Translations of

MATHEMATICAL MONOGRAPHS

Volume 149

Riemannian Geometry

Takashi Sakai

American Mathematical Society

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Riemannian Geometry

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Volume 149

Riemannian Geometry

Takashi Sakai

Translated by
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リーマン幾何学

RĪMAN KIKAGAKU

(Riemannian Geometry)
by Takashi Sakai

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ABSTRACT. The aim of this textbook is to provide to advanced undergraduate and graduate students an introduction to modern Riemannian geometry that could also serve as a reference. The book begins with an explanation of the fundamental notions of Riemannian geometry. Special emphasis is placed on understandability and readability, to guide students who are new to this area. The remaining chapters deal with various topics in Riemannian geometry, with the main focus on comparison methods and their applications.

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Preface to the English Edition

This volume is an English translation of my textbook on Riemannian geometry originally written in Japanese and published in 1992 by Shokabo, Tokyo. I wrote the Japanese edition mainly because at that time there were no textbooks written in Japanese that introduced modern Riemannian geometry to advanced undergraduate and graduate students and that could also serve as a reference. On the other hand, there are many textbooks and monographs on Riemannian geometry written in Western languages at various levels and treating a variety of topics. I have consulted them, and I have been influenced especially by the books by M. Berger and A. Besse, J. Cheeger and D. G. Ebin, and W. Klingenberg.

Now let me mention the points on which I put emphasis in the present volume.

(1) After reviewing fundamentals on differentiable manifolds in Chapter I, I try to explain the fundamental notions and results of Riemannian geometry in Chapters II and III with particular emphasis placed on understandability and readability, since, in my teaching experience, many students find it difficult to grasp Riemannian geometry on their first try.

(2) In the remaining chapters, among various topics in Riemannian geometry I am mainly concerned with the comparison methods and their applications. I take an approach using Jacobi fields to comparison methods in Chapter IV, and give their applications to the relation between the curvature and topology, geometric inequalities, and spectral geometry in Chapters V and VI.

In principle, I faithfully translated the Japanese edition, except for correcting small errors and adding a few comments on further developments. However, Appendix 6 on Gromov's convergence theorem and the collapsing of Riemannian manifolds has been expanded and revised considerably. I also added more references and notes on the references to each chapter, although they are still far from being complete.

I would like to express my gratitude to K. Grove, H. Karcher, A. Katsuda, W. Klingenberg, R. Porter, and W. Tuschmann for useful suggestions and advice. I also thank K. Shimakawa for helping me with the $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX typesetting.

Takashi Sakai
May, 1995

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Preface

In this volume we give an exposition of the fundamental concepts and results of Riemannian geometry, and explain especially the ideas called comparison methods and their applications, assuming some fundamentals on differentiable manifolds.

First we briefly mention the birth of Riemannian geometry. In his "Elements" (Stoicheia), Euclid (Eukleides) systematically arranged many facts of elementary geometry that had long been known, taking an axiomatic viewpoint for the first time. Namely, defining the notions of point, line, plane, angle, etc., and describing some of the most fundamental relationships among them as the axioms (or postulates), he systematically deduced, through strict logic, other marvelous geometric facts (propositions, theorems) based on the axioms. From an axiomatic viewpoint it had been suspected ever since the age of Euclid that the fifth postulate, which is equivalent to the statement that for a given line l and a point p in the plane there exists a *unique* line parallel to l through p, could be proven from the other axioms. After various attempts over more than 2,000 years, some people began to suspect that a new geometry might be developed by the denying the fifth postulate and leaving the remaining axioms as they stand. János Bolyai (1832) and N. I. Lobachevsky (1830) were the first who published their new geometry. Gauss himself also reached the same conclusion, but did not publish since he feared that false controversies might be caused by misunderstandings.

The discovery of non-Euclidean geometry brought about serious examinations of the foundations of geometry and the concept of space. For instance, Gauss measured the inner angles of a triangle whose vertices where the summits of three high mountains far apart in Germany, and tried to judge which geometry reflects the real world.

Under these circumstances G. F. B. Riemann proposed in 1854 an epoch-making view in his Habilitationsschrift, "Über die Hypothesen, welche der Geometrie grundliegen", submitted to Göttingen University. Namely, instead of taking an axiomatic viewpoint, he proposed to consider more general "Mannigfaltigkeiten" (manifolds), which are locally homeomorphic to Euclidean space of a fixed dimension and "spread out" manifold. Then he discussed how to measure the length of curves, the distance between two points, the angle between vectors, etc., on a given manifold, and introduced the notion of a Riemannian metric inspired by the surface theory of Gauss. Further, Riemann defined the notion of the (sectional) curvature of a Riemannian metric in terms of the Gauss curvature of a surface. Then he noted that the sectional curvatue of a Riemannian metric is constant if and only if figures are freely movable in a manifold without expansion or contraction. He also pointed out that, for manifolds of constant curvature k, the flat case (i.e., $k = 0$) describes Euclidean geometry, and the negative constant curvature case describes the non-Euclidean geomtery of Bolyai and Lobachevsky. Manifolds of positive constant curvature correspond to the elliptic non-Euclidean geometry of Riemann. It was reported that old Gauss, who attended Riemann's lecture, was deeply touched.

