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Heavy Traffic Limits for Multiphase Queues

F. I. Karpelevich
A. Ya. Kreinin

American Mathematical Society

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Heavy Traffic Limits for Multiphase Queues

F. I. Karpelevich
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American Mathematical Society
Providence, Rhode Island

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МНОГОФАЗОВЫЕ СИСТЕМЫ МАССОВОГО ОБСЛУЖИВАНИЯ В РЕЖИМЕ БОЛЬШОЙ НАГРУЗКИ

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Abstract. This book is devoted to the analysis of several types of queueing systems arising in network theory and communication theory. Numerous methods and results from the theory of stochastic processes are used. The main emphasis is made to problems of diffusion approximation of stochastic processes in queueing systems and to the results based on the applications of the hydrodynamic limit method.

This book will be useful to researchers working in the theory and applications of queueing theory and the theory of stochastic processes.

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PREFACE

This monograph deals with the analysis of random processes in multiphase service systems operating in critical regimes. Multiphase systems are widely met in practice. They play an important role in queueing theory and have been the subject of research for a long time. As a rule, in this class of systems, it is impossible to obtain explicit formulas for the operation parameters as functions of the distribution of the control sequence elements.

Usually, computer simulation is applied in these cases. However, in operation regimes close to the critical one, this approach shows low efficiency. At the same time, analytical methods provide explicit analytic formulas in a limit corresponding to the critical mode.

The material of the book is arranged in two parts. In the first part (Chapter I and Chapter II), we study waiting time processes in heavy traffic and their diffusion approximations. In the second part (Chapter III), we consider queue length processes and departure flows of customers in multiphase systems.

The operation of multiphase systems is described by random processes in polyhedral cones with reflections at their faces. The corresponding construction, which generalizes the well-known Lindley equation, is presented in the first chapter of the monograph. Systems satisfying the Kleinrock condition on the independence of service times in different phases, as well as systems with identical service, are considered uniformly using the processes with reflections.

In heavy traffic, processes with reflections are approximated by diffusion processes. To validate the corresponding limit in multiphase systems, we use the theory of weak convergence of random processes, and, in particular, the Donsker-Prohorov invariance principle.

The unified treatment has certain advantages, at the cost of disregarding some approaches, however. In particular, the martingale approach is not considered in the book although it appears to be very effective in many cases.

The second part of the book deals with the queue length processes and departure flows in multiphase systems. This part is a kind of survey. There are two reasons that made us choose the survey style. The first reason is that the queue length processes are very close to the waiting time processes. The second reason is that the results associated with the approximation of a departure flow in heavy traffic have not yet received a conventional treatment in the modern literature.

The theory of heavy traffic in service systems accumulates the efforts of many mathematicians. Unfortunately, we are in no position to give a full list of specialists in the field and to appreciate their individual contributions to the development of the theory. Here we mention only Donsker, Prohorov, Borovkov, Harrison, Reiman, and Whitt.

The results of our studies in the field have been presented many times at seminars given by B. Gnedenko, Yu. Belyaev, and A. Soloviev at the Moscow State University and at seminars given by R. Dobrushin and Yu. Suhov at the Institute for Problems of Information Transmission. We are deeply grateful to the participants of the seminars for valuable remarks and criticism.

The work on writing and translating the book was inspired by S. Gindikin and S. Gelfand. We are very glad to commend the work of A. Vainshtein, which cannot be overestimated. We are also very grateful to the American Mathematical Society for their attention to this project.

F. I. Karpelevich
A. Ya. Kreinin

FREQUENTLY USED NOTATION

$\mathbb{Z}^d$	Integer d-dimensional lattice
$C_p^m\,(X)$	Space of all m times differentiable bounded functions on X
$C\,(X)$	Space of all continuous functions on X
$D(X)$	Space of all left continuous on X functions having right limits
h^T	The matrix transposed to h
π_k	Projection to $\mathbb{R}^k$
$\xrightarrow{w}$	Weak convergence of measures
$\xrightarrow{\mathrm{P}}$	Convergence in probability of stochastic elements
$\xrightarrow{d}$	Convergence in distribution of random variables
$\mathbf{E}\xi$	Mean value of a random variable ξ
$\mathbf{D}\xi$	Variation of a random variable ξ
$\mathcal{F}_n$	σ-algebra of a process with discrete time
$\mathbf{E}(\xi_n \mid \mathcal{F}_{n-1})$	Conditional expectation of a stochastic process ξ_n
$\mathcal{A}_\xi$	Infinitesimal operator of a process ξ

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APPENDIX I

WEAK CONVERGENCE IN FUNCTION SPACES

In this appendix, several important facts of the weak convergence theory of stochastic processes are collected. This theory is presented in many monographs (see, e.g, [11], [108]).

§1. Convergence in distribution

Let $\mathcal{X}$ be a complete and separable metric space. Let ξ^λ be a family of random points of $\mathcal{X}$ depending on a parameter $\lambda \geqslant 0$. We say that the family ξ^λ *converges in distribution* to a random point ξ^0 as $\lambda \to 0$ (and denote this convergence by $\xrightarrow{d}$),

(A1.1.1) $$\xi^\lambda \xrightarrow{d} \xi^0 \quad \text{as} \quad \lambda \to 0,$$

if for each bounded continuous function $f : \mathcal{X} \to \mathbb{R}$,

(A1.1.2) $$\lim_{\lambda \to 0} \mathbf{E} f(\xi^\lambda) = \mathbf{E} f(\xi^0).$$

Observe that we want relation (A1.1.2) to be satisfied for real-valued functions f. On the other hand, if $\xi^\lambda \xrightarrow{d} \xi^0$, then (A1.1.2) is satisfied also for complex functions f. If, for instance, $\mathcal{X} = \mathbb{R}$, that is, ξ^λ is a random variable, then using the function $f(x) = e^{itx}$, we obtain the convergence of the characteristic functions (c.f.) of the random variables ξ^λ to that of ξ^0.

Let P^λ $(\lambda > 0)$ be the distribution of a random point ξ^λ of the space $\mathcal{X}$ (or the measure corresponding to ξ^λ).

Definition. The measures P^λ $(\lambda > 0)$ are said to be *weakly convergent* to the measure P^0 as $\lambda \to 0$,

$$P^\lambda \xrightarrow{w} P^0,$$

if $\xi^\lambda \xrightarrow{d} \xi^0$ $(\lambda \to 0)$.

Clearly, we can rewrite (A1.1.2) in the following form:

(A1.1.3) $$\lim_{\lambda \to 0} \int_{\mathcal{X}} f(x) P^\lambda(dx) = \int_{\mathcal{X}} f(x) P^0(dx).$$

We mention here some features of the convergence introduced above. A brilliant systematic presentation of the theory of weak convergence is given in [11].

THEOREM A1.1.1 ([11, Theorem 2.1]). *If condition* (A1.1.2) *is satisfied for all bounded and uniformly continuous functions f on $\mathcal{X}$, then it provides the convergence of* (A1.1.1).

We denote by $d_0(x, y)$ the distance in the space $\mathcal{X}$. Assume that random points ξ^λ and η^λ, $\lambda > 0$, are defined on the common probability space. Then the random variable $d_0(\xi^\lambda, \eta^\lambda)$ is also defined on the same probability space.

THEOREM A1.1.2 ([11, Theorem 4.1]). *If $\xi^\lambda \xrightarrow{d} \xi^0$ and $d_0(\xi^\lambda, \eta^\lambda) \xrightarrow{\mathrm{P}} 0$ as $\lambda \to 0$, then $\eta^\lambda \xrightarrow{d} \xi^0$.*

Consider a family of stochastic elements $\xi_T^\lambda, \xi^\lambda \in \mathcal{X}$ defined on the common probability space with support $\mathcal{X}$ depending on two parameters $\lambda \geqslant 0$, $T > 0$. Let $\mathcal{X}$ be a metric space with distance d_0.

THEOREM A1.1.3 ([11, Theorem 4.2]). *Assume that for each $T > 0$, $\xi_T^\lambda \xrightarrow{d} \xi_T^0$ as $\lambda \to 0$, and $\xi_T^0 \xrightarrow{d} \xi^0$ as $T \to \infty$. Let for each $\varepsilon > 0$ there exist $T_0 > 0$ and $\lambda_0 > 0$ such that for all $T > T_0$ and for each $\lambda \in (0, \lambda_0)$*

$$\Pr\{d_0(\xi_T^\lambda, \xi^\lambda) \geqslant \varepsilon\} < \varepsilon. \tag{A1.1.4}$$

Then $\xi^\lambda \xrightarrow{d} \xi^0$ as $\lambda \to 0$.

Let $\mathcal{X}'$ be a metric space and $\varphi\colon \mathcal{X} \to \mathcal{X}'$ a measurable mapping. Denote by A ($A \in \mathcal{X}$) the discontinuity set for the map φ. Then A is a Borel set (see [11]).

THEOREM A1.1.4 ([11, §5]). *Let ξ^λ, $\lambda > 0$, be random points in the space $\mathcal{X}$, and $\xi^\lambda \xrightarrow{d} \xi^0$. If $\Pr\{\xi^0 \in A\} = 0$, then $\varphi(\xi^\lambda) \xrightarrow{d} \varphi(\xi^0)$.*

Let $(L, \mathcal{L})$ be a measurable space (not necessarily a metric space), and let α be a random element of $(L, \mathcal{L})$ defined on some probability space. If f is a measurable function on L, then the random variable $f(\alpha)$ is defined on the same probability space.

THEOREM A1.1.5. *Assume that α is a random element in the space $(L, \mathcal{L})$ and $\{f_T\}$, $T > 0$, is a family of measurable functions that converge at any point of the space L to the function f as $T \to \infty$. Then $f_T(\alpha) \xrightarrow{d} f(\alpha)$.*

PROOF. Let P_α be the distribution of the random element α and φ be a continuous, bounded, real-valued function on $\mathbb{R}$. If $\xi_T = f_T(\alpha)$, then

$$\mathbf{E}\varphi(\xi_T) = \int_L h_T(l) P_\alpha(dl),$$

where $h_T(l) = \varphi(f_T(l))$. The functions h_T are uniformly bounded on T and converge to $h(l) = \varphi(f(l))$ as $T \to \infty$. Therefore, passing to the limit and integrating in the latter formula, we get

$$\lim_{T\to\infty} \mathbf{E}\varphi(\xi_T) = \int_L h(l) P_\alpha(dl) = \mathbf{E}\varphi(\xi),$$

and the proof is completed. □

§2. Convergence in the space D

Let $\mathbb{T}$ be an arbitrary interval $[0, T]$, or the nonnegative half-line $t \geqslant 0$.

DEFINITION. The space $D^m(\mathbb{T}) = D^m(\mathbb{T}, \mathbb{R}^m)$ is the set of left continuous functions $x: \mathbb{T} \to \mathbb{R}^m$ having finite right limits.

The distance in the space $D^m([0, T])$ is introduced in the following way (see [11], §14). Let $\Lambda = \Lambda(0, T)$ be the set of increasing continuous functions on $\mathbb{T}$, $\lambda: [0, T] \to [0, T]$, with $\lambda(0) = 0$, $\lambda(T) = T$. For every two functions $x_1, x_2 \in D^m([0, T])$, put

$$(A1.2.1) \qquad d(x_1, x_2) = \inf_{\lambda \in \Lambda} \left(\sup_{0 \leqslant t \leqslant T} |x_2(t) - x_1(\lambda(t))| + \sup_{0 \leqslant s < t \leqslant T} \left| \ln \frac{\lambda(t) - \lambda(s)}{t - s} \right| \right),$$

where $|x| = \sqrt{\sum_{l=1}^m (x^l)^2}$, $x = (x^1, \dots, x^m) \in \mathbb{R}^m$.

With respect to this metric, the space $D^m([0, T])$ is complete and separable. The corresponding topology is equivalent to the Skorohod topology.

Observe ([11], §14) that the space $C^m([0, T]) = C([0, T], \mathbb{R}^m)$ of functions continuous on $[0, T]$ is contained in $D^m([0, T])$. The metric defined by (A1.2.1) and the usual uniform metric induce equivalent topologies on the space $C^m([0, T])$.

For each $t \in \mathbb{T}$ and an arbitrary Borel set $\mathcal{B} \in \mathbb{R}^m$, let $\Gamma(t, \mathcal{B}, \mathbb{T})$ be the set of functions in $D^m(\mathbb{T})$ satisfying the inclusion $x(t) \in \mathcal{B}$.

Denote by $\mathcal{D}^m(\mathbb{T})$ the minimal σ-algebra of subsets $\Gamma(t, \mathcal{B}, \mathbb{T})$, where $t \in \mathbb{T}$ and $\mathcal{B}$ runs over all Borel sets in $\mathbb{R}^m$.

THEOREM A1.2.1 ([11, Theorem 14.5]). *The σ-algebra of Borel sets of the space $D^m([0, T])$ coincides with the σ-algebra $\mathcal{D}^m(\mathbb{T})$.*

Consider a random process ξ on the time interval $\mathbb{T}$ taking values in the space $\mathbb{R}^m$. Assume that the sample paths of this process belong to the space $D^m(\mathbb{T})$ almost surely. Then we can consider the process ξ as an element of the space $D^m(\mathbb{T})$.

Let $\mathbb{T} = [0, T]$ and ξ^λ ($\lambda \geqslant 0$) be a family of such random processes on the space $D^m([0, T])$. The family ξ^λ is called *weakly convergent* to the process ξ^0, if the corresponding family of random points in the space $D^m([0, T])$ converges in distribution to the random point ξ^0. The weak convergence of random processes is denoted as follows:

$$\xi^\lambda \xrightarrow{w} \xi^0, \quad \lambda \to 0.$$

Thus, $\xi^\lambda \xrightarrow{w} \xi^0$ if for each bounded functional $f: D^m([0, T]) \to \mathbb{R}$ the relation

$$\lim_{\lambda \to 0} \mathbf{E} f(\xi^\lambda) = \mathbf{E} f(\xi^0)$$

holds.

The Skorohod topology on the space D^m is generated by the metric (A1.2.1). Besides, we use the uniform metric defined by

$$(A1.2.2) \qquad \rho(x_1, x_2) = \sup_{0 \leqslant t \leqslant T} |x_2(t) - x_1(t)|.$$

where x_1, x_2 are functions belonging to D^m. It is clear that

$$d(x_1, x_2) \leqslant \rho(x_1, x_2). \tag{A1.2.3}$$

For each function $\lambda \in \Lambda = \Lambda(0, T)$ we can define the operator

$$\hat{\lambda}\colon D^m([0, T]) \to D^m([0, T]),$$

acting as a time change: $\hat{\lambda}x(t) = x(\lambda(t))$, $x \in D^m([0, T])$. We denote the set of these operators by $\hat{\Lambda}$.

We say that a map $A\colon D^m([0, T]) \to D^n([0, T])$ *commutes with time changes* (or, briefly, commutes with $\hat{\Lambda}$), if $A\hat{\lambda} = \hat{\lambda}A$ for any $\hat{\lambda} \in \hat{\Lambda}$. Therefore, A commutes with $\hat{\Lambda}$ if $(A(\hat{\lambda}x))(t) = (Ax)(\lambda(t))$ for each function $\lambda \in \Lambda$. Below we provide several examples of such operators.

A trivial one is generated by an arbitrary continuous function $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$:

$$Ax(t) = \varphi(x(t)). \tag{A1.2.4}$$

Another example is provided by the following map. Consider the map S that transforms a bounded function x on $\mathbb{T} = [0, T]$ by the formula

$$Sx(t) = \sup_{0 \leqslant s \leqslant T} x(s), \quad t \in \mathbb{T}. \tag{A1.2.5}$$

Let us show that S takes the space $D^1([0, T])$ to $D^1([0, T])$, and that this operator commutes with $\hat{\Lambda}$. Indeed, if $y = Sx$, $x \in D^1([0, T])$, then y is a nondecreasing and bounded function on $\mathbb{T}$. Hence, this function has finite right limits and is left continuous on $\mathbb{T}$. Since the function $x(t)$ is right continuous, the function y is right continuous as well. Therefore, S takes $D^1([0, T])$ to $D^1([0, T])$. Finally, the relation

$$(S(\hat{\lambda}x))(t) = \sup_{0 \leqslant s \leqslant t} \hat{\lambda}x(s) = \sup_{0 \leqslant s \leqslant t} x(\lambda(s)) = \sup_{0 \leqslant s \leqslant \lambda(t)} x(s) = (S(x))(\lambda(t))$$

implies the commutativity property for the operators S and $\hat{\Lambda}$. Obviously, the operator I defined by

$$Ix(t) = \inf_{0 \leqslant s \leqslant t} x(s), \quad t \in \mathbb{T}, \tag{A1.2.6}$$

also takes $D^1([0, T])$ to $D^1([0, T])$ and commutes with $\hat{\Lambda}$.

The set of operators commuting with $\hat{\Lambda}$ is an algebra: linear combinations and products also belong to the set of operators commuting with time changes. Therefore, combinations of the operators S and I with the operators (A1.2.6) provide new examples of such operators. Consider, for instance, the operator $B\colon D^3([0, T]) \to D^1([0, T])$ defined by

$$Bx(t) = B(x^1, x^2, x^3) = S(S(x^3 - x^2) + x^2 - x^1).$$

Observe that for this functional

$$Bx(t) = \sup_{0 \leqslant s_2 \leqslant s_1 \leqslant t} (x^2(s_1) - x^1(s_1) + x^3(s_2) - x^2(s_2)). \tag{A1.2.7}$$

THEOREM A1.2.2. *If a map* $A\colon D^m([0,T]) \to D^k([0,T])$ *commutes with time changes and is continuous with respect to the uniform metric, then* A *is continuous in the metric defined by* (A1.2.1) *in the space* $D^m([0,T])$.

