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L. Ya. Adrianova

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10.1090/mmono/146

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MATHEMATICAL
MONOGRAPHS

Volume 146

Introduction to Linear Systems of Differential Equations

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American Mathematical Society
Providence, Rhode Island

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ВВЕДЕНИЕ В ТЕОРИЮ ЛИНЕЙНЫХ СИСТЕМ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Translated from the Russian by Peter Zhevandrov
Translation edited by Simeon Ivanov

1991 *Mathematics Subject Classification.* Primary 34A30, 34Dxx

ABSTRACT. The structure and behavior of solutions of autonomous and periodic systems of ordinary differential equations is considered. Among the properties considered in the book are reducibility of systems, stability of solutions, estimates of solution growth in terms of the coefficients, the influence of a small exponent to the properties of solutions, and central exponents. A complete proof of the necessary and sufficient conditions for the stability of characteristic exponents of two-dimensional diagonal systems is presented.

The book can be used by researchers and graduate students working in differential equations, control theory, and mechanics.

Library of Congress Cataloging-in-Publication Data

Adrianova, L. I͡A.

[Vvedenie v teorii͡u lineĭnykh sistem different͡sial′nykh uravneniĭ. English]

Introduction to linear systems of differential equations / L. Ya. Adrianova; [translated from the Russian by Peter Zhevandrov].

p. cm. — (Translations of mathematical monographs, ISSN 0065-9282; v. 146)

Includes bibliographical references (p. –) and index.

ISBN 0-8218-0328-X (acid-free paper)

1. Differential equations, Linear. I. Title. II. Series.

QA372.A3313 1995

515′.252—dc20 95-35164

CIP

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10 9 8 7 6 5 4 3 2 1 00 99 98 97 96 95

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Preface

This textbook is the outgrowth of a lecture course on the theory of linear systems taught by the author at St. Petersburg University the last several years for fourth year students specializing in differential equations.

Linear differential systems are of interest for mathematicians both *per se* and as a tool for studying nonlinear equations by means of the method of linearization. The theory of such systems is rich in problems and methods for their solution [**23**].

It is impossible to include all the basic results and methods in a one-year special course. Indeed, this was not our aim while writing this book, although the material contained here is larger than that given in the lectures. Our goal is to provide an introduction to the theory of linear systems, to acquaint the reader with the basic notions, terms, and definitions of the theory, to show the interrelations among them, and to present some fundamental results and their proofs, in short, to prepare the reader for the study of more involved and specialized parts of the theory.

First we consider properties of solutions of systems with constant and periodic coefficients; this forms the basis for understanding the subsequent material. Here we pay special attention to the construction of a real basis in the case of real coefficients.

Further, by means of the method of characteristic exponents, we study the structure of the space of solutions of a linear system, investigate the properties of reducibility and almost reducibility, and introduce and consider in detail regular systems.

The next part of the book is devoted to the impact of perturbations of the initial data and of the coefficients on the behavior of the solutions. We study various types of stability, perturbations of the coefficients admissible for them, and give estimates of the growth of solutions. One of the most complicated problems of the theory of linear systems is the study of the impact of small perturbations of the coefficients on the characteristic exponents. In order to acquaint the reader with the basic methods for solving this problem, we dwell on the notions of upper and lower functions, central exponents, and integral separateness of a system. Finding necessary and sufficient conditions for the stability of characteristic exponents is one of the fundamental and technically subtle results of recent years, the completion of which relies on the method of rotations due to Millionshchikov [**28, 29**]. Here we give a proof of this result in the case of a two-dimensional diagonal system; this enables us to sufficiently simplify the problem from the technical point of view, preserving the main ideas of the considerations for the general case.

The classical theory of linear systems assumes that the coefficients are bounded. We indicate results that remain valid even when this requirement is weakened. For regular systems this is done in Chapter III, §7. We have tried to emphasize each, athough rare, possibility to connect the properties of solutions with those of the coefficients of a system (see, e.g., Chapter IV, §6, and Chapter VI). There are many examples that precede and conclude the arguments; this is intended to simplify the study of the book.

