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D. Vassiliev

The Asymptotic Distribution of Eigenvalues of Partial Differential Operators

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The Asymptotic Distribution of Eigenvalues of Partial Differential Operators

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Д. ВАСИЛЬЕВ И Ю. САФАРОВ

АСИМПТОТИЧЕСКОЕ РАСПРЕДЕЛЕНИЕ
СОБСТВЕННЫХ ЗНАЧЕНИЙ
ДИФФЕРЕНЦИАЛЬНЫХ ОПЕРАТОРОВ
В ЧАСТНЫХ ПРОИЗВОДНЫХ

Translated by the authors from an original Russian manuscript

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ABSTRACT. The book studies the eigenvalues of elliptic linear boundary value problems. The principal result of the book is a collection of asymptotic formulae which describe the distribution of eigenvalues with high sequential numbers. The use of these asymptotic formulae is illustrated on standard eigenvalue problems of mechanics and mathematical physics. The book is intended for pure as well as applied mathematicians specialising in partial differential equations. The book is mostly self-contained and provides a basic introduction to all the necessary mathematical concepts and tools, such as microlocal analysis, billiards, symplectic geometry and Tauberian theorems.

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Preface

Spectral asymptotics for partial differential operators have been the subject of extensive research for over a century. It has attracted the attention of many outstanding mathematicians and physicists.

As a characteristic example let us consider the following spectral problem:

$$-\Delta v = \lambda^2 v \quad \text{in} \quad M\,, \qquad v|_{\partial M} = 0\,, \tag{0.0.1}$$

where M is a bounded domain in $\mathbb{R}^3$, and Δ is the Laplace operator. The problem (0.0.1) has nontrivial solutions v only for a discrete set of $\lambda = \lambda_k$, which are called eigenvalues. Let us enumerate the eigenvalues in increasing order: $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$. In the general case the eigenvalues λ_k cannot be evaluated explicitly. Moreover, for large k it is difficult to evaluate them numerically. So it is natural to look for asymptotic formulae for λ_k as $k \to \infty$.

However, for a number of reasons it is traditional in such problems to deal with the matter the other way round, i.e., to study the sequential number k as a function of λ. Namely, let us introduce the counting function $N(\lambda)$ defined as the number of eigenvalues λ_k less than a given λ. Then our asymptotic problem is reformulated as the study of the asymptotic behaviour of $N(\lambda)$ as $\lambda \to +\infty$. The derivation of asymptotic formulae for $N(\lambda)$ is the main subject of this book.

It is well known that for the problem (0.0.1)

$$N(\lambda) = \frac{V}{6\pi^2}\,\lambda^3 + o(\lambda^3)\,, \qquad \lambda \to +\infty\,, \tag{0.0.2}$$

where V is the volume of M. The asymptotic formula (0.0.2) has been known for a long time; it appeared already in the works of Rayleigh. Written in a slightly different form it is known in theoretical physics as the Rayleigh–Jeans law.

Rayleigh [**Ra**] arrived at (0.0.2) by considering the case when the domain M is a cube of side a. Then, solving the problem (0.0.1) by separation of variables one obtains

$$N(\lambda) = \#\{\vec{q} \in \mathbb{N}^3 : |\vec{q}| < R\}\,,$$

where $R = a\lambda/\pi$. In other words, $N(\lambda)$ is the number of integer lattice points in an octant of a ball of radius R. Clearly, for large R we have

$$N(\lambda) \approx \frac{1}{8}\,(4\pi R^3/3) = \frac{a^3}{6\pi^2}\,\lambda^3 = \frac{V}{6\pi^2}\,\lambda^3\,.$$

Now physical arguments suggest that the same formula should hold for a domain of arbitrary shape.

Formula (0.0.2) is remarkable not only for its role in the development of theoretical physics, but also for the fact that Rayleigh made a mistake by writing it without the coefficient $1/8$. This mistake was corrected by J. H. Jeans. As pointed out in [**Ja**], Jeans's only contribution to the "Rayleigh–Jeans" law was the statement: "It seems to me that Lord Rayleigh has introduced an unnecessary factor 8 by counting negative as well as positive values of his integers," [**Je,** p. 98].