PROOF. We have to show that if $d(x_n, x_0) \to 0$ ($x_n \in D^m([0,T])$, $n = 0, 1, \dots$) as $n \to \infty$ and $y_n = Ax_n$, $n = 0, 1, \dots$, then $d(y_n, y_0) \to 0$. Choose $\varepsilon > 0$. Since A is continuous in the uniform metric, there exists $\delta > 0$ such that the inequality $\rho(Ax, Ax_0) < \varepsilon$ holds whenever $\rho(x, x_0) < \delta$.

Choose an integer $n_0 > 1$. For each $n > n_0$ there exists a function $\lambda_n^* \in \Lambda$ satisfying the following inequality:

$$\sup_{0\leqslant t\leqslant T} |x_0(t) - x_n(\lambda(t))| + \sup_{0\leqslant s<t\leqslant T} \left| \ln \frac{\lambda_n(s) - \lambda_n(t)}{s - t} \right| < \delta.$$

Therefore, $\rho(\hat{\lambda}_n x_n, x_0) < \delta$ for $n > n_0$. Hence,

$$\rho(A\hat{\lambda}_n x_n, y_0) < \varepsilon. \tag{A1.2.8}$$

The relation $A\hat{\lambda}_n x_n = \hat{\lambda}_n A x_n = \hat{\lambda}_n y_n$, together with inequality (A1.2.8), means that

$$\sup_{0\leqslant t\leqslant T} |y_0(t) - y_n(\lambda_n(t))| < \varepsilon.$$

On the other hand,

$$\sup_{0\leqslant s<t\leqslant T} \left| \ln \frac{\lambda_n(s) - \lambda_n(t)}{s - t} \right| < \delta.$$

Without loss of generality, we can assume that $\delta < \varepsilon$. Then $d(y_0, y_n) < 2\varepsilon$, and the proof is completed. □

§3. The Invariance Principle

Let $\beta_1^\lambda, \dots, \beta_n^\lambda, \dots$ ($\lambda > 0$) be a sequence of independent identically distributed random vectors in $\mathbb{R}^m$. Denote by β^λ the random vector in this space having the same distribution as β_n^λ. Assume that

$$\mathbf{E}\beta^\lambda = \lambda^2 a + o(\lambda^2), \qquad \mathbf{D}\beta^\lambda = \lambda^2 d + o(\lambda^2) \quad \text{as} \quad \lambda \to 0, \tag{A1.3.1}$$

where $a = (a^1, \dots, a^m) \in \mathbb{R}^m$, $\mathbf{D}\beta^\lambda$ is the matrix of the second central moments of the vector β^λ (or the covariance matrix), and d is a positive definite and symmetric matrix of order m.

Denote by P^λ the distribution of the vector β^λ. We also assume that this distribution satisfies the Lindeberg–Feller condition:

$$\int_{|x|>\varepsilon} |x|^2 P^\lambda(dx) = o(\lambda^2). \tag{A1.3.2}$$

Put

$$S_n^\lambda = \sum_{i=1}^n \beta_i^\lambda.$$

(We assume that if the lower limit is greater than upper limit, then the corresponding sum is equal to 0.)

Consider the stochastic process w^λ on the space $D^m([0,T])$ defined by the formula

$$w^\lambda = S_n^\lambda \quad \text{for} \quad n = [t/\lambda^2], \quad 0 \leqslant t \leqslant T.$$

THEOREM A1.3.1 (The Donsker–Prohorov Invariance Principle, [11], [95]). *If conditions* (A1.3.1), (A1.3.2) *are satisfied, then the family of processes* w^{λ} *converges weakly to the process*

$$w(t) = hB(t) + ta, \quad 0 \leqslant t \leqslant T,$$

as $\lambda \to 0$, *where* B *is a Brownian motion in* $\mathbb{R}^m$, *and the matrix* h *is given by the relation* $d = hh^T$.

APPENDIX II

WEAK CONVERGENCE OF FUNCTIONALS OF STOCHASTIC PROCESSES

§1. Approximated functionals

Let us consider the space $D^m([0,\infty))$. Restricting the domain of a function $x \in D^m([0,\infty))$ to the interval $[0,T]$, we obtain a function $x_T \in D^m([0,T])$. The function x_T is called the *restriction* of the function x to the interval $[0,T]$. We denote the restriction operator by Π_T. Thus, $x_T = \Pi_T x$.

Let ξ^λ, $\lambda \geqslant 0$, be a family of stochastic elements of the space $D^m([0,\infty))$. Each of them can be considered as a stochastic process with sample paths belonging to the space $D^m([0,\infty))$. Let us choose $T \geqslant 0$ and consider the restriction $\xi_T^\lambda = \Pi_T \xi^\lambda$. Assume that

$$\text{(A2.1.1)} \qquad \xi_T^\lambda \xrightarrow{w} \xi_T^0, \quad \lambda \to 0,$$

for all $T \geqslant 0$. The following question arises naturally in the weak convergence theory: when do the processes ξ^λ converge weakly on the whole half-line to the process ξ^0?

If the sample paths of the processes ξ^λ have finite limits as $t \to \infty$, then there is an opportunity to apply Theorem A1.1.3.

However, in a large number of applications, the limits of paths are infinite or do not exist at all. For this reason, we present here another approach to the problem of weak convergence.

The scheme of the proof of (A2.1.1) discussed in this appendix provides a useful tool for proving weak convergence results for different stochastic models. The construction does not depend on the form of the limit process and on the form of the processes ξ_t^λ.

To establish the convergence $\xi^\lambda \xrightarrow{w} \xi^0$ on the half-line, we introduce a class of functionals, which we call *approximated functionals*, or simply, *A-functionals.* For this class of functionals, (A1.1.4) and several other conditions provide the convergence in distribution.

Assume that the paths of the processes ξ^λ $(\lambda > 0)$ belong to some subspace $D \subset D^m([0,\infty))$. Let $D_T = \Pi_T D$. We assume that D_T is closed in the space $D^m([0,\infty))$ for all $T > 0$. Introducing the metric (A1.2.1), we turn D_T into a complete and separable space. The σ-algebra $\mathcal{D}_T$ of the Borel sets coincides with the σ-algebra of the sets $A \cap D_T$, where $A \in D^m([0,T])$.

The natural σ-algebra $\mathcal{D}$ of the subsets of the space D consists of the sets $A \cap D$, where $A \in D^m$. Clearly, if $B \in \mathcal{D}$, then $\Pi_T B \in \mathcal{D}_T$.

A real-valued functional $f: D \to \mathbb{R}$ is called *measurable*, if $f^{-1}(B) \in \mathcal{D}$ for all Borel subsets $B \in \mathbb{R}$.

We say that a functional f_T on D *depends only on the restrictions* of functions, if there exists a functional $\hat{f}_T$ on the space D_T such that

$$f_T(x) = \hat{f}_T(x_T) \tag{A2.1.2}$$

for all $x \in D$.

Let us denote by Φ_T the set of functionals depending only on the restrictions to the interval $[0, T]$. Observe that a functional $f_T \in \Phi_T$ is measurable if and only if the corresponding functional $\hat{f}_T$ is measurable on the space D_T.

DEFINITION. A functional f on D is said to be *approximated* (or an *A-functional*) if there exists a family of functionals $\{f_T\}$, $f_T \in \Phi_T$, on D such that

$$\lim_{T\to\infty} f_T(x) = f(x) \tag{A2.1.3}$$

for each $x \in D$. The family $\{f_T\}$ is said to be an *approximating family* for the functional f.

The set of A-functionals is denoted by $\mathbf{A_F}$. Observe that if f_T are measurable for all $T > 0$, then the functional f is also measurable, as a limit of measurable functionals.

Let us consider some useful examples of A-functionals. The functional

$$f(x) = \sup_{t\geqslant 0} x(t), \quad x \in D, \tag{A2.1.4}$$

belongs to the class $\mathbf{A_F}$, provided D is the space of upper bounded functions in $D^1([0, \infty])$. An approximating family is given by

$$f_T(x) = \sup_{0\leqslant t\leqslant T} x(t).$$

If φ is an arbitrary real bounded Lebesgue measurable function on $\mathbb{R}$ and $p \in L^1[0, \infty)$, then the functional

$$f(x) = \int_0^\infty \varphi(x(t))p(t)\,dt, \quad x \in D^1([0, \infty]),$$

belongs to the class $\mathbf{A_F}$. The corresponding approximating family is given by

$$f_T(x) = \int_0^T \varphi(x(t))p(t)\,dt, \quad x \in D^1([0, \infty)).$$

If φ is a bounded Lebesgue measurable function on $\mathbb{R}^2$ and $p \in L^1[0, \infty)$, then the functional on $D^1([0, \infty))$ given by

$$f(x) = \int_0^\infty \varphi(x(t), x(1/t))p(t)\,dt$$

also belongs to the class $\mathbf{A_F}$, and the corresponding approximating family is represented by

$$f_T(x) = \int_{1/T}^T \varphi(x(t), x(1/t))p(t)\,dt.$$

§2. Unapproximated functionals

Here we give an example of a functional that does not belong to the class of approximated functionals. This functional $f \notin \mathbf{A_F}$ is defined on the space $D^1([0,\infty))$.

Let us consider the functional

$$f_*(x) = \begin{cases} 0, & \text{if } \lim_{t\to\infty} x(t) \text{ exists and is finite,} \\ 1, & \text{otherwise.} \end{cases}$$

PROPOSITION A2.2.1. *The functional f_* does not belong to the class $\mathbf{A_F}$.*

PROOF. Assume, to the contrary, that $f_* \in \mathbf{A_F}$. Let $\{f_T\} \in \Phi$ be an approximating family. Consider the function $x_1(t) = \theta(t)$, $t \geqslant 0$, where θ is the indicator function of the set $[0,\infty)$. Thus, $\theta(t) = 1$ for $t \geqslant 0$ and $\theta(t) = 0$ for $t < 0$. Clearly, $f_*(x_1) = 0$. Therefore, there exists $T_1 > 1$ such that $f_{T_1}(x_1) < 1/3$. Let

$$x_2(t) = (1 - \theta(t - T_1))x_1(t) + \theta(t - T_1)\cos(2\pi t).$$

Observe that $x_2(t)$ is a continuous function on the half-line $[0,\infty)$. We prove the proposition by induction. Assume that a finite sequence of functions $x_1, x_2, \dots, x_n$, $(n \geqslant 2)$ satisfying the conditions below is constructed. The conditions are as follows:

(1) All the functions in the sequence are continuous on $[0,\infty)$.
(2) There exist nonnegative integers $T_0 = 0, T_1, \dots, T_{n-1}$, $T_i > 1 + T_{i-1}$, $i = 1, 2, \dots, n-1$, such that $x_i(t) = x_{i-1}(t)$ for $t \in [0, T_{i-1}]$, $i = 2, 3, \dots, n$.
(3) If $i = 2l+1$, then $x_i(t) = 1$ for $t \geqslant T_{i-1}$ $(1 \leqslant i \leqslant n)$. If $i = 2l$, then $x_i(t) = \cos(2\pi t)$ for $t \geqslant T_{i-1}$ $(1 \leqslant i \leqslant n)$.
(4) If $i = 2l+1$, then $f_{T_i}(x_i) < 1/3$, $1 \leqslant i \leqslant n-1$. If $i = 2l$, then $f_{T_i}(x_i) > 2/3$, $1 \leqslant i \leqslant n-1$.

The functions x_1, x_2 and the number T_1 satisfy these conditions. Now we are going to construct the number T_n and the function x_{n+1}.

Let n be an even number. Then the function $x_n(t)$ coincides with $\cos(2\pi t)$ for $t \geqslant T_{n-1}$ and does not have a limit as $t \to \infty$. Therefore, $f(x_n) = 1$. Then, there exists $T_n > T_{n-1} + 1$ such that $f_{T_n}(x_n) > 2/3$. The function x_{n+1} is defined by the relation

$$x_{n+1}(t) = (1 - \theta(t - T_n))x_n(t) + \theta(t - T_n).$$

Let n be an odd number. Then $x_n(t) = 1$ for $t \geqslant T_{n-1}$ and $f(x_n) = 0$. Hence, there exists $T_n > T_{n-1} + 1$ such that $f_{T_n}(x_n) < 1/3$. The function x_{n+1} in this case is defined by

$$x_{n+1}(t) = (1 - \theta(t - T_n))x_n(t) + \theta(t - T_n)\cos(2\pi t).$$

Clearly, the function $x_{n+1}(t)$ is continuous and satisfies, together with the number T_n, conditions (1)–(4).

Now let us consider the function

$$x_0(t) = x_n(t) \quad \text{for} \quad t \in [T_{n-1}, T_n), \quad n = 1, 2, \dots.$$

It follows from condition (2) that $x_0(t) = x_n(t)$ for $t \in [0, T_n)$. Therefore, we see that x_0 is continuous on $[0, T_n]$. Hence, $x_n(t) = x_0(t)$, $t \in [0, T_n]$. The function $x_0(t)$ does not have a limit as $t \to \infty$. Hence, $f(x_0) = 1$. Therefore, $\lim_{T\to\infty} f_T(x_0) = 1$. On the other hand, for odd n we have $f_{T_n}(x_0) = f_{T_n}(x_n) < 1/3$, which contradicts the assumption $f_* \in \mathbf{A_F}$. The proposition is proved. □

Observe that the property of a functional to be approximated depends on the space D where the functional is defined. If D coincides with the set of functions $x(t)$ having limits as $t \to \infty$, then $f_* \in \mathbf{A_F}$. However, if this functional is defined on the space $C([0,\infty))$, then, as it was shown, $f_* \notin \mathbf{A_F}$.

§3. Convergence in distribution for approximated functionals

Let us consider a family of random elements ξ_T^λ. Assume condition (A2.1.1):

$$\xi_T^\lambda \xrightarrow{w} \xi_T^0, \quad \lambda \to 0,$$

where $\xi_T^\lambda = \Pi_T \xi^\lambda$.

DEFINITION. We say that a functional $f_T \in \Phi_T$ is *admissible*, if the following conditions are satisfied:

(1) The functional f_T is measurable.
(2) The family of random variables $X_T^\lambda = f_T(\xi^\lambda)$ converges in distribution to the random variable $X_T^0 = f_T(\xi^0)$.

A simple property of admissible functionals is provided by the following statement.

PROPOSITION A2.3.1. *Let $\xi_T^\lambda = \Pi_T \xi^\lambda$, $\lambda > 0$, and $\xi_T^\lambda \xrightarrow{w} \xi_T^0$ as $\lambda \to 0$. Assume that $\hat{f}_T$ is measurable on D_T and A is the set of its discontinuity points. If*

$$\Pr\{\xi_T^0 \in A\} = 0,$$

then the corresponding functional defined by (A2.1.2) *belongs to Φ_T and is admissible.*

PROOF. Clearly, the functional f_t is measurable. Relation (A2.1.2) implies $X_T^\lambda = \hat{f}_T(\xi_T^\lambda)$. Now condition (2) follows from Theorem A1.1.4. □

THEOREM A2.3.1. *Assume that condition* (A2.1.1) *is satisfied and the functional $f \in \mathbf{A_F}$ on D can be approximated by a family $\{f_T\}$, where f_T is admissible for each $T \geqslant 0$. Let for any $\varepsilon > 0$ there exist $T_0 > 0$ and $\lambda_0 > 0$ such that the inequality*

$$\Pr\left\{|f_T(\xi^\lambda) - f(\xi^\lambda)| \geqslant \varepsilon\right\} < \varepsilon \tag{A2.3.1}$$

holds for all $T > T_0$ and $\lambda \in (0, \lambda_0)$. Then

$$f(\xi^\lambda) \xrightarrow{d} f(\xi^0) \quad as \quad \lambda \to 0.$$

PROOF. Consider the random variables $X_T^\lambda = f_T(\xi^\lambda)$ and $X^\lambda = f(\xi^\lambda)$ for $\lambda \geqslant 0$. Since the functionals $\{f_T\}$ are admissible for each $T > 0$, we get $X_T^\lambda \xrightarrow{d} X_T^0$. Next, relation (A2.1.3) and Theorem A1.1.5 with $\alpha = \xi^\lambda$ imply

$$X_T^\lambda \xrightarrow{d} X^\lambda \quad \text{as } T \to \infty$$

for any $\lambda \geqslant 0$. Now the assertion of the theorem follows from Theorem A1.1.3 with the random variable X_T^λ used instead of ξ_T^λ and X_λ instead of ξ^λ, $\lambda \geqslant 0$. □

APPENDIX III

AUXILIARY RESULTS

§1. Bounds for semimartingales

Here we have gathered several auxiliary facts not related directly to the subject of the book.

LEMMA A3.1.1. *Assume that a nonnegative process* $\xi = \{\xi_n\}$ *with discrete time* $n = 0, 1, \dots, N$ *can be represented as the sum of two processes* $\eta = \{\eta_n\}$ *and* $\zeta = \{\zeta_n\}$, $\xi = \eta + \zeta$, *such that one of them, say* η, *is a supermartingale, while the other,* ζ, *a submartingale. Then the inequality*

$$\Pr\left\{\sup_{0 \leqslant n \leqslant N} \xi_n \geqslant R\right\} \leqslant \frac{1}{R}(\mathbf{E}\eta_0 + \mathbf{E}\zeta_N) \tag{A3.1.1}$$

holds for any number $R > 0$.