Naturally, this book borrows some material from basic monographs on the general theory of linear systems **[9, 19]**. At the same time, a number of results given here can be found only in papers published in specialized journals and are not contained in other monographs on differential equations.

The numbering of formulas, theorems, lemmas, etc., is triple (number of chapter, number of section, number of object within section).

N. A. Izobov, a corresponding member of the Belarus Academy of Sciences, has helped the author a lot during the preliminary stage of the preparation of the special course; the collaboration with him has been of great value for the author. Important remarks concerning the plan of the book were furnished by Professor V. A. Pliss, to whom the author is also grateful for his constant support.

Principal Notation

$\mathbb{C}^n$ is the n-dimensional complex vector space,
$\mathbb{R}^n$ is the n-dimensional Euclidean space (we write $\mathbb{R}$ instead of $\mathbb{R}^1$),
$\mathbb{Z}$ is the set of integers,
$\mathbb{Z}_+$ is the set of nonnegative integers,
$\mathbb{R}_+$ is the set of nonnegative reals,
$\mathbb{N}$ is the set of naturals,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \text{ is a vector,}$$

$$x^\top = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}^\top = (x_1, x_2, \dots, x_n),$$

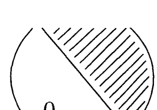
$\|x\|$ is the norm,
$A = \{a_{ij}\}$, $i = 1, \dots, n$, $j = 1, \dots, n$, is the $n \times n$ matrix with the elements a_{ij},
$\operatorname{Sp} A = \sum_{k=1}^{n} a_{kk}$ is the trace of a matrix,
A^* is the Hermitian adjoint for the matrix A,
$\|A\| = \max_{\|x\|=1} \|Ax\|$ is the norm of a matrix A,
A_m is an $m \times m$ matrix,
E is the identity $n \times n$ matrix,
E_m is the identity $m \times m$ matrix,

$$J_\nu(\alpha) = \begin{pmatrix} \alpha & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & \alpha \end{pmatrix} \text{ is the } \nu \times \nu \text{ Jordan block,}$$

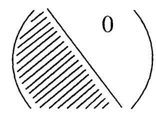
$$C = \operatorname{diag}[c_1, \dots, c_r] = \begin{pmatrix} c_1 & & 0 \\ & c_2 & \\ 0 & & c_r \end{pmatrix} \text{ is a block-diagonal matrix,}$$

$A_d = \operatorname{diag}[a_{11}, a_{22}, \dots, a_{nn}]$,

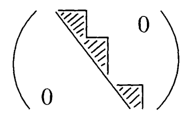
an upper-triangular matrix:



a lower-triangular matrix:



a block-triangular matrix:



$X(t) = \{x_1(t), \dots, x_n(t)\}$ is the matrix whose columns are vectors $x_1(t), \dots, x_n(t)$,
$X^{-1}(t)$ is the matrix inverse for $X(t)$,
$X(t, \tau) = X(t)X^{-1}(\tau)$,
Det X is the determinant of a matrix X,
$A \in C(I)$, $x \in C(I)$ means that the elements of the matrix A and vector x are continuous on the interval I,
dim L denotes the dimension of a lineal $L \in \mathbb{C}^n$,
$[t]$ is the integer part of a number $t \in \mathbb{R}$,
σ_X is the sum of characteristic exponents of a basis X,
$\bar{\lambda} = \overline{\alpha + i\beta} = \alpha - i\beta$ denotes complex conjugation,
$\prec$ is the sign of comparison of growths of functions,
$\chi[\,\cdot\,]$ is the characteristic exponent,
δ_{ij} is the Kronecker symbol, $\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$
$\dot{x} = \dfrac{dx}{dt}$ denotes the derivative.