Thus a completely new and huge field of geometry opened. Riemann's idea was first developed by G. Ricci, T. Levi-Civita, and other people as an absolute differential calculus for tensors, which seemed rather formal. However, such tensor calculus turned out to provide a needed mathematical tool when Einstein established his general theory of relativity with a gravitation field in 1916, and Riemannian geometry was highlighted.

Subsequently Hermann Weyl and Élie Cartan took a more general view of the connection, and unified Riemann's idea and F. Klein's program interpreting geometries in terms of transformation groups. S. Cohn-Vossen, W. Blaschke, and others studied the global properties relating the metric invariants to the topology of the surface. H. Poincaré, G. D. Birkhoff, M. Morse, J. Hadamard, E. Hopf, and others worked on various properties of geodesics from different standpoints. H. Hopf studied the global properties of spaces of constant curvature, and É. Cartan originated and made an extensive study of the symmetric spaces, a remarkable class of Riemannian manifolds. Through all this essential work Riemannian geometry was linked to various fields of mathematics (e.g., dynamical systems, calculus of variations, topology), and it was recognized that the relation between local properties (e.g., curvature) determined by the metrics and global properties related to the whole structure of manifolds are important objects of the investigation. Also the notion of differentiable manifolds was defined rigorously in the terminology of modern mathematics by H. Weyl and H. Whitney, and the fundamental concepts of manifolds and Riemannian geometry were consolidated. For instance, H. Hopf and W. Rinow defined the notion of completeness of a Riemannian metric, through which the global notions were established.

In the present book, after reviewing fundamentals on differentiable manifolds in Chapter I, we treat with care some fundamental concepts and results of Riemannian geometry in Chapters II and III. Especially, we explain the notions of geodesic, Jacobi fields, and curvature together with many examples in Chapter II, and some global concepts and results of Riemannian geometry, which are mainly related to geometry of geodesics, in Chapter III. I hope that the reader may grasp Riemannian geometry in outline through Chapters II and III.

Modern Riemannian geometry has been developed in many branches from various viewpoints mainly as geometry on manifolds, and it is impossible to cover all topics in a textbook. In the present volume we are mainly concerned with the comparison methods and their applications in Chapters IV, V, and VI. A complete simply connected Riemannian manifold of positive constant curvature δ is isometric to the sphere of radius $1/\sqrt{\delta}$. H. Hopf conjectured that a complete simply connected Riemannian manifold whose sectional curvature is not necessarily equal to a positive constant but remains close to a positive constant is still topologically a sphere. Then H. E. Rauch established this fact in his epoch-making paper in 1951. M. Berger and W. Klingenberg improved and developed Rauch's idea, and got the best possible sphere theorem for the case where the ratio of the minimal and the maximal value of the sectional curvature is greater than $1/4$. Through their work and work of D. Gromoll, J. Cheeger, E. Ruh, K. Shiohama, P. Eberlein, K. Grove,

H. Karcher, and other geometers, great progress has been made in studying the relation between metrical invariants and global properties of Riemannian manifolds. In particular, comparison methods, which compare a given Riemannian manifold with a standard Riemannian manifold of constant curvature in terms of some geometric invariants, were developed. In Chapter IV we state these comparison methods in a unified manner in terms of Jacobi fields. Then we apply these methods to the relation between curvature and topology of Riemannian manifolds in Chapter V, and to the inequalities among geometric invariants and spectral geometry in Chapter VI. On the other hand, since the fields treated in Chapters V and VI are still in rapid progress, we cannot state in detail the front line of current research in this textbook. However, in Appendix 6 we mention some of M. Gromov's ideas, which have been one of the main sources promoting the recent development of Riemannian geometry, and have inspired many excellent young geometers.

On the other hand, we cannot state in detail the applications of dynamical systems, partial differential equations, etc. to Riemannian geometry, e.g., minimal submanifold, harmonic map, heat flow, etc. For these topics the reader may consult, e.g., Hajime Urakawa's book [Ur-2].

I would like to express my gratitude to Professor S. Murakami, who invited me to write this book, and to Mr. S. Hosoki of Shokabo Publishing Company for his kind cooperation.

In concluding the preface, I would like to remember the late Professor Shigeo Sasaki, under whose guidance I began to take an interest in Riemannian geometry. Professor Sasaki was one of the pioneers of modern differential geometry in Japan, and emphasized the importance of studying global problems that are also related to other fields of mathematics. He himself did much pioneering research on Riemannian geometry. He passed away in the summer of 1987, when I began to prepare the present book. During the writing I often wished that he were still alive to advise me, and often recalled his enthusiasm for mathematics and his great personality.

Takashi Sakai
April, 1992

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The author's intent behind this book was to provide to advanced undergraduate and graduate students an introduction to modern Riemannian geometry that could also serve as a reference. The book begins with an explanation of the fundamental notion of Riemannian geometry. The remaining chapters deal with various topics in Riemannian geometry, with the main focus on comparison methods and their applications. A special feature of the book is an appendix, which has been considerably revised and expanded as compared to the original Japanese edition, on the collapsing of Riemannian manifolds and Gromov's convergence theorem.

With special emphasis placed on understandability and readability and with many problems included, the book is suitable as a text for a course in differential geometry for well-prepared and motivated graduate students. It can also serve as a good reference for a practicing mathematician interested in Riemannian geometry.

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