PROOF. The proof of the lemma is similar to the proof of the Kolmogorov inequality for nonnegative semimartingales. Namely, let τ be the moment when the process ξ exceeds the level R for the first time,

$$\tau = \min(n\colon 0 \leqslant n \leqslant N,\ \xi_n \geqslant R).$$

If $\xi_n < R$ for all $n = 0, \dots, N$, we set $\tau = N$. Evidently,

$$\Pr\left\{\sup_{0 \leqslant n \leqslant N} \xi_n \geqslant R\right\} = \Pr\{\xi_\tau \geqslant R\}.$$

By the Chebyshev inequality for nonnegative variables,

$$\Pr\{\xi_\tau \geqslant R\} \leqslant \frac{1}{R}\mathbf{E}\xi_\tau = \frac{1}{R}(\mathbf{E}\eta_\tau + \mathbf{E}\zeta_\tau).$$

Since τ is a stopping time, and $0 \leqslant \tau \leqslant N$, well-known inequalities for semimartingales imply $\mathbf{E}\eta_\tau \leq \mathbf{E}\eta_0$ and $\mathbf{E}\zeta_\tau \leqslant \mathbf{E}\zeta_N$, and thus (A3.1.1) follows. □

§2. Bounds for processes with special structure

Other necessary auxiliary results involve processes with a special structure described below.

Let $\{\mathcal{F}_n\}$ $(n = 0, 1, \dots)$ be a sequence of nondecreasing σ-algebras, $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. A process $U = \{U_n\}$ $(n = 0, 1, \dots)$ is said to be *consistent* with the sequence $\{\mathcal{F}_n\}$ if each random variable U_n $(n = 0, 1, \dots)$ is measurable with respect to the σ-algebra $\mathcal{F}_n$.

Let us consider an arbitrary process $X = \{X_n\}$ with $X_0 = 0$ that is consistent with the sequence $\{\mathcal{F}_n\}$. Assume that X is represented as the sum of m processes consistent with the sequence $\{\mathcal{F}_n\}$:

$$X = Y + \sum_{i=1}^{m-1} Z^i, \tag{A3.2.1}$$

and $Y_0 = Z_0^i = 0$, $i = 1, \dots, m-1$.

LEMMA A3.2.1. *Let there exist positive numbers a, D_1, and D_2 such that for the process X the following conditions hold with probability* 1:[1]

(1) $\mathbf{E}(Y_n \mid \mathcal{F}_{n-1}) - Y_{n-1} \leqslant -a$, $n = 1, 2, \dots$;
(2) $\mathbf{E}[(Y_n - \mathbf{E}(Y_n \mid \mathcal{F}_{n-1}))^2 \mid \mathcal{F}_{n-1}] \leqslant D_1$, $n = 1, 2, \dots$;
(3) $\mathbf{E}(L_n \mid \mathcal{F}_{n-1}) \leqslant D_2 + L_{n-1}$, $n = 1, 2, \dots$, *where*

$$L_n = \sum_{i=1}^{m-1} (Z_n^i)^2.$$

Then the following inequalities hold for any natural N, any $R > 0$, and any $C \in [0, a(N+1))$:

$$\Pr\left\{\sup_{0 \leqslant n \leqslant N} X_n \geqslant R\right\} \leqslant \frac{mND}{R^2}, \tag{A3.2.2}$$

$$\Pr\left\{\sup_{N < n < \infty} X_n \geqslant -C\right\} \leqslant \frac{(N+1)2mD}{((N+1)a - C)^2}. \tag{A3.2.3}$$

$$\Pr\left\{\sup_{0 \leqslant n < \infty} X_n \geqslant R\right\} \leqslant \frac{2\sqrt{2}mD}{aR}. \tag{A3.2.4}$$

where $D = D_1 + D_2$.

PROOF. Let us put

$$\begin{aligned} \widetilde{Y}_n &= \sum_{j=1}^{n} (\mathbf{E}(Y_j \mid \mathcal{F}_{j-1}) - Y_{j-1}), \quad n \geqslant 1, \quad \widetilde{Y}_0 = 0, \\ \hat{Y}_n &= \sum_{j=1}^{n} (Y_j - \mathbf{E}(Y_j \mid \mathcal{F}_{j-1})), \quad n \geqslant 1, \quad \hat{Y}_0 = 0. \end{aligned} \tag{A3.2.5}$$

[1] Those readers who are acquainted with the modern martingale theory (see, e.g., [86]) will undoubtedly see that conditions 1 and 2 are restrictions imposed on the compensator and the quadratic characteristic of the process Y. However, since in what follows we do not use these notions, we do not discuss the relations of the result stated to this deep and advanced theory.

Then $Y = \widetilde{Y} + \hat{Y}$. Condition (1) of the lemma implies $\widetilde{Y} \leqslant -na$; hence,

$$X_n \leqslant -na + \hat{Y}_n + \sum_{i=1}^{m-1} Z_n^i.$$

Therefore,

$$\text{(A3.2.6)} \qquad \begin{aligned} \Pr\left\{\sup_{0\leqslant n\leqslant N} X_n \geqslant R\right\} &\leqslant \Pr\left\{\sup_{0\leqslant n\leqslant N}\left(\hat{Y}_n + \sum_{i=1}^{m-1} Z_n^i\right) \geqslant R\right\} \\ &\leqslant \Pr\left\{\sup_{0\leqslant n\leqslant N} \xi_n \geqslant R^2/m\right\}, \end{aligned}$$

and

$$\text{(A3.2.7)} \qquad \begin{aligned} \Pr\left\{\sup_{n>N} X_n \geqslant -C\right\} &\leqslant \Pr\left\{\sup_{n>N} \frac{1}{n}\left(\hat{Y}_n + \sum_{i=1}^{m-1} Z_n^i\right) \geqslant a - \frac{C}{N+1}\right\} \\ &\leqslant \Pr\left\{\sup_{n>N} \frac{1}{n^2}\xi_n \geqslant \frac{1}{m}\left(a - \frac{C}{N+1}\right)^2\right\}, \end{aligned}$$

where $\xi_n = \hat{Y}_n^2 + L_n$.

Relation (A3.2.5) implies $\hat{Y}_n - \hat{Y}_{n-1} = Y_n - \mathbf{E}(Y_n \mid \mathcal{F}_{n-1})$. The variable $\hat{Y}_{n-1}$ is measurable with respect to the σ-algebra $\mathcal{F}_{n-1}$. Therefore, $\mathbf{E}(\hat{Y}_n \mid \mathcal{F}_{n-1}) = \hat{Y}_{n-1}$, and thus

$$\mathbf{E}[(\hat{Y}_n - \hat{Y}_{n-1})^2 \mid \mathcal{F}_{n-1}] = \mathbf{E}(\hat{Y}_n^2 \mid \mathcal{F}_{n-1}) - \hat{Y}_{n-1}^2.$$

Taking into account conditions (2) and (3) of the lemma, we infer $\mathbf{E}(\hat{Y}_n^2 \mid \mathcal{F}_{n-1}) \leqslant D_1 + \hat{Y}_{n-1}^2$ and

$$\text{(A3.2.8)} \qquad \mathbf{E}(\xi_n \mid \mathcal{F}_{n-1}) \leqslant D + \xi_{n-1}.$$

Now let us consider the processes $\eta_n = \xi_n - nD$ and $\zeta_n = nD$. Evidently, the process $\{\zeta_n\}$ is a submartingale. Relation (A3.2.8) for the process $\{\eta_n\}$ implies

$$\mathbf{E}(\eta_n \mid \mathcal{F}_{n-1}) = \mathbf{E}(\xi_n - nD \mid \mathcal{F}_{n-1}) \leqslant \xi_{n-1} + (n-1)D.$$

Therefore, the process $\{\eta_n\}$ is a supermartingale with respect to the sequence $\{\mathcal{F}_n\}$. Applying Lemma A3.1.1 (with R replaced by R^2/m) and taking into account the relation $\eta_0 = 0$, we get

$$\Pr\left\{\sup_{0\leqslant n\leqslant N} \xi_n \geqslant \frac{R^2}{m}\right\} \leqslant \frac{DNm}{R^2}.$$

The latter inequality together with (A3.2.6) yields (A3.2.2).

To prove (A3.2.3), we consider the process

$$U_n = \frac{1}{n^2}\xi_n + D\sum_{r=n+1}^{\infty} \frac{1}{r^2}.$$

Relation (A3.2.8) implies

$$\mathbf{E}(U_n \mid \mathcal{F}_{n-1}) \leqslant \frac{1}{n^2}(D + \xi_{n-1}) + D \sum_{r=n+1}^{\infty} \frac{1}{r^2} = \frac{1}{n^2}\xi_{n-1} + D \sum_{r=n}^{\infty} \frac{1}{r^2} \leqslant U_{n-1}.$$

Therefore, the process U_n is a supermartingale with respect to the sequence $\{\mathcal{F}_n\}$. Relying on the Kolmogorov inequality for nonnegative supermartingales, we infer

$$\begin{aligned} \Pr\left\{\sup_{n>N} \frac{1}{n^2}\xi_n \geqslant R(a,m,N,C)\right\} &\leqslant \Pr\left\{\sup_{n>N} U_n \geqslant R(a,m,N,C)\right\} \\ &\leqslant \frac{\mathbf{E}U_{N+1}}{R(a,m,N,C)}, \end{aligned} \tag{A3.2.9}$$

where $R(a,m,N,C) = \frac{1}{m}\left(a - \frac{C}{N+1}\right)^2$.

Now let us bound $\mathbf{E}U_{N+1}$. It follows from (A3.2.8) that $\mathbf{E}\xi_n \leqslant D + \mathbf{E}\xi_{n-1}$. Taking into account that $\xi_0 = 0$, we get $\mathbf{E}\xi_{N+1} \leqslant (N+1)D$, and thus

$$\mathbf{E}U_{N+1} \leqslant \frac{D}{N+1} + D \sum_{r=N+2}^{\infty} \frac{1}{r^2} \leqslant \frac{2D}{N+1}.$$

Introducing this to (A3.2.9) and taking into account (A3.2.7), we get inequality (A3.2.3).

To prove inequality (A3.2.4), let us choose an arbitrary natural N. Since

$$\sup_{n\geqslant 0} X_n = \max\left(\sup_{0\leqslant n\leqslant N} X_n, \sup_{n>N} X_n\right),$$

we have

$$\begin{aligned} \Pr\left\{\sup_{n\geqslant 0} X_n \geqslant R\right\} &\leqslant \Pr\left\{\sup_{0\leqslant n\leqslant N} X_n \geqslant R\right\} + \Pr\left\{\sup_{n>N} X_n \geqslant R\right\} \\ &\leqslant \Pr\left\{\sup_{0\leqslant n\leqslant N} X_n \geqslant R\right\} + \Pr\left\{\sup_{n>N} X_n \geqslant 0\right\}. \end{aligned}$$

The first term in the right-hand side is estimated with the help of (A3.2.2), and the second one with the help of (A3.2.3) (with $C = 0$); thus, we get

$$\Pr\left\{\sup_{n\geqslant 0} X_n \geqslant R\right\} \leqslant mD\left(\frac{N}{R^2} + \frac{2}{(N+1)a^2}\right).$$

To get (A3.2.4), it suffices to set $N = [\sqrt{2}R/a]$ in the latter inequality. The proof is completed. □

COROLLARY A3.2.1. *Let* $\pi_1, \dots, \pi_n, \dots$ *be a sequence of independent identically distributed random variables, and*

$$\sigma_n = \sum_{j=1}^{n} \pi_j, \; n \geqslant 1, \quad \sigma_0 = 0.$$

If $\mathbf{E}\pi_n = -a < 0$, *then the following inequalities hold for any natural* N, *an arbitrary* $R > 0$, *and an arbitrary* $C \in [0, (N+1)a)$:

$$\text{(A3.2.10)} \qquad \Pr\left\{ \sup_{0 \leqslant n \leqslant N} \sigma_n \geqslant R \right\} \leqslant \frac{ND}{R^2},$$

$$\text{(A3.2.11)} \qquad \Pr\left\{ \sup_{n > N} \sigma_n \geqslant -C \right\} \leqslant \frac{2D(N+1)}{(a(N+1) - C)^2},$$

$$\text{(A3.2.12)} \qquad \Pr\left\{ \sup_{n \geqslant 0} \sigma_n \geqslant R \right\} \leqslant \frac{2\sqrt{2}D}{aR},$$

where $D = \mathbf{D}\pi_n$.

PROOF. Let us put $Y_n = \sigma_n$ and $m = 1$ in (A3.2.1). In this case, all the processes Z^i disappear, and L_n vanishes. Condition (3) of Lemma A3.2.1 holds for $D_2 = 0$. The σ-algebra generated by π_j, $j = 1, \dots, n$, is regarded as the σ-algebra $\mathcal{F}_n$. Let us verify conditions (1) and (2) of Lemma A3.2.1. Evidently, $\sigma_n = \pi_n + \sigma_{n-1}$. Therefore,

$$\mathbf{E}(\sigma_n \mid \mathcal{F}_{n-1}) = \mathbf{E}\pi_n + \sigma_{n-1} = -a + \sigma_{n-1},$$

and, hence, condition (1) holds. Condition (2) holds as well, since $\sigma_n - \mathbf{E}(\sigma_n \mid \mathcal{F}_{n-1}) = \pi_n - \mathbf{E}\pi_n$ and

$$\mathbf{E}[(\sigma_n - \mathbf{E}(\sigma_n \mid \mathcal{F}_{n-1}))^2 \mid \mathcal{F}_{n-1}] = \mathbf{D}\pi_n = D.$$

Applying now Lemma A3.2.1, we get (A3.2.10), (A3.2.11), and (A3.2.12) from (A3.2.2), (A3.2.3), and (A3.2.4), respectively. □

Observe that inequality (A3.2.12) implies the following

COROLLARY A3.2.2. *Under the assumptions of Corollary* A3.2.1,

$$\Pr\{ \sup_{0 \leqslant n < \infty} \sigma_n < \infty \} = 1.$$

APPENDIX IV

TANDEM QUEUES IN HEAVY TRAFFIC

In this appendix we solve problem (2.4.12)–(2.4.15) for tandem queues. We find an explicit formula for the joint limit distribution function of waiting times in heavy traffic.

§1. Reduction to a boundary value problem

The equation describing the stationary joint distribution of waiting times is an elliptic equation with constant coefficients. Relations (2.4.12)–(2.4.15) provide a problem with obligue derivative for a two-dimensional elliptic differential equation. Similar problems have been studied by numerous authors; however, our case seems to be different from those analyzed in the literature (see Gakhov [35], Gakhov and Chersky [36]).

Below we find a solution using the reduction to a certain boundary value problem for functions of a complex variable.

To simplify the notation, we use the functions $\Phi(x)$ and $\Phi(x, y)$ instead of the functions $u_1(x)$ and $u_2(\mathbf{x})$, respectively. Recall (see Chapter II, §4 and also Introduction) that the limit waiting time distribution $\Phi(x)$ at the first phase of our queueing system satisfies the equation

$$-a_1 \frac{d\Phi}{dx} + \frac{1}{2}(d + d_1)\frac{d^2\Phi}{dx^2} = 0, \quad -\infty < x < 0,$$

and the boundary conditions

$$\lim_{x \to -\infty} \Phi(x) = 1, \qquad \lim_{x \to 0-} \Phi(x) = 0.$$

It is obvious that

$$\Phi(x) = \begin{cases} 0, & \text{if } x \geqslant 0, \\ 1 - \exp\dfrac{2a_1 x}{d + d_1}, & \text{if } x < 0\,. \end{cases}$$

Now we reformulate problem (2.4.12)–(2.4.15) for the case of tandem queues.

Basic two-dimensional problem. Find a function $\Phi(x, y)$ that is continuous on the entire plane and satisfies the following conditions:

(1) $\Phi(x, y) = 0$ if $x \geqslant 0$;
(2) $\Phi(x, y)$ does not depend on x for $x < y$, $x < 0$;

(3) for $y \leqslant x < 0$, the function $\Phi(x, y)$ is a solution of the elliptic differential equation

$$\mathbf{L}\Phi(x, y) = 0, \tag{A4.1.1}$$

$$\frac{\partial\Phi(x, x)}{\partial x} = 0, \tag{A4.1.2}$$

$$\Phi(0, y) = 0; \tag{A4.1.3}$$

(4) the following relations holds on the entire plane uniformly in x:

$$\lim_{y\to-\infty}(\Phi(x, y) - \Phi(x)) = 0, \tag{A4.1.4}$$

where

$$\mathbf{L}\Phi = -a_1\frac{\partial\Phi}{\partial x} - a_2\frac{\partial\Phi}{\partial y} + \frac{1}{2}\left((d + d_1)\frac{\partial^2\Phi}{\partial x^2} + 2d\frac{\partial^2\Phi}{\partial x\partial y} + (d + d_2)\frac{\partial^2\Phi}{\partial y^2}\right).$$

Observe that the drift coefficients and the elements of the diffusion matrix satisfy the inequalities

$$a_i > 0 \quad (i = 1, 2), \qquad d_1d_2 + d(d_1 + d_2) > 0. \tag{A4.1.5}$$

In what follows, we refer to the problem defined by relations (A4.1.1)–(A4.1.4) as problem (**BP**).

