 0,$$

which is called the *Levi-Civita connection* determined by g. Let $\{g_{ij}\}$ be the local expression of g. Then the local expression $\{\Gamma_{jk}{}^{i}\}$ of the Levi-Civita connection Γ is given by

$$\Gamma_{jk}{}^{i} = \frac{1}{2} \sum_{h=1}^{n} g^{ih} \left(\frac{\partial g_{hj}}{\partial x^k} + \frac{\partial g_{hk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h} \right),$$

where $(g^{ij})_{1 \leqq i, j \leqq n}$ denotes the inverse of the matrix $(g_{ij})_{1 \leqq i, j \leqq n}$. If $(x^1, \dots, x^n)$ is a normal coordinate system with respect to the Levi-Civita connection, we have

$$\frac{\partial g_{ij}}{\partial x^k}(0) = 0, \qquad 1 \leqq i, j, k \leqq n.$$

Let f be a C^∞ function on M. The *gradient* $\operatorname{grad} f$ of f is a vector field which is determined by the condition that $g(\operatorname{grad} f, X) = Xf$ for any vector field X. Let X be a vector field on M. Let $(x^1, \ldots, x^n)$ be a local coordinate system on an open set U of M, and let $\{g_{ij}\}$ be the local expression of g with respect to this coordinate system. Let us consider the following differential form of degree n on U:

$$v = \sqrt{\mathbf{g}}\, dx^1 \wedge \cdots \wedge dx^n, \qquad \mathbf{g} = \left|\det(g_{ij})_{1 \leqq i, j \leqq n}\right|.$$

Then since v does not vanish anywhere on U, a C^∞ function $\operatorname{div} X$ on U is defined by $\theta(X)v = (\operatorname{div} X)v$, where $\theta(X)$ denotes the Lie differentiation relative to X. Actually, $\operatorname{div} X$ is determined independently of the choice of local coordinate system, and hence, a C^∞ function $\operatorname{div} X$, which we call the *divergence* of X, is defined on the whole of M.

Moreover, we assume that g is a *Riemannian metric*, that is, g_x is positive definite for every $x \in M$. Then M is called a *Riemannian manifold*. Further, we assume that M is connected. For any two points $x, y \in M$, let $d(x, y)$ be the infimum of the lengths of all piecewise smooth curves joining x to y. Then d satisfies the axioms of distance, and the topology of M determined by d is identical to the original one. The Riemannian manifold M is called *complete* if the metric space (M, d) is complete. If M is complete, any two points can be joined by a geodesic with respect to the Levi-Civita connection. A diffeomorphism of M is called an *isometry* if it leaves g invariant. Then the group $I(M)$ of isometries of M is a Lie transformation group with respect to the compact open topology. If $I(M)$ acts transitively on M, then M is complete. For these refer to Kobayashi and Nomizu [17].

10. For root systems of semisimple Lie algebras and weights of representations refer to Humphreys [13]. For fundamental groups, fibre bundles, covering homotopy theorem, and homotopy exact sequences, refer to Steenrod [24].

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