Appendix

1. The Gronwall-Bellman lemma and its generalization.

THE GRONWALL-BELLMAN LEMMA [19]. *Let functions $u(t) \geqslant 0$, $v(t) \geqslant 0$ be defined and continuous for $t \geqslant t_0$ and*

$$u(t) \leqslant \lambda + \int_{t_0}^{t} u(\tau)v(\tau)\,d\tau, \tag{A.1}$$

where $\lambda \in \mathbb{R}_+$. Then for $t \geqslant t_0$ we have

$$u(t) \leqslant \lambda e^{\int_{t_0}^{t} v(\tau)\,d\tau}. \tag{A.2}$$

PROOF. Denote the right-hand side of the inequality (A.1) by $g(t)$; hence,

$$g'(t) = u(t)v(t) \leqslant g(t)v(t),$$

or $g'(t) - v(t)g(t) \leqslant 0$. Multiplying the last inequality by $\exp\left(-\int_{t_0}^{t} v(\tau)\,d\tau\right)$, we write

$$d\left[g(t)\exp\left(-\int_{t_0}^{t} v(\tau)\,d\tau\right)\right] \leqslant 0.$$

Integrating the result from t_0 to t, we obtain

$$g(t)\exp\left(-\int_{t_0}^{t} v(\tau)\,d\tau\right) - g(t_0) \leqslant 0,$$

or $g(t) \leqslant \lambda \exp \int_{t_0}^{t} v(\tau)\,d\tau$. Taking into account that $u(t) \leqslant g(t)$, we obtain (A.2). □

COROLLARY. *If in the conditions of the lemma $\lambda = 0$, then $u(t) \equiv 0$ for $t \geqslant t_0$.*

A GENERALIZATION OF THE GRONWALL-BELLMAN LEMMA. *Let the function $u(t)$ be positive and continuous for $t \in (a, b)$ and let it satisfy the integral inequality*

$$u(t) \leqslant u(\tau) + \left|\int_{\tau}^{t} u(z)v(z)\,dz\right| \tag{A.3}$$

for any $t, \tau \in (a, b)$, where $v \in C(a, b)$ and $v(t) \geqslant 0$, $t \in (a, b)$. Then the two-sided estimate

$$u(t_0)e^{-\int_{t_0}^{t} v(z)\,dx} \leqslant u(t) \leqslant u(t_0)e^{\int_{t_0}^{t} v(z)\,dz} \tag{A.4}$$

is valid for $a < t_0 \leqslant t < b$.

PROOF. 1) $t \geqslant \tau$. In this case the inequality (A.3) has the form

$$u(t) \leqslant u(\tau) + \int_\tau^t u(z)v(z)\,dz$$

and the right-hand side of the estimate (A.4) follows from the last inequality by the Gronwall-Bellman lemma for $\tau = t_0$.

2) $t \leqslant \tau$. In this case the inequality (A.3) can be rewritten in the following way:

$$u(t) \leqslant u(\tau) + \int_t^\tau u(z)v(z)\,dz. \tag{A.5}$$

Denoting the right-hand side of (A.5) by $g(t)$, we have

$$g'(t) = -u(t)v(t) \geqslant -v(t)g(t),$$

or

$$g'(t) + v(t)g(t) \geqslant 0.$$

Multiplying the last inequality by

$$\exp\left(\int_\tau^t v(z)\,dz\right),$$

we obtain

$$d\left(g(t)\exp\int_\tau^t v(z)\,dz\right) \geqslant 0. \tag{A.6}$$

Let us integrate (A.6) from t to τ; then

$$g(\tau) - g(t)\exp\int_\tau^t v(z)\,dz \geqslant 0, \qquad \text{or} \qquad g(\tau) \geqslant g(t)\exp\int_\tau^t v(z)\,dz.$$

Taking into account that $g(\tau) = u(\tau)$ and $g(t) \geqslant u(t)$, we have

$$u(\tau) \geqslant u(t)\exp\int_\tau^t v(z)\,dz = u(t)\exp\left(-\int_t^\tau v(z)\,dz\right). \tag{A.7}$$

Recall that $\tau \geqslant t$. Changing t to t_0 and τ to t in (A.7), we obtain the left-hand side of the estimate (A.4). □

2. Regularity of a linear system almost reducible to an autonomous one.

THEOREM 3.5.2. *A linear system almost reducible to a system with constant coefficients is regular.*