To solve the problem, we must rewrite the equation and the boundary conditions in the invariant metric. This metric is defined by the scalar product $\langle x, y\rangle = x\widehat{D}^{-1}y^T$, where $\widehat{D}$ is the diffusion matrix. This metric is called invariant, since the scalar product of vectors is preserved under linear transformations diagonalizing the matrix of the operator $\mathbf{L}$. Therefore, the angles between vectors do not change under these transformations.

PROPOSITION A4.1.1. *In an appropriate coordinate system x_1, x_2 with the invariant metric, problem* (**BP**) *is transformed to the following one: in terms of the angle*

$$0 < x_2\cos\alpha < x_1\sin\alpha, \quad 0 < \alpha \leqslant \pi/2,$$

find a function $\Psi(x_1, x_2)$ such that

$$\triangle\Psi + 2\frac{\partial\Psi}{\partial\mathbf{e}(\gamma)} = 0, \quad \Psi(x_1, 0) = 0, \quad \left.\frac{\partial\Psi}{\partial\mathbf{e}(\beta)}\right|_l = 0,$$

and

$$\lim_{x_1\to\infty}(\Psi(x_1, x_2) - (1 - \exp(-2\sin x_2))) = 0$$

uniformly in x_2, where l is the ray defined by the equation $x_2\cos\alpha = x_1\sin\alpha$, and $\mathbf{e}(\varphi) = (\cos\varphi, \sin\varphi)$. The angles α, β, γ satisfy the inequalities

$$-\pi/2 \leqslant \beta < 0, \quad 0 < \gamma - \beta < \pi, \quad \alpha - \beta \geqslant \pi/2.$$

PROOF. Consider the diffusion matrix

$$\widehat{D} = \begin{pmatrix} d + d_1 & d \\ d & d + d_2 \end{pmatrix}.$$

The inverse matrix is given by

$$\widehat{D}^{-1} = \frac{1}{\Delta} \begin{pmatrix} d + d_2 & -d \\ -d & d + d_1 \end{pmatrix},$$

where

$$\Delta = \det \widehat{D} = d_1 d_2 + d(d_1 + d_2) > 0.$$

The problem for the function Φ is described by the vectors

$$\mathbf{e} = (1, 0), \quad \mathbf{k} = (1, 1), \quad \mathbf{r} = (0, -1), \quad \mathbf{g} = (-a_1, -a_2).$$

The vector $\mathbf{e}$ corresponds to the reflecting direction at the line defined by $\mathbf{k}$. The vector $\mathbf{r}$ corresponds to the second side of the angle. The drift of the process is given by the vector $\mathbf{g}$.

The lengths of these vectors are as follows:

$$\begin{aligned} \Delta|\mathbf{e}|^2 &= (1, 0)\widehat{D}^{-1}(1, 0)^T = d_2 + d, \\ \Delta|\mathbf{k}|^2 &= d_2 + d_1, \\ \Delta|\mathbf{r}|^2 &= d_1 + d, \\ \Delta|\mathbf{g}|^2 &= (a_1 - a_2)^2 + a_1^2 d_2 + a_2^2 d_1. \end{aligned}$$

The scalar products of the vectors satisfy the relations

$$\begin{aligned} \Delta\langle \mathbf{e}, \mathbf{g}\rangle &= (a_2 - a_1)d - a_1 d_2, \\ \Delta\langle \mathbf{k}, \mathbf{g}\rangle &= -(a_1 d_2 + a_2 d_1), \\ \Delta\langle \mathbf{r}, \mathbf{g}\rangle &= -(a_2 d_1 + d(a_2 - a_1)), \\ \Delta\langle \mathbf{k}, \mathbf{e}\rangle &= d_2, \\ \Delta\langle \mathbf{e}, \mathbf{r}\rangle &= d, \\ \Delta\langle \mathbf{k}, \mathbf{r}\rangle &= -d_1. \end{aligned}$$

Then, for the angle between the vectors $\mathbf{k}$ and $\mathbf{r}$ we have

$$\cos(\pi - \alpha) = -\cos\alpha = \frac{\langle \mathbf{k}, \mathbf{r}\rangle}{|\mathbf{k}||\mathbf{r}|}.$$

Therefore,

$$\cos\alpha = \frac{d_1}{\sqrt{(d + d_1)(d_2 + d_1)}} \geqslant 0.$$

and $0 < \alpha \leqslant \pi/2$. For the angle between the vectors **e** and **r** we have

$$\cos\beta = \frac{d}{\sqrt{(d+d_1)(d_2+d)}} \geqslant 0.$$

Therefore $-\pi/2 < \beta < 0$. It is not difficult to show that

$$\begin{aligned} \cos\alpha &< \cos\beta, & \text{if } d &< d_1, \\ \cos\alpha &= \cos\beta, & \text{if } d &= d_1, \\ \cos\alpha &> \cos\beta, & \text{if } d &> d_1. \end{aligned}$$

The scalar product of the vectors **g** and **k** is not positive. Therefore, $0 < \gamma - \beta < \pi$. □

§2. Solution of the boundary value problem (BP)

Passing to a new unknown function $\Psi_0(\mathbf{x}) = \exp(\mathbf{e}(\gamma), \mathbf{x})\Psi(\mathbf{x})$, we get the following problem:

$$\text{(A4.2.1)} \qquad \triangle\Psi_0 = \Psi_0,$$

$$\text{(A4.2.2)} \qquad \Psi_0(x_1, 0) = 0,$$

$$\text{(A4.2.3)} \qquad \left(\frac{\partial\Psi_0}{\partial\mathbf{e}} - (\mathbf{e}, \mathbf{e}(\gamma))\Psi_0\right)\Big|_l = 0,$$

$$\text{(A4.2.4)} \qquad \lim_{x_1\to\infty}(\Psi_0(x_1, x_2) - \exp(\mathbf{e}(\gamma), \mathbf{x}) + \exp(\mathbf{e}(-\gamma), \mathbf{x}))e^{-(\mathbf{e}(\gamma).\mathbf{x})} = 0,$$

where $\mathbf{e} = \mathbf{e}(\beta)$.

The main idea of the solution is to represent the function Ψ_0 as the sum $\Psi_0(\mathbf{x}) = U(\mathbf{x}) + V(\mathbf{x})$, where $U(\mathbf{x})$ is a linear combination of exponential functions, and $V(\mathbf{x})$ has an explicit integral representation. Below we consider three cases:

$$1.\ \gamma < \pi + 2\beta; \qquad 2.\ \gamma > \pi + 2\beta; \qquad 3.\ \gamma = \pi + 2\beta.$$

2.1. Case 1: $\gamma < \pi + 2\beta$**.** We assume first that $(\pi - \gamma)/(2\alpha)$ is not an integer, and put

$$m = \left[\frac{\pi - \gamma}{2\alpha}\right].$$

For $k = 0, 1, \ldots, m$, we introduce the sequences of vectors

$$\mathbf{g}_k = \mathbf{e}(\gamma + 2k\alpha), \qquad \mathbf{g}'_k = \mathbf{e}(-\gamma - 2k\alpha).$$

Observe that the vectors $\mathbf{g}'_k$ are obtained by the reflection of the vectors $\mathbf{g}_k$ with respect to the x_1-axis.

Let us put

$$U(\mathbf{x}) = \sum_{k=0}^{m} C_k(e^{(\mathbf{g}_k.\mathbf{x})} - e^{(\mathbf{g}'_k.\mathbf{x})}).$$

Then the function $U(\mathbf{x})$ satisfies (A4.2.1) and (A4.2.2). If $C_0 = 1$, then

$$\lim_{x_1\to\infty} (U(x_1, x_2) - \exp(\mathbf{e}(\gamma), \mathbf{x}) + \exp(\mathbf{e}(-\gamma), \mathbf{x}))e^{-(\mathbf{e}(\gamma), \mathbf{x})} = 0.$$

Let us denote $\mathbf{g} = \mathbf{e}(\gamma)$ and consider the expression $(\partial U/\partial \mathbf{e} - (\mathbf{e}, \mathbf{g})U)\,|_l$. We have

$$\frac{\partial U}{\partial \mathbf{e}} - (\mathbf{e}, \mathbf{g})U = \sum_{k=0}^{m} C_k(((\mathbf{g}_k, \mathbf{e}) - (\mathbf{g}, \mathbf{e}))e^{(\mathbf{g}_k, \mathbf{x})} - ((\mathbf{g}'_k, \mathbf{e}) - (\mathbf{g}, \mathbf{e}))e^{(\mathbf{g}'_k, \mathbf{x})}).$$

It is not difficult to verify that $(\mathbf{g}_k, \mathbf{x})\,|_l = (\mathbf{g}'_{k-1}, \mathbf{x})\,|_l$. Therefore,

$$\left(\frac{\partial U}{\partial \mathbf{e}} - (\mathbf{e}, \mathbf{g})U\right)\bigg|_l$$
$$= \sum_{k=1}^{m} e^{(\mathbf{g}_k, \mathbf{x})}(C_k(\mathbf{g}_k - \mathbf{g}, \mathbf{e}) - C_{k-1}(\mathbf{g}'_{k-1} - \mathbf{g}, \mathbf{e}))\,|_l - C_m(\mathbf{g}'_m - \mathbf{g}, \mathbf{e})e^{(\mathbf{g}'_m, x)}\,|_l\,.$$

Let us set

(A4.2.5) $$C_k(\mathbf{g}_k - \mathbf{g}, \mathbf{e}) = C_{k-1}(\mathbf{g}'_{k-1} - \mathbf{g}, \mathbf{e});$$

it follows from the condition $\gamma < \pi + 2\beta$ that $(\mathbf{g}_k - \mathbf{g}, \mathbf{e}) \neq 0$ for $k = 1, \ldots, m$. Then (A4.2.5) and the condition $C_0 = 1$ define the function $U(\mathbf{x})$ uniquely, with

$$C_k = \prod_{j=1}^{k} \frac{(\mathbf{g}'_{j-1} - \mathbf{g}, \mathbf{e})}{(\mathbf{g}_j - \mathbf{g}, \mathbf{e})}, \qquad k = 1, \ldots, m,$$

and

$$\left(\frac{\partial U}{\partial \mathbf{e}} - (\mathbf{e}, \mathbf{g})U\right)\bigg|_l = Ae^{(\mathbf{g}'_m, \mathbf{x})},$$

where $A = -C_m(\mathbf{g}'_m - \mathbf{g}, \mathbf{e})$. The second term in the representation of the solution, the function $V(\mathbf{x})$, satisfies the equation

(A4.2.6) $$\triangle V = V$$

with the boundary conditions

(A4.2.7) $$V(x_1, 0) = 0,$$

(A4.2.8) $$\left(\frac{\partial V}{\partial \mathbf{e}} - (\mathbf{e}, \mathbf{e}(\gamma))V\right)\bigg|_l = -Ae^{(\mathbf{g}'_m, \mathbf{x})},$$

and

(A4.2.9) $$\lim_{x_1\to\infty} e^{-(\mathbf{g}, \mathbf{x})}V(\mathbf{x}) = 0$$

uniformly in x_2. The function $V(x_1, x_2)$ is sought in the form

(A4.2.10)
$$V(x_1, x_2) = \int_{-\infty}^{\infty} f(s) \exp(-x_1\sqrt{1+t^2} + ix_2 t)\,ds,$$

where the variables s and t satisfy the relation $t = \sinh(2\alpha/\pi \operatorname{arcsinh} s)$, while the function $f(s)$ is continuous, odd, and satisfies the inequality

(A4.2.11)
$$\int_{-\infty}^{\infty} |tf(s)|\,ds < \infty.$$

It is not difficult to verify that the function V defined in this way satisfies (A4.2.6), (A4.2.7), and (A4.2.9). Condition (A4.2.8) can be rewritten in the form

(A4.2.12)
$$\int_{-\infty}^{\infty} f(s)Q(s)e^{-\rho\zeta(s)} = Ae^{-\rho a}, \qquad \rho > 0,$$

where

$$Q(s) = \cos(\beta - \gamma) + \sqrt{1+t^2}\cos\beta - it\sin\beta,$$
$$\rho = \sqrt{x_1^2 + x_2^2}, \qquad a = -(\mathbf{e}(\alpha), \mathbf{g}'_m),$$

and

$$\zeta(s) = \sqrt{1+t^2}\cos\alpha - it\sin\alpha = \cosh\frac{2\alpha}{\pi}\operatorname{arccosh}\frac{s}{i},$$

where the principal branch of the function arccosh is assumed.

Let us consider the function $\zeta(s)$ as a point in the complex plane $x_1 + ix_2$. When s varies from $-\infty$ to ∞, the point $\zeta(s)$ runs over the right branch of the hyperbola

(A4.2.13)
$$\frac{x_1^2}{\cos^2\alpha} - \frac{x_2^2}{\sin^2\alpha} = 1,$$

while the point $Q(s)$ runs over the right branch of the hyperbola

(A4.2.14)
$$\frac{x_1^2 - \cos(\beta-\gamma)}{\cos^2\beta} - \frac{x_2^2}{\sin^2\beta} = 1.$$

Observe that $a = -(\mathbf{g}'_m, \mathbf{e}(\alpha)) > \cos\alpha$, and hence the point $(a, 0)$ lies inside the area bounded by the right branch of hyperbola (A4.2.13). The function $\zeta(s)$ is analytic in the interior of the right branch of the hyperbola (A4.2.13). Therefore, (A4.2.12) is satisfied if the product $f(s)Q(s)$ is represented as

(A4.2.15)
$$f(s)Q(s) = \frac{\Psi_+(s)}{ip - s},$$

where $\Psi_+(s)$ is the boundary value of a function analytic in the upper half-plane and increasing as $s \to \infty$ at most as a power function of s,

$$p = \cosh\frac{\pi}{2\alpha}\operatorname{arccosh} a = \cos\frac{\pi \arccos a}{2\alpha} > 0.$$

and $\Psi_+(ip) = -A/(2\pi i)$. Using the fact that the function $f(s)$ is odd, we get

$$-\frac{Q(s)}{Q(-s)} = \frac{\Psi_+(s)(ip+s)}{\Psi_+(s)(ip-s)}.$$

It is easy to see that $Q(-s) = \overline{Q}(s)$. Therefore, denoting by $\varphi(s)$ the continuous branch of $\operatorname{Arg} Q(s)$, $\operatorname{Im} s = 0$, determined by the condition $\lim_{s\to\infty} \varphi(s) = -\beta$, we can rewrite the latter equation as

$$\text{(A4.2.16)} \qquad -e^{2i\varphi(s)} = \frac{\Psi_+(s)(ip+s)}{\Psi_+(s)(ip-s)}.$$

The condition $\gamma < \pi+2\beta$ guarantees that the vertex of the right branch of the hyperbola (A4.2.14) is located on the positive real semi-axis. Therefore, the function $\varphi(s)$ is odd, and

$$\text{(A4.2.17)} \qquad \lim_{s\to-\infty} \varphi(s) = \beta.$$

Let us put

$$\varphi_+(s) = \frac{s-ip}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi(\sigma)}{(\sigma-s)(\sigma-ip)}d\sigma \quad \text{for} \quad \operatorname{Im} s > 0,$$
$$\varphi_-(s) = \frac{s-ip}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi(\sigma)}{(\sigma-s)(\sigma-ip)}d\sigma \quad \text{for} \quad \operatorname{Im} s < 0.$$

The functions φ_+ and φ_- are analytic in the half-planes $\operatorname{Im} s > 0$ and $\operatorname{Im} s < 0$, respectively, and $\varphi(ip) = 0$. Using the fact that the function $\varphi(\sigma)$ is odd, one gets that $\varphi_-(\sigma) = \varphi_+(-\sigma)$ for $\operatorname{Im}\sigma < 0$. Therefore, according to the Sokhotsky formula (see, e.g., [35]),

$$\text{(A4.2.18)} \qquad \varphi_+(s) - \varphi_+(-s) = \varphi(s),$$

$$\text{(A4.2.19)} \qquad \varphi_+(s) + \varphi_+(-s) = \frac{s-ip}{\pi i}\int_{-\infty}^{\infty} \frac{\varphi(\sigma)}{(\sigma-s)(\sigma-ip)}d\sigma$$

for $\operatorname{Im} s = 0$, where the integral is understood in the sense of principal value. It follows easily from (A4.2.17) that $\varphi_+(s) \equiv \frac{\beta}{\pi}\log|s|$ as $s \to \infty$ ($\operatorname{Im} s \geqslant 0$).

Relation (A4.2.18) implies

$$-e^{2i\varphi(s)} = \frac{s e^{2i\varphi_+(s)}}{(-s)2i\varphi_+(-s)}.$$

Therefore, we can take

$$\Psi_+(s) = \frac{Ai}{\pi}\frac{s e^{2i\varphi_+(s)}}{ip+s}$$

in (A4.2.16). Indeed, since $\varphi(ip) = 0$, we have $\Psi(ip) = -\frac{A}{2\pi i}$, and $|\Psi_+(s)| = O(|s|^{\beta_0})$ as $s \to \infty$ ($\operatorname{Im} s \geqslant 0$), where $\beta_0 = 2\beta/\pi$. Therefore, (A4.2.15) yields

$$f(s)Q(s) = \frac{A}{i\pi}\frac{se^{2i\varphi_+(s)}}{p^2+s^2}. \tag{A4.2.20}$$

Finally, it follows from the asymptotic behavior of $\Psi_+(s)$ that $|tf(s)| = O(|s|^{\beta_0-1})$ as $s \to \infty$. Therefore, relation (A4.2.13) is satisfied.

