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Traveling Wave Solutions of Parabolic Systems

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Traveling Wave Solutions of Parabolic Systems

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ABSTRACT. Traveling wave solutions of parabolic systems describe a wide class of phenomena in combustion physics, chemical kinetics, biology, and other natural sciences. The book is devoted to the general mathematical theory of such solutions. The authors describe in detail such questions as existence and stability of solutions, properties of the spectrum, bifurcations of solutions, approach of solutions of the Cauchy problem to waves and systems of waves. The final part of the book is devoted to applications to combustion theory and chemical kinetics.

The book can be used by graduate students and researchers specializing in nonlinear differential equations, as well as by specialists in other areas (engineering, chemical physics, biology), where the theory of wave solutions of parabolic systems can be applied.

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Preface

The theory of traveling wave solutions of parabolic equations is one of the fast developing areas of modern mathematics. The history of this theory begins with the famous mathematical work by Kolmogorov, Petrovskii, and Piskunov and with works in chemical physics, the best known among them by Zel’dovich and Frank-Kamenetskiĭ in combustion theory and by Semenov, who discovered branching chain flames.

Traveling wave solutions are solutions of special type. They can be usually characterized as solutions invariant with respect to translation in space. The existence of traveling waves appears to be very common in nonlinear equations, and, in addition, they often determine the behavior of the solutions of Cauchy-type problems.

From the physical point of view, traveling waves usually describe transition processes. Transition from one equilibrium to another is a typical case, although more complicated situations can arise. These transition processes usually “forget” their initial conditions and reflect the properties of the medium itself.

Among the basic questions in the theory of traveling waves we mention the problem of wave existence, stability of waves with respect to small perturbations and global stability, bifurcations of waves, determination of wave speed, and systems of waves (or wave trains). The case of a scalar equation has been rather well studied, basically due to applicability of comparison theorems of a special kind for parabolic equations and of phase space analysis for the ordinary differential equations. For systems of equations, comparison theorems of this kind are, in general, not applicable, and the phase space analysis becomes much more complicated. This is why systems of equations are much less understood and require new approaches. In this book, some of these approaches are presented, together with more traditional approaches adapted for specific classes of systems of equations and for a more complete analysis of scalar equations. From our point of view, it is very important that these mathematical results find numerous applications, first and foremost in chemical kinetics and combustion. The authors understand that the theory of traveling waves is far from being complete and hope that this book will help in its development.

This book was basically written when the authors worked at the Institute of Chemical Physics of the Soviet Academy of Sciences. This scientific school, created by N. N. Semenov, Director of the Institute for a long time, by Ya. B. Zeldovich, who worked there, and by other outstanding personalities, has a strong tradition
of collaboration among physicists, chemists, and mathematicians. This special atmosphere had a strong influence on the scientific interests of the authors and was very useful to us. We would like to thank all our colleagues with whom we worked for many years and without whom this book could not have been written.

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