PROOF. Let a system

$$\dot{x} = A(t)x, \qquad A \in C(\mathbb{R}_+), \tag{A.8}$$

be almost reducible to a system $\dot{y} = By$, where B is a constant matrix. According to Theorem 3.4.3, the system (A.8) is also almost reducible to a diagonal system, whose diagonal consists of the real parts of the eigenvalues of the matrix B. Denote them by $\lambda_1, \lambda_2, \dots, \lambda_n$. Thus, for any $\delta > 0$ there exists a Lyapunov transformation $x = L_\delta(t)z$ reducing the system (A.8) to the system

$$\dot{z} = \operatorname{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]z + \Phi(t)z, \tag{A.9}$$

where $\sup \|\Phi(t)\| \leqslant \delta$. By the stability of characteristic exponents of systems with constant coefficients the spectrum of the system (A.9) is δ-close to $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, and, by the properties of Lyapunov transformations, it coincides with the spectrum of the system (A.8).

Let $X(t)$ and $Z(t)$ be normal fundamental matrices of the systems (A.8) and (A.9) such that $X(t) = L_\delta(t)Z$; therefore,

$$\operatorname{Det} X(t) = \operatorname{Det} L_\delta(t) \operatorname{Det} Z(t).$$

Hence, by the Ostrogradskiĭ-Liouville formula, we have

$$\begin{aligned}\operatorname{Det} X(t_0) \exp \int_{t_0}^{t} \operatorname{Sp} A(\tau)\, d\tau \\ = \operatorname{Det} L_\delta(t) \operatorname{Det} Z(t_0) \exp \left(\int_{t_0}^{t} \sum_{i=1}^{n} \lambda_i \, d\tau + \int_{t_0}^{t} \operatorname{Sp} \Phi(\tau)\, d\tau \right).\end{aligned}$$

Passing to absolute values in both sides of the last equality, taking logarithms, and dividing by t, we obtain

$$\begin{aligned}\frac{1}{t} \int_{t_0}^{t} \operatorname{Re} \operatorname{Sp} A(\tau)\, d\tau \\ = \frac{1}{t} \ln |\operatorname{Det}(L_\delta(t) Z(t_0) X^{-1}(t_0))| + \frac{t - t_0}{t} \sum_{i=1}^{n} \lambda_i + \frac{1}{t} \int_{t_0}^{t} \operatorname{Re} \operatorname{Sp} \Phi(\tau)\, d\tau.\end{aligned}$$

Note that $|\operatorname{Sp} \Phi(t)| \leqslant n\delta$. Thus,

$$\begin{aligned}\frac{t - t_0}{t} \sum_{i=1}^{n} \lambda_i + \frac{1}{t} \ln |\operatorname{Det}(L_\delta(t) Z(t_0) X^{-1}(t_0))| - n\delta \frac{t - t_0}{t} \\ \leqslant \frac{1}{t} \int_{t_0}^{t} \operatorname{Re} \operatorname{Sp} A(\tau)\, d\tau \\ \leqslant \frac{1}{t} \ln |\operatorname{Det} L_\delta(t) Z(t_0) X^{-1}(t_0)| + n\delta \frac{t - t_0}{t} + \frac{t - t_0}{t} \sum_{i=1}^{n} \lambda_i.\end{aligned}$$

In the last inequality we pass to the limit as $t \to \infty$, taking into account that $\sup_{t \in \mathbb{R}_+} |\operatorname{Det} L_\delta(t)|$ is bounded. By the squeeze convergence principle, we obtain that the limit

$$\lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{Re} \operatorname{Sp} A(\tau)\, d\tau$$

exists and

$$\sum_{i=1}^{n} \lambda_i - n\delta \leqslant \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{Re} \operatorname{Sp} A(\tau)\, d\tau \leqslant \sum_{i=1}^{n} \lambda_i + n\delta.$$

Since $\delta > 0$ is arbitrary, we have

$$\lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{Re} \operatorname{Sp} A(\tau)\, d\tau = \sum_{i=1}^{n} \lambda_i;$$

this, by Lemma 3.5.1, shows that system (A.8) is regular. □

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