It follows from (A4.2.18)–(A4.2.20) that

$$f(s) = \frac{A}{i\pi}\frac{se^{q(s)}}{p^2+s^2}\frac{1}{|Q(s)|}, \tag{A4.2.21}$$

where

$$q(s) = \frac{s-ip}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(\sigma)}{(\sigma-s)(\sigma-ip)}d\sigma.$$

Observe that the expression for the function $q(s)$ can be rewritten as follows:

$$q(s) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(\varphi(\sigma)-\varphi(s))(\sigma s+p^2)}{(\sigma-s)(\sigma^2+p^2)}d\sigma. \tag{A4.2.22}$$

Indeed, we have

$$\frac{s-ip}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(\sigma)d\sigma}{(\sigma-ip)(\sigma-s)} = \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(\sigma)\left(\frac{1}{\sigma-s}-\frac{\sigma}{\sigma^2+p^2}-\frac{ip}{\sigma^2+p^2}\right)d\sigma,$$

and therefore, since the function $\varphi(\sigma)$ is odd,

$$\pi q(s) = \int_{-\infty}^{\infty}\varphi(\sigma)\left(\frac{1}{\sigma-s}-\frac{\sigma}{\sigma^2+p^2}\right)d\sigma.$$

Since

$$\int_{-\infty}^{\infty}\left(\frac{1}{\sigma-s}-\frac{\sigma}{\sigma^2+p^2}\right)d\sigma = 0$$

in the sense of principal value, we get (A4.2.22).

Relations (A4.2.12), (A4.2.21), and (A4.2.22) provide a representation for the function $V(\mathbf{x})$ in the case when $(\pi-\gamma)/(2\alpha)$ is not an integer.

If the number $(\pi-\gamma)/(2\alpha)$ is integer, then $p = 0$. The function $V(\mathbf{x})$ is again determined by relations (A4.2.12) and (A4.2.21), and the integral in (A4.2.22) is understood as its limit value as $p \to 0+$.

2.2. Case 2: $\gamma > \pi + 2\beta$. Along with the sequences of vectors $\mathbf{g}_k$ and $\mathbf{g}'_k$, we consider the sequences $\mathbf{r}_k$ and $\mathbf{r}'_k$, where

$$\mathbf{r}_k = \mathbf{e}(2\pi+2\beta-\gamma-2k\alpha), \qquad \mathbf{r}'_k = \mathbf{e}(\gamma-2\pi-2\beta-2k\alpha), \quad k = 0, 1 \dots.$$

We assume that

$$\frac{\pi-\gamma}{2\alpha}, \quad \frac{\gamma-\pi-2\beta}{2\alpha}, \quad \frac{\pi+\beta-\gamma}{2\alpha}$$

are not integers, and put

$$m = \left[\frac{\pi - \gamma}{2\alpha}\right], \qquad n = \left[\frac{\gamma - 2\beta - \pi}{2\alpha}\right].$$

Observe that the vectors $\mathbf{r}'_k$ are obtained by reflection of the vectors $\mathbf{r}_k$ with respect to the x_1-axis.

Let us put

$$U(\mathbf{x}) = \sum_{k=0}^{m} C_k (e^{(\mathbf{g}_k.\mathbf{x})} - e^{(\mathbf{g}'_k.\mathbf{x})}) + \sum_{k=0}^{n} D_k (e^{(\mathbf{r}_k.\mathbf{x})} - e^{(\mathbf{r}'_k.\mathbf{x})}). \tag{A4.2.23}$$

Then the function $U(\mathbf{x})$ satisfies (A4.2.1) and (A4.2.2). If $C_0 = 1$, then

$$\lim_{x_1 \to \infty} (U(x_1, x_2) - \exp(\mathbf{e}(\gamma), \mathbf{x}) + \exp(\mathbf{e}(-\gamma), \mathbf{x}))e^{-(\mathbf{e}(\gamma).\mathbf{x})} = 0.$$

Observe that $(\mathbf{g}, \mathbf{e}) = (\mathbf{r}_0, \mathbf{e})$. Therefore, if the constants C_k are given by

$$C_k = \prod_{j=1}^{k} \frac{(\mathbf{g}'_{j-1} - \mathbf{g}, \mathbf{e})}{(\mathbf{g}_j - \mathbf{g}, \mathbf{e})}, \quad k = 1, \dots, m,$$

and D_k satisfy the equation

$$D_k(\mathbf{r}_k - \mathbf{g}.\mathbf{e}) = D_{k-1}(\mathbf{r}'_{k-1} - \mathbf{g}.\mathbf{e}), \quad k = 1, 2, \dots, n, \tag{A4.2.24}$$

then

$$\left(\frac{\partial U}{\partial \mathbf{e}} - (\mathbf{e}.\mathbf{g})U\right)\Big|_l = (Ae^{(\mathbf{g}'_m.\mathbf{x})} + Be^{(\mathbf{r}'_n.\mathbf{x})})\,|_l\,,$$

where

$$A = -C_m(\mathbf{g}'_m - \mathbf{g}.\mathbf{e}), \qquad B = -D_n(\mathbf{r}'_n - \mathbf{g}.\mathbf{e}),$$

and D_k are given by

$$D_k = D_0 \prod_{j=1}^{k} \frac{(\mathbf{r}'_{j-1} - \mathbf{g}.\mathbf{e})}{(\mathbf{r}_j - \mathbf{g}.\mathbf{e})}, \quad k = 1, \dots, n.$$

The constant D_0 will be determined later.

The second term in the representation of the solution, the function $V(\mathbf{x})$, satisfies the equation $\triangle V = V$ with the boundary conditions

$$V(x_1, 0) = 0,$$
$$\left(\frac{\partial V}{\partial \mathbf{e}} - (\mathbf{e}.\mathbf{e}(\gamma))V\right)\Big|_l = -(Ae^{(\mathbf{g}_m{}'.\mathbf{x})} + Be^{(\mathbf{r}_n{}'.\mathbf{x})})\,|_l. \tag{A4.2.25}$$

As before, the function $V(x_1, x_2)$ is sought in the form

$$V(x_1, x_2) = \int_{-\infty}^{\infty} f(s) \exp(-x_1\sqrt{1 + t^2} + ix_2 t)ds.$$

where the variables s and t satisfy the relation $t = \sinh(2\alpha/\pi \operatorname{arcsinh} s)$, while the function $f(s)$ is continuous and satisfies the inequality

$$\int_{-\infty}^{\infty} |tf(s)|ds < \infty.$$

According to (A4.2.25),

$$\int_{-\infty}^{\infty} f(s)Q(s)e^{-p\zeta(s)}ds = Ae^{-pa} + Be^{-pb}, \quad p > 0, \tag{A4.2.26}$$

where $b = -(\mathbf{r}'_n, \mathbf{e}(\alpha))$. Observe that the points $(a, 0)$, $(b, 0)$ lie inside the area bounded by the right branch of the hyperbola

$$\frac{x_1^2}{\cos^2\alpha} - \frac{x_2^2}{\sin^2\alpha} = 1.$$

Then equation (A4.2.26) is satisfied, provided

$$f(s)Q(s) = \Psi_+(s)\frac{1}{ip - s}\frac{1}{iq - s},$$

where

$$p = \cosh\frac{\pi}{2\alpha}\operatorname{arccosh} a = \cos\frac{\pi \arccos a}{2\alpha} > 0,$$
$$q = \cosh\frac{\pi}{2\alpha}\operatorname{arccosh} b = \cos\frac{\pi \arccos b}{2\alpha} > 0,$$

and $\Psi_+(s)$ is the boundary value of a function analytic in the upper half-plane and increasing as $s \to \infty$ at most as a power function of s. Moreover,

$$\Psi_+(ip) = -\frac{A}{2\pi}(q - p), \qquad \Psi_+(iq) = -\frac{B}{2\pi}(p - q).$$

Let us consider once more the function $\zeta(s)$ as a point in the complex plane $x_1 + ix_2$. Since the function $f(s)$ is odd, one can rewrite equation (A4.2.16) in the form

$$-e^{2i\varphi(s)} = \frac{\Psi_+(s)(ip + s)(iq + s)}{\Psi_+(s)(ip - s)(iq - s)}. \tag{A4.2.27}$$

It follows from the condition $\gamma > \pi + 2\beta$ that the origin is located within the right branch of the hyperbola (A4.2.14). Therefore, the function $\varphi(s)$ in this case is not odd, but instead satisfies the equation $\varphi(s) + \varphi(-s) = -2\pi$. Since

$$\int_{-\infty}^{\infty} \frac{1}{(\sigma - s)(\sigma - ip)}d\sigma = 0$$

for $\operatorname{Im} s > 0$, the function $\varphi(s)$ satisfies, for $\operatorname{Im} s > 0$, the relations

$$\varphi_+(s) - \varphi_+(-s) = \varphi(s) + \pi,$$

(A4.2.28)
$$\varphi_+(s) + \varphi_+(-s) = \frac{s - ip}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi(\sigma) + \pi}{(\sigma - s)(\sigma - ip)} d\sigma,$$

where the integral is understood in the sense of principal value. Since

$$\lim_{s \to \pm\infty} (\varphi(s) + \pi) = \pm(\beta + \pi),$$

it is easy to show that

$$\varphi_+(s) \equiv \frac{\pi + \beta}{i\pi} \log|s| \quad \text{as} \quad s \to \infty \ (\operatorname{Im} s \geqslant 0).$$

Thus, we see that one can take

(A4.2.29)
$$\Psi_+(s) = \frac{A(q^2 - p^2)}{i\pi} s \frac{e^{2i\varphi_+(s)}}{(ip + s)(iq + s)}$$

in (A4.2.27). Obviously, $|\Psi_+(s)| = O(|s|^{\beta_0 + 1})$, $s \to \infty$, where $\beta_0 = 2\beta/\pi$. Therefore, equation (A4.2.11) is satisfied.

Now let us determine the constant D_0. It follows from (A4.2.29) that

$$\Psi_+(iq) = -\frac{A(q - p)}{2\pi} e^{2i\varphi_+(iq)}.$$

On the other hand,

$$\Psi_+(iq) = -\frac{B}{2\pi}(p - q).$$

Therefore, we get

(A4.2.30)
$$D_0 = e^{2i\varphi_+(iq)} \frac{A(p + q)}{(\mathbf{r}'_n - \mathbf{g}, \mathbf{e})} \prod_{j=1}^{n} \frac{(\mathbf{r}_j - \mathbf{g}, \mathbf{e})}{(\mathbf{r}'_{j-1} - \mathbf{g}, \mathbf{e})}.$$

We point out that

$$\varphi_+(iq) = \frac{q - p}{2\pi} \int_{-\infty}^{\infty} \frac{\varphi(\sigma)}{(\sigma - iq)(\sigma - ip)} d\sigma = \frac{q - p}{2\pi} \int_{-\infty}^{\infty} \frac{\varphi(\sigma) + \pi}{(\sigma - iq)(\sigma - ip)} d\sigma.$$

The function $\varphi(\sigma) + \pi$ is odd. Hence, we can rewrite the latter relation as

$$\varphi_+(iq) = \frac{i(q^2 - p^2)}{2\pi} \int_{-\infty}^{\infty} \frac{\sigma\varphi(\sigma)}{(\sigma^2 + q^2)(\sigma^2 + p^2)} d\sigma.$$

Therefore, the constant D_0 is real. Finally, for the function $f(s)$ we obtain the following expression:

(A4.2.31)
$$f(s) = \frac{A(q^2 + p^2)}{i\pi} \frac{s e^{q(s)}}{(p^2 + s^2)(q^2 + s^2)} \frac{1}{|Q(s)|},$$

where $q(s)$ is determined by

$$q(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\varphi(\sigma) - \varphi(s))(\sigma s + p^2)}{(\sigma - s)(\sigma^2 + p^2)} d\sigma.$$

The cases when

$$\frac{\pi - \gamma}{2\alpha}, \quad \frac{\gamma - \pi - 2\beta}{2\alpha}, \quad \frac{\pi + \beta - \gamma}{2\alpha}$$

are integers are considered using passages to the limit as $p \to 0+$, $q \to 0+$, $p \to q$, respectively.

2.3. Case 3: $\gamma = 2\beta + \pi$. In this case the right branch of the hyperbola (A4.2.14) passes through the origin. The argument of the function $Q(s)$ is discontinuous at the point $s = 0$. The solution in this case can also be obtained by passing to the limit. Taking, for example, $\varphi(0-) = -\pi/2$, $\varphi(0+) = \pi/2$, which corresponds to the limit value for $\gamma < \pi + 2\beta$, we find that the solution $\Psi_0(x_1, x_2)$ is expressed as $\Psi_0(\mathbf{x}) = U(\mathbf{x}) + V(\mathbf{x})$, where the function U is given by the formula

$$U(\mathbf{x}) = \sum_{k=0}^{m} C_k (e^{(\mathbf{g}_k, \mathbf{x})} - e^{(\mathbf{g}'_k, \mathbf{x})}),$$

$$C_k = \prod_{j=1}^{k} \frac{(\mathbf{g}'_{j-1} - \mathbf{g}, \mathbf{e})}{(\mathbf{g}_j - \mathbf{g}, \mathbf{e})}, \quad k = 1, \dots, m,$$

and the function V is given by the formula

$$V(x_1, x_2) = \int_{-\infty}^{\infty} f(s) \exp(-x_1\sqrt{1+t^2} + ix_2 t)ds,$$

$$f(s) = \frac{A}{i\pi} \frac{se^{q(s)}}{p^2 + s^2} \frac{1}{Q(s)},$$

$$q(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\varphi(\sigma) - \varphi(s))(\sigma s + p^2)}{(\sigma - s)(\sigma^2 + p^2)} d\sigma.$$

Observe that the function $f(s)$ has a singularity as $s \to 0$: $f(s) = O(s^{-1})$. The integral in the formula for the function $V(x_1, x_2)$ must be understood in the sense of principal value. Since f is odd, we can rewrite V in the form

$$V(x_1, x_2) = 2i \int_{-\infty}^{\infty} f(s) e^{-x_1\sqrt{1+t^2}} \sin tx_2 ds.$$

§3. Solutions of the problem in particular cases

Let $\beta = -k\alpha$, where k is an integer. Then the term $V(x_1, x_2)$ vanishes, and the function Ψ_0 is given by

$$\Psi_0(\mathbf{x}) = \sum_{j=0}^{k} C_j (e^{(\mathbf{g}_j, \mathbf{x})} - e^{(\mathbf{g}'_j, \mathbf{x})}). \tag{A4.3.1}$$

where the constants C_j satisfy the relation

$$C_l = \prod_{j=1}^{l} \frac{(\mathbf{g}'_{j-1} - \mathbf{g}, \mathbf{e})}{(\mathbf{g}_j - \mathbf{g}, \mathbf{e})}, \quad l = 1, \dots, k,$$

and $C_0 = 1$. Indeed, it is not difficult to verify that the function Ψ_0 satisfies the equation $\triangle\Psi_0 = \Psi_0$, the boundary condition $\Psi_0(x_1, 0) = 0$, and the asymptotic relation

$$\lim_{x_1 \to \infty} (\Psi_0(x_1, x_2) - \exp(\mathbf{e}(\gamma), \mathbf{x}) + \exp(\mathbf{e}(-\gamma), \mathbf{x}))e^{-(\mathbf{e}(\gamma), \mathbf{x})} = 0.$$

while it follows from $C_l(\mathbf{g}_l - \mathbf{g}, \mathbf{e}) = C_{l-1}(\mathbf{g}'_{l-1} - \mathbf{g}, \mathbf{e})$ that

$$\begin{aligned}\left(\frac{\partial \Psi_0}{\partial \mathbf{e}} - (\mathbf{e}, \mathbf{g})\Psi_0\right)\Big|_l &= \sum_{j=1}^{k} e^{(\mathbf{g}_j, \mathbf{x})}(C_j(\mathbf{g}_j - \mathbf{g}, \mathbf{e}) - C_{j-1}(\mathbf{g}'_{j-1} - \mathbf{g}, \mathbf{e}))\big|_l - C_k(\mathbf{g}'_k - \mathbf{g}, \mathbf{e}) \\ &= -C_k(\cos(\gamma + \beta + 2k\alpha) - \cos(\beta - \gamma)).\end{aligned}$$

However, $\cos(\gamma + \beta + 2k\alpha) = \cos(\beta - \gamma)$. Therefore, Ψ_0 satisfies the relation

$$\left(\frac{\partial \Psi_0}{\partial \mathbf{e}} - (\mathbf{e}, \mathbf{g})\Psi_0\right)\Big|_l = 0.$$

The case $k = 1$ was indicated in [42], [58] and corresponds to a queueing system with identical coefficients of variation of the arriving flow and service time at the first phase. This condition is necessary and sufficient for a solution to have a product form (for the joint limit distribution function of waiting times to have independent components). In the case $k > 1$, the probabilistic meaning of the arising relations is unclear. For example, for $k = 2$,

$$\frac{v_1^2}{v^2} - \sqrt{1 + \frac{v_1^2}{v^2}}\sqrt{1 + \frac{v_2^2}{v^2}} = 1,$$

where v, v_1, and v_2 are the coefficients of variation of interarrival times and service times at the first and second stations, respectively. In this case the solution is also found explicitly, showing that the density of the sojourn distribution is not separable in general.

Observe that solutions similar to (A4.3.1) appear also in problems for Reflected Brownian Motions with a constant drift vector and a constant diffusion matrix in planar regions of various shapes (see [47], [48], [59]).