The first rigorous proof of (0.0.2) was given by H. Weyl [**We1**]. Later, R. Courant and D. Hilbert included a proof of (0.0.2) in their classical textbook [**CouHilb**], which stimulated the study of asymptotic formulae of this type. The list of mathematicians who have contributed to this field includes S. Agmon, V. M. Babich, P. H. Bérard, M. S. Birman, T. Carleman, Y. Colin de Verdière, J. Duistermaat, B. V. Fedosov, L. Gårding, V. W. Guillemin, L. Hörmander, V. Ya. Ivrii, M. Kac, B. M. Levitan, R. B. Melrose, G. Métivier, Å. Pleijel, R. T. Seeley, M. A. Shubin, M. Z. Solomyak, A. Weinstein, and many others. An extensive bibliographical review can be found in [**RoSoSh**]. Physicists also worked on spectral asymptotics and have made essential contributions. Being less familiar with the physical literature we shall only mention the names of M. V. Berry, P. Debye, and L. Onsager; see also [**BaHilf**] for further references.

The asymptotic formula (0.0.2) is remarkably simple: the asymptotic coefficient is determined only by the volume of the domain and is independent of its shape. Moreover, a similar one-term asymptotic formula has been established in a very general setting, namely, for an elliptic self-adjoint partial differential operator with variable coefficients acting on a manifold subject to reasonably good boundary conditions.

However, this simplicity and high degree of generality indicate the weaknesses of (0.0.2) and its analogues. First, such formulae involve only the most basic geometric characteristics of M: say, the eigenvalues of the problem (0.0.1) for a cube and a long narrow parallelepiped of the same volume are obviously quite different, but (0.0.2) does not feel this difference. Second, one-term asymptotic formulae do not depend on the boundary conditions: say, if we replace in (0.0.1) the Dirichlet boundary condition by the Neumann one the eigenvalues will change substantially, which cannot be noticed from (0.0.2). These deficiencies motivated the search for sharper results.

In 1913 H. Weyl put forward [**We2**] a conjecture concerning the existence of a second asymptotic term. Namely, he predicted that for the problem (0.0.1)

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^3 - \frac{S}{16\pi}\lambda^2 + o(\lambda^2), \qquad \lambda \to +\infty, \tag{0.0.3}$$

where S is the surface area of ∂M. Formula (0.0.3) became known as *Weyl's conjecture*. It was finally justified, under a certain condition on periodic billiard trajectories, by Ivrii [**Iv1**] and Melrose [**Me**] only in 1980. This revived interest to such problems. In particular, in subsequent years Ivrii extended his result on two-term asymptotics to much more general classes of boundary value problems. As our book does not aim to provide a full bibliographic review and reflects the research interests of its authors, we refer only to Ivrii's publications [**Iv2**]–[**Iv4**] where the reader can find further references.

Our contribution to the problem concerns the following aspects.

First, we are interested in deriving two-term asymptotic formulae for higher order differential operators.

Second, we study the case when the condition on periodic billiard trajectories, which guarantees the existence of a classical second term in Weyl's formula, fails.

In this case the second asymptotic term may contain an oscillating function, which depends on the structure of the set of periodic billiard trajectories.

Third, we obtain two-term asymptotic formulae for the spectral function. In this case one has to deal with loops instead of periodic billiard trajectories.

The basic idea which we use for the derivation of spectral asymptotics is due to Levitan [**Ltan**]. It involves the study of the singularities of the corresponding evolutionary problem (say, in the case of (0.0.1) this would be the wave equation), and the subsequent application of Fourier Tauberian theorems. This approach produces the sharpest possible results. Levitan's method was developed by Hörmander, Duistermaat, Guillemin, and Melrose (see [**Hö1**], [**DuiGui**], [**DuiGuiHö**], [**Me**]). The most advanced version of this method is due to Ivrii [**Iv1**]–[**Iv4**]. Our approach, however, is somewhat different from that of Ivrii, even in the case of the Laplace operator.

