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Methods for Analysis of Nonlinear Elliptic Boundary Value Problems

I. V. Skrypnik

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Methods for Analysis of Nonlinear Elliptic Boundary Value Problems

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МЕТОДЫ ИССЛЕДОВАНИЯ НЕЛИНЕЙНЫХ ЭЛЛИПТИЧЕСКИХ ГРАНИЧНЫХ ЗАДАЧ
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ABSTRACT. This monograph deals with the investigation of boundary value problems for nonlinear elliptic equations of arbitrary order. Monotone operator methods are presented. A broad range of applications of topological methods to several topics of nonlinear differential equations are given: solvability, estimation of the number of solutions, and the branching of solutions of nonlinear equations. A priori estimates and the regularity of solutions of nonlinear elliptic equations of arbitrary order are established by various procedures. Methods of homogenization of nonlinear elliptic problems in perforated domains are given.

The book is intended for mathematicians, specialists, as well as for students, in advanced university courses interested in differential equations and nonlinear functional analysis.

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Contents

Preface

The theory of nonlinear elliptic equations of arbitrary order is nowadays one of the most actively developing branches of the theory of partial differential equations. The study of nonlinear second-order elliptic equations has a history of almost one century and the main trends of investigations, namely, the regularity of solutions and the solvability of boundary value problems, are assigned by Hilbert's nineteenth and twentieth problems. The research of S. N. Bernstein, J. Leray, J. Schauder, C. B. Morrey, E. De Giorgi, J. Moser, O. A. Ladyzhenskaya, N. N. Ural'tseva, and other authors has contributed not only to the solution of Hilbert's problems, but also to creating many methods playing fundamental roles both in the theory of differential equations and in related parts of mathematics. A presentation of the basic results of these investigations can be found in the monograph **[49]**.

Nevertheless, the rich experience accumulated in the study of second order equations proved inefficient in the study of nonlinear elliptic equations of arbitrary order. The fact is that, in many cases, the methods for obtaining a priori estimates (namely, such estimates play a key role in the nonlinear theory) have turned out to be inapplicable to higher order equations. This led to the necessity of working out new techniques for nonlinear elliptic partial differential equations of arbitrary order. This monograph is dedicated to the presentation of these methods, contained to a considerable extent in journal papers. Basically, the following topics are treated:

- methods of monotone operator theory and their application to the study of nonlinear elliptic boundary value problems;
- methods for introducing topological characteristics for general nonlinear problems both for divergent and nondivergent elliptic equations;
- broad applications of topological arguments to various classes of equations: from essentially nonlinear to weakly nonlinear ones; applications of these methods not only to proving the solvability but also to estimating the number of solutions and branching of solutions of nonlinear problems;
- various methods for obtaining a priori estimates and the proof of the regularity of the solutions of nonlinear elliptic equations of arbitrary order;
- homogenization methods for nonlinear elliptic problems in perforated domains.

Starting with the sixties, the study of higher order quasilinear elliptic equations witnessed a broad development. The first results concerning the solvability of boundary value problems by the modified Galerkin method are due to M. I. Vishik. The subsequent progress is connected with the application of the theory of monotone and more general classes of operators. These methods were first developed by F. E. Browder, and then by J. Leray, J. P. Lions, Yu. A. Dubinskiĭ, S. I. Pokhozhaev and others. Their results are presented in Chapter 1 of the monograph. Alongside the solvability of coercive problems and those with odd operators, we briefly survey results for the equations with rapidly growing coefficients and for equations of infinite order.

The substantial progress in the treatment of nonlinear boundary value problems is connected with the creation of topological methods for mappings of monotone type. The foundation of these methods has its origin in the paper **[16b]** of F. E. Browder and W. V. Petryshyn on the degree of A-proper mappings and in the author's paper **[77a]** on the degree of mappings of class $(S)_+$. These studies continue and develop the celebrated results of J. Leray and J. Schauder concerning the degree of completely continuous perturbations of the identity map. On the basis of the degree of mappings of class $(S)_+$, the topological characteristics of boundary value problems for general nonlinear elliptic equations with general nonlinear boundary conditions are determined naturally.

Five chapters of the monograph (starting with Chapter 2), are devoted to topological methods of investigation of nonlinear problems. The definition of the degree of mappings of class $(S)_+$ as well as its properties are given in Chapter 2. We present a formula for the index of a critical point; it plays an important role in applications. Chapter 3 deals with various methods of reducing differential boundary value problems to equations involving operators satisfying condition $(S)_+$. This is done by means of a priori L_p-estimates for linear elliptic operators. In the case of the Dirichlet problem, an important role is played by the coercive estimates for pairs of linear elliptic operators; these estimates are proved in Chapter 3.

Various applications of topological methods to the investigation of nonlinear boundary value problems are given in Chapter 4. There we prove the solvability of the Neumann problem for quasilinear second- order elliptic equations and the solvability of the Dirichlet problem for the Monge-Ampère equation. In Chapter 5 we establish the solvability of the general nonlinear Dirichlet problem in a narrow strip. Here, in the proof of a priori estimates of solutions and in the application of the degree of a mapping, decisive is the choice of the function space which permits us to obtain estimates that are uniform (with respect to a given family of domains). In Chapter 6 we consider nonlinear problems with a linear principal part. Attention is mainly paid to solvability conditions in the resonance case, i.e., when the linear principal part is noninvertible. The pioneering work of Landesman and Lazer has been followed by a large number of papers in this direction.

Chapter 7 deals with the regularity of generalized solutions and with a priori estimates of solutions of boundary value problems. In this respect, we present the contributions of J. Nečas, the author, S. I. Pokhozhaev, C. B. Morrey, M. Giaquinta, S. Campanato, A. I. Koshelev, A. I. Shishkov, and others, who have made substantial advancements in a number of difficult problems. Here, in particular, Gehring's lemma is used to prove the increase of the exponent of summability of the derivatives of the solution, a complete solution of the problems of regularity in the two- dimensional case is obtained and a partial regularity of any generalized solution is established. We note the exposition of S. I. Pokhozhaev's interpolation methods for a priori estimates.

The last three chapters are dedicated to the study of the behavior of the solutions of the Dirichlet problem for quasilinear elliptic equations near a nonsmooth boundary and in a family of perforated domains. The sharp estimates of the solutions of model nonlinear problems established by the author are the main tool in obtaining results characterizing the pointwise behavior of the solutions. In Chapter 8, we investigate the Wiener regularity of a boundary point for a second-order quasilinear elliptic equation, i.e., we establish conditions on the boundary of the domain, guaranteeing the continuity of the solutions at a given boundary point. In particular, if the energy space for quasilinear equations is the space $W_2^1(\Omega)$ then a necessary and sufficient condition for the regularity of a boundary point for such an equation coincides with Wiener's criterion for the Laplace equation.

Chapters 9 and 10 deal with homgenization of a family of quasilinear boundary value problems in domains with a fine-grained boundary and in domains with channels. In the linear situation such problems were studied in the paper **[59]** of V. A. Marchenko and E. Ya. Khruslov. We note the importance of these studies for many problems in nonlinear mechanics of strongly nonhomogeneous media and composite materials.

It is shown in the monograph that the solutions of quasilinear Dirichlet problems in a family of perforated domains are close to the solution of a certain averaged quasilinear problem in a nonperforated domain. Conditions for the existence of the averaged problem are elucidated and concrete methods for its construction are presented.

In the monograph, as a rule, methods and proofs of the results are presented in detail, save for some cumbersome proofs of the regularity of generalized solutions, published in the monographs **[43, 77a]**.

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