Finally, we observe that tandem queues provide the simplest example of a queueing network, so any analytical insights obtained for tandem queues should be valuable for analytic approaches to heavy traffic approximations of general networks.

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APPENDIX V

JOINT DISTRIBUTIONS IN POISSONIAN TANDEM QUEUES

In this appendix a tandem queueing system $M/M/1/\infty \to M/1/\infty$ is considered. We study the joint stationary distribution of the waiting times W_i $(i = 1, 2)$ of the customer at the ith station and the joint stationary distribution of the number of customers at the arrival epoch. Our aim is to prove Theorem 2.5.3.

The appendix is arranged as follows. In §1 we describe the queueing system we are interested in. In §2 we introduce a birth-and-death process describing the behavior of the second station and study its characteristics. The relation between this process and the joint stationary distribution of the queue lengths at the arrival epochs is given in the same section. The generating function of the joint queue length distribution at the arrival epochs is treated in §3. The waiting time and the sojourn time distributions are considered in §4. The heavy traffic transient behavior is considered in §5.

§1. Description of the queue

Consider a pair of single server queues arranged in series. We assume that customers arrive individually from outside and queue up for service at the first station. Having completed service there, they proceed to a queue in front of the second station, and after completing service at the second station they depart the system. Service discipline is FIFO at each station.

We assume that the sequences of interarrival times and service times form three mutually independent sequences of identically distributed (i.d.) random variables. The interarrival times I_j have the exponential distribution with parameter λ

$$\Pr\{I_j < t\} = 1 - \exp(-\lambda t), \quad t > 0,$$

and service times at the ith station $(i = 1, 2)$ have the exponential distribution with parameter μ_i. As usual, we denote the traffic intensities by $\rho_i = \lambda/\mu_i$ and say that the system is stable if and only if $\rho_i < 1$ for $i = 1, 2$.

Tandem queues are the simplest among the systems having the product form of the stationary queue length distribution. We reformulate the corresponding result, usually called the Jackson Theorem [55], in the form adapted to our particular case.

Let $P(n_1, n_2)$ denote the stationary probability distribution

$$P(n_1, n_2) = \Pr\{N_1 = n_1, N_2 = n_2\},$$

where N_i is the number of customers at the ith station $(i = 1, 2)$. Then N_1 and N_2 are independent random variables and

$$P(n_1, n_2) = q_1 q_2 \rho_1^{n_1} \rho_2^{n_2}, \tag{A5.1.1}$$

where $q_i = 1 - \rho_i$, $i = 1, 2$.

Let L_i be the number of customers at the ith station ($i = 1, 2$) at the arrival epoch. It was shown by Reich [11] that in the stationary regime

$$\Pr\{L_1 = 0, L_2 = 0\} = \Pr\{L_1 = 0\} \Pr\{L_2 = 0\} \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 - \lambda}.$$

It is clear that these random variables are dependent and the joint distribution of L_1 and L_2 does not have the product form. It was also shown by Burke [21] and Loynes [87] that the stationary waiting times W_i of the customer at the ith station are dependent (this follows from the equality $\Pr\{L_1 = 0, L_2 = 0\} = \Pr\{W_1 = 0, W_2 = 0\}$), while the sojourn times U_1, U_2 are independent. The output streams from the first and from the second service units are Poisson (see also Kelly [68] and Yashkov [134]).

§2. Auxiliary process $X(t)$ and second queue behavior

Consider a "birth-and-death" process $X(t)$ with birth intensities μ_1 and death intensities μ_2 at each state m of the process $X(t)$, $m = 0, 1, \ldots$. We assume that the initial distribution of the process $X(t)$ is given by

$$\Pr\{X(0) = m\} = (1 - \rho_2)\rho_2^m, \quad m = 0, 1, 2, \ldots. \tag{A5.2.1}$$

This process describes the stationary regime of the second station of the system during a busy period of the first service unit.

Let

$$P_{i,k}(t) = \Pr\{X(t) = k, Y(t) = i\},$$

where $Y(t)$ is the number of births during the time interval $[0, t]$ for the process $X(t)$.

Our aim is to obtain a relation that relates the characteristics of the process $X(t)$ and the stationary distribution of the vector (L_1, L_2). We denote this stationary distribution by $\pi_{i,k}$: $\pi_{i,k} = \Pr\{L_1 = i, L_2 = k\}$.

The following lemma establishes the relation between the probabilities $P_{i,k}(t)$ and $\pi_{i,k}$.

LEMMA A5.2.1. *If $\rho_1 < 1$ and $\rho_2 < 1$, then*

$$\pi_{i,k} = \mu_1 q_1 \rho_1^i \int_0^\infty P_{i,k}(t)\, dt. \tag{A5.2.2}$$

PROOF. It is more convenient to introduce the conditional probabilities

$$Q_{i,k}(t) = \Pr\{X(t) = k \mid Y(t) = i\}.$$

Observe that

$$P_{i,k}(t) = Q_{i,k}(t) e^{-\mu_1 t} \frac{(\mu_1 t)^i}{i!}. \tag{A5.2.3}$$

The probability $\pi_{i,k}$ can be represented as the product $\pi_{i,k} = p_i c_k(i)$, where p_i is the probability of finding i customers at the arrival epoch at the first node and $c_k(i)$

is the conditional probability of finding k customers at the arrival epoch in the second node. However, (A5.1.1) implies

$$p_i = q_1\rho_1^i \quad \text{and} \quad c_k(i) = \Pr\{L_2 = k \mid L_1 = i\} = \int_0^\infty Q_{i.k}(t)\mu_1 e^{-\mu_1 t}\frac{(\mu_1 t)^i}{i!}\,dt.$$

Therefore,

$$\pi_{i.k} = \mu_1 q_1 \rho_1^i \int_0^\infty Q_{i.k}(t)\mu_1 e^{-\mu_1 t}\frac{(\mu_1 t)^i}{i!}\,dt = \mu_1 q_1 \rho_1^i \int_0^\infty P_{i.k}(t)\,dt,$$

as required. □

We define the generating functions

$$\pi(z_1, z_2) = \mathbf{E}(z_1^{L_1} z_2^{L_2}) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\pi_{i.k} z_1^i z_2^k,$$

$$P(z_1, z_2, t) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} P_{i.k}(t) z_1^i z_2^k,$$

where $|z_j| < 1$, $j = 1, 2$. Then it follows from Lemma A5.2.1 that

$$\pi(z_1, z_2) = \mu_1 q_1 \widetilde{P}(\rho_1 z_1, z_2, 0), \tag{A5.2.4}$$

where

$$\widetilde{P}(z_1, z_2, s) = \int_0^\infty e^{-st} P(z_1, z_2, t)\,dt.$$

Our next step is to find the Laplace transform of the function $P(z_1, z_2, t)$.

Lemma A5.2.2. *The Laplace transform $\widetilde{P}(z_1, z_2, s)$ of the generating function $P(z_1, z_2, t)$ is given by*

$$\widetilde{P}(z_1, z_2, s) = \frac{\mu_2(1 - z_2)\widetilde{P}(z_1, 0, s)(1 - \rho_2 z_2) - q_2 z_2}{(\mu_1 z_1 z_2^2 - (\mu_1 + \mu_2 + s)z_2 + \mu_2)(1 - \rho_2 z_2)}. \tag{A5.2.5}$$

Proof. It is not difficult to show that the following relations hold for all $t > 0$:

$$\begin{aligned}
\dot{P}_{i.k}(t) &= -(\mu_1 + \mu_2)P_{i.k}(t) + \mu_2 P_{i.k+1}(t) + \mu_1 P_{i-1.k-1}(t) && \\
& && \text{for } i > 0, k > 0,\\
\dot{P}_{i.0}(t) &= -\mu_1 P_{i.0}(t) + \mu_2 P_{i.1}(t) && \text{for } i \geqslant 0,\\
\dot{P}_{0.k}(t) &= -(\mu_1 + \mu_2)P_{0.k}(t) + \mu_2 P_{0.k+1}(t) && \text{for } k \geqslant 1.
\end{aligned} \tag{A5.2.6}$$

Then, by (A5.2.6), the generating function $P(z_1, z_2, t)$ satisfies the relation

$$\frac{\partial P(z_1, z_2, t)}{\partial t} = -(\mu_1 + \mu_2)P(z_1, z_2, t) + \frac{\mu_2}{z_2}P(z_1, z_2, t) + \mu_1 z_1 z_2 P(z_1, z_2, t) + \mu_2(1 - z_2^{-1})\sum_{i=0}^{\infty} P_{i.0}(t) z_1^i.$$

Taking the Laplace transforms, we get

$$\begin{aligned} s\widetilde{P}(z_1, z_2, s) =& P(z_1, z_2, 0) + \widetilde{P}(z_1, z_2, s)R(z_1, z_2) \\ &+ \mu_2 \widetilde{P}(z_1, 0, s)(1 - 1/z_2), \end{aligned} \tag{A5.2.7}$$

where

$$R(z_1, z_2) = \frac{\mu_2}{z_2} - \mu_1 - \mu_2 + \mu_1 z_1 z_2.$$

Observing that

$$P(z_1, z_2, 0) = \sum_{k=0}^{\infty} P_{0,k}(0) z_2^k = \frac{q_2}{1 - \rho_2 z_2},$$

we get (A5.2.5). □

§3. The joint queue length distribution

Let us denote

$$Q(z_1, z_2, s) = \mu_1 z_1 z_2^2 - (\mu_1 + \mu_2 + s) z_2 + \mu_2.$$

The generating function $\pi(z_1, z_2)$ is described by the following

THEOREM A5.3.1. *Let* $\pi(z_1, z_2) = \mathbf{E}(z_1^{L_1} z_2^{L_2})$, $|z_i| \leqslant 1$, $i = 1, 2$. *Then*

$$\pi(z_1, z_2) = \frac{\mu_1(1-\rho_1)(1-\rho_2)(z_* - z_2)(1 - \rho_2 z_2 z_*)}{Q(\rho_1 z_1, z_2, 0)(1 - z_*)(1 - \rho_2 z_*)(1 - \rho_2 z_2)}, \tag{A5.3.1}$$

where $z_* = z_*(z_1)$ *and* $z^* = z^*(z_1)$ *are the roots of the equation*

$$\lambda z_1 z^2 - (\mu_1 + \mu_2) z + \mu_2 = 0. \tag{A5.3.2}$$

These roots are continuous functions of z_1 *and satisfy the inequalities*

$$|z_*| < 1, \quad |z^*| > 1.$$

PROOF. By (A5.2.4), we should find the Laplace transform $\widetilde{P}(\rho_1 z_1, z_2, 0)$. Using the Rouché theorem, we see that the equation

$$Q(z_1, z_2, s) = 0 \tag{A5.3.3}$$

has a single root in the unit disk $D_{z_2} = \{z_2 : |z_2| \leqslant 1\}$ when $s \geqslant 0$ and $|z_1| \leqslant 1$. We denote this root by z_*. Note that this root depends on z_1 and s: $z_* = z_*(z_1, s)$. We denote the second root of equation (A5.3.3) by $z^* = z^*(z_1, s)$, $|z^*| \geqslant 1$. If $s \geqslant 0$ and $|z_1| \leqslant 1$ are fixed, then $\widetilde{P}(z_1, z_2, s)$ is an analytic function of z_2 in the disk D_{z_2}, and Lemma A5.2.2 yields

$$\widetilde{P}(z_1, 0, s) = \frac{q_2 z_*}{\mu_2(1 - z_*)(1 - \rho_2 z_*)}. \tag{A5.3.4}$$

From (A5.3.4) and (A5.2.5) we get

$$\widetilde{P}(z_1, z_2, s) = \frac{q_2(1-\rho_2 z_* z_2)(z_* - z_2)}{Q(z_1, z_2, s)(1-z_*)(1-\rho_2 z_*)(1-\rho_2 z_2)}. \tag{A5.3.5}$$

Observe that for $s = 0$ the roots of equation (A5.3.3) satisfy the relations

$$z^* + z_* = \frac{\mu_1 + \mu_2}{\mu_1 z_1}, \tag{A5.3.6}$$

$$z^* z_* = \frac{\mu_2}{\mu_1 z_1}, \tag{A5.3.7}$$

and thus we get $\widetilde{P}(z_1, z_2, 0)$.

Observe that

$$\widetilde{P}(\rho_1 z_1, z_2, s) = \frac{q_2(1-\rho_2 z_* z_2)(z_* - z_2)}{Q(\rho_1 z_1, z_2, s)(1-z_*)(1-\rho_2 z_*)(1-\rho_2 z_2)},$$

where $z_* \in D_{z_2}$ is the root of the equation $Q(\rho_1 z_1, z_2, s) = 0$.

Setting $s = 0$ in the formula for $\widetilde{P}(\rho_1 z_1, z_2, 0)$ and using (A5.2.4), we find that the joint generating function $\pi(z_1, z_2)$ satisfies (A5.3.1). Theorem A5.3.1 is thus proved. □

COROLLARY A5.3.1. *The generating function of* L_2 *is given by*

$$\pi(z_2) = \frac{q_2}{1-\rho_2 z_2}. \tag{A5.3.8}$$

PROOF. Substituting $z_1 = 1$ in (A5.3.2) we get rather easily that

$$z^* + z_* = \frac{\mu_1 + \mu_2}{\lambda}.$$

$$z^* z_* = \frac{\mu_2}{\lambda}.$$

Besides,

$$(1-z_*)(1-\rho_2 z_*) = \frac{\mu_1}{\mu_2} q_1 z_*$$

and

$$\rho_2 z_*(z^* - z_2) = 1 - \rho_2 z_* z_2.$$

Substituting these terms in (A5.3.1), we obtain (A5.3.8). □

COROLLARY A5.3.2. *The covariance of the random variables* L_1 *and* L_2 *is given by*

$$\operatorname{cov}(L_1. L_2) = \mathbf{E}(L_1 L_2) - \mathbf{E}L_1 \mathbf{E}L_2 = \frac{z_0}{(z_0 - 1)^2}. \tag{A5.3.9}$$

where

$$z_0 = \frac{\mu_1 + \mu_2 + \sqrt{(\mu_1 + \mu_2)^2 - 4\lambda\mu_2}}{2\lambda}.$$

PROOF. The proof of the corollary is based on formulas (A5.2.5), (A5.3.5)–(A5.3.9). Setting $z_1 = z_2 = 1$, $s = 0$ in (A5.2.5), we find

$$P(\rho_1, 1, 0) = 1, \tag{A5.3.10}$$

$$\frac{\partial P(\rho_1, 1, 0)}{\partial z_1} = \frac{1}{\mu_1 q_1^2}, \tag{A5.3.11}$$

$$\frac{\partial P(\rho_1, 1, 0)}{\partial z_2} = \frac{\rho_2}{\mu_1 q_1 q_2}. \tag{A5.3.12}$$

Let us denote

$$A(z_1, z_2) = \frac{1}{z^* - z_2} + \frac{\rho_2}{1 - \rho_2 z_2} - \frac{\rho_2 z_*}{1 - \rho_2 z_2 z_*},$$

$$B(z_1, z_2) = \frac{1}{(z^* - z_2)^2}\frac{\partial z^*}{\partial z_2} + \frac{\rho_2}{(1 - \rho_2 z_2 z_*)^2}\frac{\partial z_*}{\partial z_1}.$$

Then it follows from (A5.3.8) that

$$\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}P(z_1, z_2, 0) = A(z_1, z_2)\frac{\partial P(z_1, z_2, 0)}{\partial z_1} - B(z_1, z_2)\frac{\partial P(z_1, z_2, 0)}{\partial z_2}. \tag{A5.3.13}$$

Setting $z_1 = \rho_1$, $z_2 = 1$ in (A5.3.5) and (A5.3.6), we find that

$$\frac{\partial z^*}{\partial z_1} + \frac{\partial z_*}{\partial z_1} = -\frac{\mu_1 + \mu_2}{\mu_1 \rho_1^2}, \tag{A5.3.14}$$

$$\frac{\partial z^*}{\partial z_1}z_* + \frac{\partial z^*}{\partial z_2}z^* = -\frac{1}{\rho_1 \rho_2}. \tag{A5.3.15}$$

Now from (A5.3.10)–(A5.3.15) we derive a formula for the second derivative of the generating function:

$$\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}P(\rho_1, 1, 0) = \frac{\rho_2}{\mu_1 q_2 q_1^2} + \frac{\mu_2}{\lambda^2 q_1 z_*(z^* - 1)^2}. \tag{A5.3.16}$$

Observe that

$$\mathbf{E}L_i = \frac{\rho_i}{q_i}, \quad i = 1, 2.$$

Hence, (A5.3.9) follows from (A5.3.16) and (A5.2.4). □

COROLLARY A5.3.3. *The stationary probability* $\pi_{00} = \Pr\{L_1 = 0, L_2 = 0\}$ *is given by*

$$\pi_{00} = q_1 q_2 \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 - \lambda}. \tag{A5.3.17}$$

PROOF. The probability π_{00} satisfies the relation $\pi_{00} = \pi(0, 0)$. The root z_* of equation (A5.3.2) is given by

$$z_* = \frac{\mu_2}{\mu_1 + \mu_2} \tag{A5.3.18}$$

for $z_1 = 0$. Substituting $z_1 = 0$, $z_2 = 0$ and (A5.3.18) in (A5.3.1) we obtain (A5.3.17). □

§4. Waiting and sojourn times distributions

In this subsection we consider the joint distribution of the waiting times W_1 and W_2 and also the sojourn time distribution in the $M/M/1/\infty \to M/1/\infty$ queueing system. The second main result of this appendix is given by

THEOREM A5.4.1. *Let $t_1 > 0$, $t_2 > 0$. Then the Laplace transform*

$$\Psi(t_1, t_2) = \mathbf{E}e^{-t_1 W_1 - t_2 W_2}$$

of the joint stationary waiting time distribution is given by

$$\Psi(t_1, t_2) = \pi_{00} + q\left(\frac{\lambda\mu_1}{\mu_2 q_2 + t_2} + \frac{\lambda\mu_2}{\mu_1 q_1 + t_1} + \frac{\lambda\mu_1\mu_2}{(\mu_1 q_1 + t_1)(\mu_2 q_2 + t_2)}\right), \tag{A5.4.1}$$

where

$$q_1 = 1 - \rho_1, \quad q_2 = 1 - \rho_2, \ \textit{and}\ q = \frac{q_1 q_2}{\mu_1 + \mu_2 - \lambda}.$$

PROOF. The proof of this theorem is arranged as follows. We introduce the stationary distribution of the process $(L_1, L_2, S^{(1)}, W_1)$. After that, we establish the relations between this process and the probabilities $P_{ik}(t)$ introduced in Lemma A5.2.1.

Let us denote

$$P_{ik}(s_1, s) = \Pr\{L_1 = i,\ L_2 = k,\ S^{(1)} < s,\ W_1 \leqslant s_1\},$$

where $S^{(i)}$ is the service time at the ith service station ($i = 1, 2$). Let

$$p_{ik}(s_1, s) = \frac{\partial^2 P_{ik}(s_1, s)}{\partial s \partial s_1}, \quad i > 0, \quad k \geqslant 0,$$

and, for $i = 0$,

$$p_{0k}(s) = \frac{\partial P_{0k}(s_1, s)}{\partial s}, \quad s > 0, \quad k \geqslant 0.$$

If $i = 0$, then the waiting time W_1 vanishes. Thus, all the probabilities $P_{0k}(s_1, s)$ vanish for $s_1 < 0$ and are independent of s_1 for $s_1 > 0$.

The number of customers L_2 at the second station at the arrival epoch leads to the distribution of the waiting time at the station. We have

$$\Pr\{W_1 \leqslant s_1, W_2 \leqslant s_2\} = \lim_{s\to\infty} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} P_{ik}(s_1, s) H_k(s_2),$$

where

$$H_k(t) = \Pr\left\{\sum_{j=1}^{k} S_j^{(2)} \leqslant t\right\}$$

and $S_j^{(2)}$ is the residual service time of the jth customer at the second station. It is not difficult to observe that all the random variables $S_j^{(2)}$ are mutually independent and have the exponential distribution with parameter μ_2. Then

$$H_k(x) = 1 - \exp(-\mu_2 x) \sum_{j=0}^{k-1} \frac{(\mu_2 x)^j}{j!}, \quad x \geqslant 0, \quad k \geqslant 1.$$

Now we can establish a relationship between the densities $p_{ik}(s_1, s)$ and the probabilities $P_{ik}(t)$ introduced in Lemma A5.2.1.

LEMMA A5.4.1. *The probabilities* $P_{ik}(t)$ *and the densities* $p_{ik}(s_1, s)$ *are related as follows*:

(A5.4.2)
$$p_{ik}(s_1, s) = \mu_1^2 q_1 \rho_1^i \sum_{j=k-1}^{\infty} P_{i-1,j}(s_1) e^{-(\mu_1+\mu_2)s} \frac{(\mu_2 s)^{j+1-k}}{(j+1-k)!}$$
for $i \geqslant 1,\ k \geqslant 1$;

(A5.4.3)
$$p_{i0}(s_1, s) = \mu_1^2 q_1 \rho_1^i \sum_{j=0}^{\infty} P_{i-1,j}(s_1) e^{-(\mu_1+\mu_2)s} \sum_{m=j+1}^{\infty} \frac{(\mu_2 s)^m}{m!}$$
for $i \geqslant 1$;

(A5.4.4)
$$p_{0k}(s) = \mu_1 q_1 q_2 \rho_2^k \exp\left(-(\mu_1 + \mu_2 - \lambda)s\right)$$
for $k \geqslant 1$;

(A5.4.5)
$$p_{00}(s) = \mu_1 q_1 \exp(-\mu_1 s)(1 - \rho_2 \exp(-(\mu_2 - \lambda)s)).$$

PROOF. We start the proof of the lemma with equality (A5.4.2). For $k \geqslant 1,\ i \geqslant 1$ we have

(A5.4.6)
$$\begin{aligned} &\Pr\{L_1 = i,\ L_2 = k,\ s < S^{(1)} \leqslant s + ds,\ s_1 < W_1 \leqslant s_1 + ds_1\} \\ &= q_1 \rho_1^i \sum_{j=k-1}^{\infty} P_{i-1,j}(s_1) \mu_1 ds_1 \mu_1 \exp(-\mu_1 s) ds \exp(-\mu_2 s) \frac{(\mu_2 s)^{j+1-k}}{(j+1-k)!}, \end{aligned}$$

where the term $q_1 \rho_1^i$ represents the probability of the event $\{L_1 = i\}$; $P_{i-1,j}(s_1)$ is the probability that $(i-1)$ customers being served during time s_1 and the ith customer finds j others at the second service station; $\mu_1 ds_1$ is the probability that the ith customer finishes its service time during time ds_1,

$$\Pr\{s < S^{(1)} \leqslant s + ds\} = \mu_1 \exp(-\mu_1 s) ds.$$

and

$$\exp(-\mu_2 s) \frac{(\mu_2 s)^{j+1-k}}{(j+1-k)!}$$

is the probability of the event $\{N_s = j + 1 - k\}$, where N_s is the number of customers that received their service during time s at the second station. One the other hand,

$$\Pr\{L_1 = i,\ L_2 = k,\ s < S^{(1)} \leqslant s + ds,\ s_1 < W_1 \leqslant s_1 + ds_1\} = p_{ik}(s_1, s) ds ds_1.$$

Now (A5.4.2) follows from this equation and (A5.4.6). Formulas (A5.4.3) and (A5.4.4) are proved similarly.

The next step is to find the distribution of the vector $\{L_1, L_2, W_1, S^{(1)}\}$. To do this, we define, for $|z_1| \leqslant 1$, $|z_2| \leqslant 1$, the joint generating function

$$\Pi(\vec{z}, \vec{t}) = \mathbf{E}(z_1^{L_1} z_2^{L_2} \exp(-tS^{(1)} - t_1 W_1)),$$

where $\vec{z} = (z_1, z_2)$, $\vec{t} = (t, t_1)$, $t \geqslant 0$, $t_1 \geqslant 0$, and denote the Laplace transform of the densities $p_{ik}(s_1, s)$ by

$$\widetilde{p}_{ik}(t, t_1) = \int_0^\infty \int_0^\infty e^{-s_1 t_1 - st} p_{ik}(s_1, s)\, ds\, ds_1 \quad \text{for } k \geqslant 0, \quad i \geqslant 1,$$
$$\widetilde{p}_{0k}(t) = \int_0^\infty e^{-st} p_{0k}(s)\, ds \quad \text{for } k \geqslant 0.$$

Then

$$\Pi(\vec{z}, \vec{t}) = \sum_{i=1}^\infty \sum_{k=0}^\infty z_1^i z_2^k \widetilde{p}_{ik}(t, t_1) + \sum_{k=0}^\infty z_2^k \widetilde{p}_{0k}(t). \tag{A5.4.7}$$

LEMMA A5.4.2. *For $t \geqslant 0$, $t_1 \geqslant 0$, $|z_1| \leqslant 1$, $|z_2| \leqslant 1$ one has*

$$\begin{aligned}\Pi(\vec{z}, \vec{t}) = & \frac{\mu_1 q_1 q_2}{\mu_1 + t} \frac{\mu_1 + t + \mu_2}{\mu_1 + t + \mu_2 - \lambda} + \frac{\mu_1 q_1 q_2}{\mu_1 + t + \mu_2 - \lambda} \frac{\rho_2 z_2}{1 - \rho_2 z_2} \\ & + \frac{\mu_1^2 \mu_2 q_1 \rho_1 z_1}{(\mu_1 + t)(\mu_1 + t + \mu_2)} \cdot P\left(\rho_1 z_1, \frac{\mu_2}{\mu_1 + t + \mu_2}, t_1\right) \\ & + \frac{\mu_1^2 q_1 \rho_1 z_1}{(\mu_1 + t)(\mu_1 + t + \mu_2)} \frac{z_2}{z_2 - \frac{\mu_2}{\mu_1 + t + \mu_2}} \cdot \mathfrak{P}(z_1, z_2, t_1),\end{aligned} \tag{A5.4.8}$$

where

$$\mathfrak{P}(z_1, z_2, t_1) = z_2 P(\rho_1 z_1, z_2, t_1) - \frac{\mu_2}{\mu_1 + t + \mu_2} P\left(\rho_1 z_1, \frac{\mu_2}{\mu_1 + t + \mu_2}, t_1\right).$$

PROOF. We use the Laplace transforms of relations (A5.4.2)–(A5.4.5) to prove formula (A5.4.8) and to compute directly the generating function $\Pi(\vec{z}, \vec{t})$. It can be easily seen that the waiting time at the second station is given by

$$W_2 = \sum_{j=1}^{L_2} S_j^{(2)}.$$

Let us put $\Psi(t_1, t_2) = \mathbf{E}\exp(-t_1 W_1 - t_2 W_2)$; then

$$\Psi(t_1, t_2) = \mathbf{E}\exp\left(-t_1 W_1 - t_2 \sum_{j=1}^{L_2} S_j^{(2)}\right) = \Pi\left(1, \frac{\mu_2}{t_2 + \mu_2}, 0, t_1\right).$$

It follows from (A5.2.5), (A5.3.4), and (A5.3.5) that

$$P(\rho_1, z_2, t_1) = \frac{q_2}{(1-\rho_2 z_2)(\mu_1 + t_1 - \lambda)}. \tag{A5.4.9}$$

Using (A5.4.9), we derive by direct calculations that

$$\begin{aligned}\Pi(\vec{z}, \vec{t}) = \pi_{00} &+ \frac{\lambda \mu_2 q_1 q_2}{\mu_1 + \mu_2 - \lambda} \frac{1}{\mu_1 - \lambda + t_1} + \frac{\mu_1 q_1 q_2}{\mu_1 + \mu_2 - \lambda} \frac{\rho_2 z_2}{1 - \rho_2 z_2} \\ &+ \frac{\mu_1 q_1 \lambda}{\mu_1 + \mu_2 - \lambda} \cdot \frac{q_2 z_2}{1 - \rho_2 z_2} \frac{1}{\mu_1 + t_1 - \lambda}.\end{aligned} \tag{A5.4.10}$$

Setting

$$z_2 = \frac{\mu_2}{\mu_2 + t_2}$$

in (A5.4.10), we obtain

$$\Psi(t_1, t_2) = \pi_{00} + q\left(\frac{\lambda\mu_1}{\mu_2 q_2 + t_2} + \frac{\lambda\mu_2}{\mu_1 q_1 + t_1} + \frac{\lambda\mu_1\mu_2}{(\mu_1 q_1 + t_1)(\mu_2 q_2 + t_2)}\right).$$

The theorem is thus proved. □

Let us consider a series of tandem queues depending on a parameter ε. Let $\lambda = \lambda(\varepsilon)$, $\mu_i = \mu_i(\varepsilon)$, and

$$\mu_i - \lambda = a_i \varepsilon + o(\varepsilon), \quad a_i > 0, \quad i = 1, 2.$$

These conditions define the heavy traffic regime in our queueing system as $\varepsilon \to 0$ and $\rho_i \to 1$.

COROLLARY A5.4.1. *The limit joint distribution of the waiting times is given by*

$$\lim_{\varepsilon \to 0} \Pr\{W_1(\varepsilon) < x, W_2(\varepsilon) < y\} = (1 - e^{-a_1 x})(1 - e^{-a_2 y}).$$

This statement can be proved by passing to the limit in (A5.4.1) as $\varepsilon \to 0$.

It can be easily seen that stationary waiting times are asymptotically independent in heavy traffic. It was shown by Harrison [42], [43] and by Karpelevich and Kreinin [58] (see also Appendix IV) that the random variables $W_1(\varepsilon)$, $W_2(\varepsilon)$ are independent in heavy traffic if and only if the variation coefficients of the service time and the interarrival time are equal at the first station. It follows from this fact that the stationary sojourn times in $GI/M/1/ \to M/1/\infty$ queues are dependent, because sojourn times are asymptotically equivalent to waiting times in heavy traffic.

§5. Heavy traffic transient behavior

In this section, we find the generating function describing the transient behavior of the tandem queue in the heavy traffic regime. Assume that initially the second queue is empty and M customers are placed at the first queueing system. Our aim is to determine the queueing length distribution of the second service station at the arrival epoch of the Mth customer. In terms of the transient distribution, we are going to study the probabilities

$$\pi_M(k) = \Pr\{L_2(\tau) = k \mid L_1(0) = M, L_2(0) = 0\},$$

where $\tau = t_M^{(2)}$ is the arrival moment of the Mth customer at the second queue. The heavy traffic regime in the queueing system is defined by the relation $\mu_1 = \mu_2 = \mu$. Thus, the intensity of the departure flow from the first node is equal to the service intensity at the second node.

PROPOSITION A5.5.1. *The generating function of the distribution* $\pi_M(k)$,

$$\pi(z_1, z_2) = \sum_{M=0}^{\infty} \sum_{k=0}^{\infty} \pi_M(k) z_1^M z_2^k,$$

is given by the relation

$$\pi(z_1, z_2) = \frac{1}{z_1(z_* - z_2)(1 - z^*)},$$

where z_* *and* z^* *are the roots of the quadratic equation*

$$z_1 z^2 - 2z + 1 = 0.$$

These roots are continuous functions of z_1 *and satisfy the inequalities*

$$|z_*| < 1, \quad |z^*| > 1.$$

The average queue length at the arrival epoch of the Mth customer at the second node satisfies the relation

$$E(M) \sim \frac{1}{\sqrt{2\pi}} \sqrt{M}.$$

PROOF. Let us consider the process $X(t)$, $t \geqslant 0$, with birth intensities $\mu_1 = \mu$ and death intensities $\mu_2 = \mu$ at each state m of the process $X(t)$, $m = 0, 1, \ldots$. We assume that the initial distribution of the process $X(t)$ is given by $\Pr\{X(0) = 0\} = 1$. This process describes the second queue behavior on the busy period of the first node. Let

$$\widetilde{Q}_{M.k}(t) = \Pr\{X(t) = k \mid Y(t) = M. X(0) = 0\}.$$

The probabilities $\pi_M(k)$ satisfy the relation

$$\begin{aligned}\pi_M(k) &= \int_0^\infty \widetilde{Q}_{M.k}(t) \mu e^{-\mu t} \frac{(\mu t)^M}{M!} dt \\ &= \mu \int_0^\infty \Pr\{X(t) = k. Y(t) = M | X(0) = 0\} dt.\end{aligned}$$

where $Y(t)$ is the number of births during the interval $[0. t]$, $t > 0$, for the process $X(t)$.

The probabilities $P_{Mk}(t) = \Pr\{X(t) = k, Y(t) = M \mid X(0) = 0\}$ satisfy the equations
(A5.5.1)

$$\begin{aligned} \dot P_{M.k}(t) &= -2\mu P_{M.k}(t) + \mu P_{M.k+1}(t) + \mu P_{i-1.k-1}(t) && \text{for } M > 0,\ k > 0, \\ \dot P_{M.0}(t) &= -\mu P_{M.0}(t) + \mu P_{i.1}(t) && \text{for } M > 0, \\ \dot P_{0.k}(t) &= 0 && \text{for } k \geqslant 1, \\ \dot P_{0.0}(t) &= -\mu P_{0.0}(t). \end{aligned}$$

We introduce the generating function

$$P(z_1, z_2, t) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} P_{i.k}(t) z_1^i z_2^k,$$

where $|z_j| < 1$, $j = 1, 2$. Then it follows from equations (A5.5.1) that this generating function satisfies the relation

$$\begin{aligned} \frac{\partial P(z_1, z_2, t)}{\partial t} = &- 2\mu(P(z_1, z_2, t) - P(z_1, 0, t)) - \mu P(z_1, 0, t) \\ &+ \frac{\mu}{z_2}(P(z_1, z_2, t) - P(z_1, 0, t)) + \mu z_1 z_2 P(z_1, z_2, t), \end{aligned}$$

which can be rewritten in the form

$$\begin{aligned} \frac{\partial P(z_1, z_2, t)}{\partial t} &- P(z_1, z_2, t)\left(\mu z_1 z_2 - 2\mu + \frac{\mu}{z_2}\right) \\ &+ \mu P(z_1, 0, t)\left(1 - \frac{1}{z_2}\right) - P(z_1, 0, t) = 0. \end{aligned} \tag{A5.5.2}$$

As above, let us denote by $\widetilde P(z_1, z_2, s)$ the Laplace transform of the generating function $P(z_1, z_2, t)$:

$$\widetilde P(z_1, z_2, s) = \int_0^\infty e^{-st} P(z_1, z_2, t) dt.$$

Taking the Laplace transform, we obtain from (A5.5.2) that

$$s\widetilde P(z_1, z_2, s) - 1 = \mu \widetilde P(z_1, z_2, s) R(z_1, z_2) + \mu \widetilde P(z_1, 0, s)(1 - 1/z_2), \tag{A5.5.3}$$

where

$$R(z_1, z_2) = \frac{1}{z_2} - 2 + z_1 z_2.$$

It follows from (A5.5.3) that

$$\widetilde P(z_1, z_2, s) = \frac{\mu(1 - z_2)\widetilde P(z_1, 0, s) - z_2}{\mu z_1 z_2^2 - (2\mu + s) z_2 + \mu}. \tag{A5.5.4}$$

Using the Rouché theorem, we see that the equation

$$\mu z_1 z_2^2 - (2\mu + s) z_2 + \mu = 0 \tag{A5.5.5}$$

has a single root in the unit disk $D_{z_2} = \{z_2\colon |z_2| \leqslant 1\}$ when $s \geqslant 0$ and $|z_1| \leqslant 1$.

We denote this root by z_*. The second root of equation (A5.5.5) is denoted by z^*, $|z^*| \geqslant 1$. If $s \geqslant 0$ and $|z_1| \leqslant 1$ are fixed, then $\widetilde{P}(z_1, z_2, s)$ is an analytic function of z_2 in the disk D_{z_2} and, setting $z_2 = z_*$ in equation (A5.5.4), we get

$$\widetilde{P}(z_1, 0, s) = \frac{z_*}{\mu(1 - z_*)},$$

where z_* and z^* depend on z_1 and s. Finally, we obtain

$$\widetilde{P}(z_1, z_2, s) = \frac{1}{\mu z_1 (z_2 - z_*)(z_* - 1)}. \tag{A5.5.6}$$

Note that for the roots of equation (A5.5.5) we have

$$z^* + z_* = \frac{1}{z_1}, \qquad z^* z_* = \frac{2\mu + s}{\mu z_1}.$$

The generating function $\pi(z_1, z_2)$ satisfies the relation $\pi(z_1, z_2) = \mu \widetilde{P}(z_1, z_2, 0)$. Hence, it follows from (A5.5.6) that

$$\pi(z_1, z_2) = \frac{1}{z_1(z_* - z_2)(1 - z^*)}.$$

This relation proves the first assertion of the proposition.

Let us consider the function

$$E(M) = \sum_{k=0}^{M} k \pi_M(k), \quad M = 1, 2, \ldots,$$

and denote by $\mathbf{E}(z)$ the generating function of this sequence:

$$\mathbf{E}(z) = \sum_{M=0}^{\infty} E(M) z^M, \quad |z| \leqslant 1.$$

Then we have

$$\mathbf{E}(z) = \left.\frac{\partial \pi(z_1, z_2)}{\partial z_2}\right|_{z_2 = 1} = \frac{1}{z(1 - z^*)(1 - z_*)^2}.$$

For $s = 0$, $z_1 = z$ the roots of equation (A5.5.5) satisfy the relations

$$z_* = \frac{1 - \sqrt{1 - z}}{z}, \qquad z^* = \frac{1 + \sqrt{1 - z}}{z}$$

and

$$(1 - z^*)(z_* - 1) = \frac{1}{z} - 1.$$

Therefore, we get

$$\text{(A5.5.7)} \qquad \mathbf{E}(z) = \frac{1}{z(1-z^*)(1-z_*)^2} = \frac{(1-z)^{-\frac{3}{2}}z}{1+\sqrt{1-z}}.$$

The asymptotic behavior of the sequence $\{E(M)\}$ is related to the behavior of the generating function for this sequence. A fundamental result in this area is the Tauberian theorem in [33] (see also [130]), which implies

$$\lim_{z\to 1} \mathbf{E}(z)(1-z)^{\beta} = \Gamma(\beta+1) \lim_{M\to\infty} \frac{E(M)}{M^{\beta-1}}.$$

In our case, $\beta = 3/2$. Therefore,

$$E(M) \sim \frac{1}{\Gamma(3/2)}\sqrt{M}.$$

This relation completes the proof of the proposition. □

BIBLIOGRAPHICAL NOTES

Chapter I.

This chapter contains basic results related to processes with reflection. The process X_n^i was studied by Klimov [75], Harrison [42], Glynn and Whitt [39], Karpelevich and Kreinin [58–61], Reiman [101], Harrison and Reiman [48], [49]. Lemma 1.1.1 was stated (in different notation) in [58]. Lemma 1.1.2 and Corollary 1.1.1 were proved in [61]. The representation of the infinitesimal operator (1.2.17) is borrowed from [60] (see also Harrison [43]). The process X is deeply related to the Skorohod Reflection Problem. This problem was considered by Reiman [101] and by Anisimov and Chabanuk [4] for Jackson networks of queues. The convergence to the process ξ is provided by limit theorems in function spaces presented in Appendix II. Conditions (1.3.3), (1.3.4), and (1.4.11), (1.4.12) generalize the corresponding relations for the waiting time process (see [58], [60]). Proposition 1.5.1 is new. Theorem 1.5.2 on the convergence of the processes ξ_T^λ is new as well. The uniform convergence in probability to the process $\bar{\xi}(x_{m-1})$ (see Theorem 1.6.1) is proved for the waiting time process in tandem queues in [59], [60].

Chapter II.

Many queues in series were considered by Glynn and Whitt [39], Harrison [42], [46], Jackson [55], Karpelevich and Kreinin [58], [61], Kelbert et al. [63], Kella and Whitt [68], Kipnis [73], Szczotka and Kelly [112]. Lemma 2.1.1 was proved in [5], [61]. Theorem 2.2.1 is borrowed from [59]. For the case of tandem queues it was proved in [58]. Theorem 2.3.3 is new. The proof is based on the approach to the weak convergence theory presented in Appendix II. Theorem 2.4.1 is new. The case $k = 1$ is well known (see [15], [40], [58]).

Theorems 2.5.1 and 2.5.2 were proved in [59], [60]. Theorem 2.5.3 is borrowed from [62]. The dependence of sojourn times on service times in tandem queues was studied by Cao [132].

Queues with identical service and Poisson input flow were studied by Boxma [19], Vinogradov [120], [121], Makarichev [88]. The heavy traffic limit theorems for queues with identical service were proved in [120], [121] for the Poisson input flow. Systems with general distributions of interarrival times were studied in [61]. Lemmas 2.6.1–2.6.4 were borrowed from [61]. Theorem 2.6.2 was also proved in [61]. Theorem 2.7.1 is due to Karpelevich and Kreinin [61]. The proof given is new. Theorem 2.7.3, Theorem 2.7.5, and Lemma 2.7.3 were proved in [61] by using martingale techniques. The proof of Theorem 2.7.5 is based on the approach to the weak convergence theory presented in Appendix II.

More precise asymptotic relations for queues with identical service were obtained by Vinogradov [120], [121]. Unlike systems satisfying the Kleinrock condition, the stationary waiting time for systems with identical service in heavy traffic is described by a degenerate process. For "rough" normalizations, as the one assumed in §§4–6 of Chapter II, the stationary sojourn time for the phases $2, \dots, m$ tends to zero in probability. However, more delicate normalizations exists, such that the limit distribution becomes nondegenerate.

The first results in this direction are due to Vinogradov [120] and apply to systems with Poisson input flow. As initial point of this study, he used the following integral relation involving the joint distribution of the sojourn times at the first phase and at phases indexed by $2, 3, \dots, k+1$. Let us denote

$$F_k(z, x) = \Pr\{U_1 < z, V_k < x\},$$

where U_1 is the sojourn time at the first phase, and V_k is the sojourn time of the same customer at the phases $2, \dots, k+1$. Besides, we denote

$$\Phi(q, x) = \int_0^x e^{-qt}\, d\Pr\{S \leqslant t\}, \qquad a(q, x, k) = \int_0^\infty e^{-qz}\, dF_k(z, x).$$

Let λ be the parameter of the Poisson input flow. Then the following relation holds (see [120]):

$$\begin{aligned} a(q, x, k) &= a(q, x/k, 1) \exp\left(-(k-1)\int_{x/k}^\infty q_\lambda(t)\, dt\right) \\ &= \frac{(1-\lambda)\Phi(q, x/k)(q_\lambda(x/k) - q)}{\lambda - q - \lambda\Phi(q, x/k)} \exp\left(-k\int_{x/k}^\infty q_\lambda(t)\, dt\right), \end{aligned}$$

where $q_\lambda(x) \geqslant 0$ is the unique root of the equation

$$\lambda - q - \lambda\Phi(q, x) = 0.$$

Let us introduce the marginal distribution

$$F_k(x) = \Pr\{V_k < x\}.$$

Then

$$\begin{aligned} F_k(x) &= F_1(x/k) \exp\left(-(k-1)\int_{x/k}^\infty q_\lambda(t)\, dt\right) \\ &= \frac{(1-\lambda)B(x/k)q_\lambda(x/k)}{\lambda\overline{B}(x/k)} \exp\left(-k\int_{x/k}^\infty q_\lambda(t)\, dt\right) \end{aligned}$$

with $\overline{B}(x) = 1 - B(x) = \Pr\{S \geqslant x\}$.

Relying on these relations, the following asymptotic results were obtained for the system with Poisson input flow in heavy traffic. Let us consider a family of multiphase systems depending on parameter λ. Assume that the input flow is Poisson with parameter λ, mean service time equal to 1, and $\lambda \to 1$. Let us introduce the random variables

$$U_1^*(\lambda) = (1-\lambda)U_{1.\lambda}, \qquad V_k^*(\lambda) = \frac{\overline{B}(V_{k.\lambda}/k)}{(1-\lambda)^2}.$$

Then the joint distribution converges weakly to the random vector

$$(U_1^*(\lambda), V_k^*(\lambda)) \underset{\lambda\to 1}{\longrightarrow} (U_1^*, V_k^*).$$

In certain cases, it is possible to write down the limit distribution of the vector $(U_1^*(\lambda), V_k^*(\lambda))$ and to find the asymptotics of $V_{k.\lambda}$. For instance, let $\overline{B}(x) = e^{-x}$, $k = 2$; then

$$\lim_{\lambda\to 1} \Pr\{V_{2.\lambda} + 2\log(1-\lambda) < x\} = \frac{2}{1+\sqrt{1+4e^{-x}}}.$$

It was shown in [121] that for $\lambda \to 1$ and $\overline{B}(x) = e^{-x}$ one has

$$\mathbf{E}V_{k.\lambda} \sim -k\log(1-\lambda)^2.$$

One can see that the limit behavior of the variables $V_{k.\lambda}$ for systems with identical service differs significantly from that for systems satisfying the Kleinrock condition.

Several stones are left unturned. For instance, the problems of heavy traffic interchangeability considered by Tsoucas and Walrand [117] for $\cdot/M/1/\infty$ queues in tandem and the stochastic ordering of tandem queues are unsolved. The comparison problem for tandem queues with dependent and independent service times was studied by Pinedo and Wolff [94], but the general situation was not considered. Rare events for queues in series were studied by Tsoucas [116].

Chapter III.

Heavy traffic approximations for queue length processes in systems with single server were studied by Reiman [101], Harrison [42–44], Harrison and Reiman [48], [49], Anisimov and Chabanuk [4]. The normalization condition (3.1.7) is very close to conditions (2.3.1). Such conditions have been used in heavy traffic limit theorems for queueing networks (see Reiman [101]), for queues with vacations (see Kella and Whitt [65], [66]), and for several other models. Theorem 3.1.2 is due to Reiman [101].

Departure moments from many queues in series were studied by Glynn and Whitt [39]. Theorem 3.2.1 was proved in [39]. The strong approximations given in Theorem 3.2.2 were based on the results of Csorgo, Komlos [25]. Strong approximations have been used previously to study queues with vacations [38], as well as other stochastic models.

Hydrodynamic limits have been studied in physical probability models and in queueing theory (see De Masi and Presutti [27], Liggett [83], Andjel [2], Andjel and Kipnis [3], Benassi and Fouque [9], Kipnis [73]). Theorem 3.2.3 was proved in [39]. The proof given in [39] is based on the ergodic theorem for subadditive processes and on the theory of stochastic ordering of associated random variables (see Stoyan [110], Barlow and Proschan [8], Baccelli and Makowski [6], Liggett [83], Berenstein et al. [10], Chang [23], Kijima [71]).

The function $\gamma(x)$ describing the hydrodynamic limit satisfies the relation $\gamma(x) = x\gamma\,(1/x)$ and the inequalities $\gamma(x) \geqslant 1$, $\gamma(x+y)-\gamma(x) \geqslant y$. These results were proved by Glynn and Whitt [39].

Strong approximations can be used to study the rate of convergence of waiting time processes in heavy traffic. Bounds for the average waiting time and for the waiting time distribution can be used to estimate the quality of approximations (see Kreinin [79], Stoyan [110]).

Theorem 3.3.1 and relation (3.3.4) were proved by the authors (unpublished). Theorem 3.3.2 is due to Glynn and Whitt [39]. The role of the max-scheme in the theory of many queues in series is extremely important. As it is shown in Chapters II and III, a combination of the max-scheme and the addition of random variables describes departure moments from multiphase queues. The max-scheme has been studied in many papers. Rachev and Ruschendorf [98] studied the rate of convergence for sums and maxima and doubly ideal metrics. Zolotorev and Rachev [137] studied the rate of convergence in limit theorems for the max-scheme. A survey of modern results can be found in Zolotorev [137]. We also mention here the monograph by Leadbeter, Lindgren, and Rootzen [81] devoted to extremes of random sequences, and Feller's course on probability theory [33] as well.

Appendix I.

Appendix I contains a brief review of the theory of weak convergence of stochastic processes. This theory is presented in many monographs (see, for example, Billingsley [11], Skorohod [108]). The ideas and basic theorems of the theory of weak convergence in function spaces were given by Prohorov [95]. Theorem A1.1.5 on the convergence in distribution is used for the proof of weak convergence of functionals.

The convergence in the space $D([0, T])$ was considered by Skorohod [107]. The distance (A1.2.1) in the space $D([0, T])$ was introduced by Lindvall [84]. The corresponding topology is equivalent to the Skorohod topology.

Other topologies have also been used in queueing theory (see [38], [65]). The set of operators $\hat{\Lambda}$ introduced in this section is new. The algebra of operators commuting with $\hat{\Lambda}$ is given in this appendix for the first time. Theorem A1.2.3 is new.

The Donsker–Prohorov invariance principle is treated in a number of books and papers [11], [95]. Several modern results can be found in Wacker [122].

Appendix II.

The weak convergence of functionals on the half-line is considered in Appendix II. The approach presented provides a new point of view on the theory of weak convergence of functionals defined on the half-line. The circle of ideas is very close to Borovkov [14]. The scheme of the proof of the convergence discussed in this appendix gives a useful tool for proving weak convergence results for various stochastic models. The construction does not depend on the form of the limit process and on the form of the processes ξ_λ. The definition of the family of approximated functionals is given here for the first time.

An example of a functional that does not belong to the class of approximated functionals is based on the so-called diagonal process. The notion of an admissible functional is new. Proposition A2.3.1 containing several important facts about admissible functionals is new as well.

Appendix III.

Several important technical results are contained in Appendix III. Lemma A3.1.1 is borrowed from [58]. The corresponding inequality is a generalization of the Kolmogorov inequality for semimartingales. Inequalities (A3.2.10)–(A3.2.12) were proved in [58], [61].

Appendix IV.

The general theory of boundary value problems for elliptic equations is presented in Bitsadze [12]. The reduction to a boundary value problem given in this section is rather standard. This technique is presented in Gakhov [35], Gakhov and Chersky [36] and is used in many problems for partial differential equations. Very similar problems were studied by Harrison [43], [44] for tandem queues and queues in series. Such problems also appear for Brownian models of queueing networks with heterogeneous populations (see Harrison [45]) and for Brownian models of open queueing networks with homogeneous customer populations (see Harrison and Williams [50]).

Proposition A4.1.1 was given in [58] without proof. Nonseparable exponential solutions were discovered in [58]. The solution of the boundary value problem is borrowed from [58]. The generalizations to many queues in series were considered by Harrison [42], Karpelevich and Kreinin [59], [60], but the problem of stationary distribution in such systems is still unsolved when the number of phases is greater than 2.

On the other hand, the progress achieved in this area stimulated the study of stationary distributions of the reflected Brownian motion in planar and multidimensional regions (see Harrison, Landau, and Shepp [47], Harrison and Reiman [48], [49], Harrison and Williams [50], and Varadhan and Williams [119]).

Appendix V.

The general theory of joint distributions in tandem Poisson queues is borrowed mainly from [62]. Jackson's theorem for queueing networks was proved in [55]. Queues with multiplicative form of the stationary queue length distribution have been studied by many authors (see Borovkov [17], Kelly [68]). The ergodicity conditions for tandem queues were proved by Loynes [87]. For queueing networks the corresponding ergodicity theorem was proved by Foss [34] and by Rybko and Stolyar [105]. The joint stationary distribution of the number of customers at the arrival epoch was studied by Reich [99], [100] and Burke [21]. Exit times were studied for series of Jackson networks by Massey [89]. Besides, Massey suggested an operator analytic approach to the study of the stationary and transient behavior of queueing networks [90].

Theorems A5.3.1 and A5.4.1 were proved in [62]. The heavy traffic transient behavior of tandem queues is discussed in section A5.5. The proof of the asymptotic formula (A5.5.8) is based on Tauberian theorems borrowed from [33]; see also Widder [130].

Proposition A5.5.1 is new. Similar problems for the relaxation time in tandem queues were studied by Blanc [13]. Yakushev [133] tried to find the joint distribution of queue lengths for tandem queues with Poisson input flow and general service times.

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