We tried to make the book self-contained and all our constructions explicit. The main results are collected in Chapter 1. Chapter 2 introduces the reader to the main technical tools; it can be regarded as a brief introduction to microlocal analysis. Chapters 3–5 are devoted to the proofs of our main results. Chapter 6 lists the basic mechanical applications; it is intended mostly for applied mathematicians and does not require a sophisticated mathematical background. The book also has a number of appendices. Some of them can be read separately from the main text, others contain cumbersome proofs. Appendix A was written by A. Holst, and Appendix B by M. Levitin.

We do not aim at achieving the highest possible degree of generality in our book. In particular, we do not discuss

1. Systems; see [**Sa4**], [**Sa5**], [**SaVa1**], [**Va4**], [**Va6**].
2. Piecewise smooth boundaries; see [**Va6**].
3. Highly nonsmooth (fractal) boundaries; see [**FlLtinVa1**], [**FlLtinVa2**], [**FlVa1**], [**FlVa2**], [**LtinVa1**], [**LtinVa2**], [**Va10**].

This book was preceded by survey papers [**GolVa**], [**Sa7**], [**SaVa2**] describing our main results.

We take this opportunity to express our gratitude to our teachers, V. B. Lidskiĭ and M. Z. Solomyak, for guiding us through our first steps in modern analysis and introducing us to the spectral theory of partial differential operators. We would also like to thank our colleagues S. Agmon, E. B. Davies, A. Holst, M. Levitin, Yu. Netrusov, L. Parnovski, A. V. Sobolev, and T. Weidl for their help and useful comments during the preparation of this manuscript. We thank our graduate students W. Nicoll and A. Roth for providing technical support. Last, but not least, we thank S. I. Gelfand for his patience and understanding in waiting all these years for our manuscript.

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Principal Notation

$\dot{}$ (dot)	denotes differentiation with respect to time t
$\hat{}$ (hat)	denotes the Fourier transform
$\widehat{}$ (wide hat)	stands for the extension from (of) our original manifold
$\simeq$	equality of formal Taylor expansions
$\lesssim$	less than or of the order of
$\{\tau\}$	$\in [0,1)$ fractional part of the real number τ
$\{\tau\}_{2\pi}$	$\in [-\pi,\pi)$ residue modulo 2π of the real number τ
$(\tau)_+$	$=\tau$ if $\tau \geqslant 0$, and $=0$ if $\tau < 0$
$\int$	absence of limits of integration implies integration over the whole space
$(M * N)(\lambda)$	$=\int M(\lambda-\mu)N(\mu)\,d\mu$ convolution
$\{f,g\}$	$=\langle f_\xi, g_x\rangle - \langle f_x, g_\xi\rangle$ Poisson brackets
∇	gradient
#	the number of elements in a finite set

$\mathbf{A}$, $\mathbf{A}^+$	1.6.3
$\mathbf{A}(z')$, $\mathbf{A}^+(z')$	5.4.2
$\mathcal{A}$	self-adjoint operator generated by the boundary value problem, 1.1.7
$\overline{\mathcal{A}}$	$=(\mathcal{A}^*)^T$
C_φ	2.1.1, 2.1.2 and Section 2.2
$C^\infty_{\bar{B}}$, C^∞_{B}, $C^\infty_{B_x}$	3.1.1
D_{x_k}	$=-i\partial/\partial x_k$
$\mathcal{D}$	$=C_0^\infty$, Schwartz space of test functions
$\mathcal{D}'$	Schwartz space of distributions (dual of $\mathcal{D}$)
$\dot{\mathcal{D}}'$, $\dot{\mathcal{D}}'_{\bar{B}}$	3.1.1
E_λ	spectral projection of the operator $\mathcal{A}^{1/(2m)}$, 1.8.1
$\mathbf{E}_\nu$, $\mathbf{E}^+_\nu$	spectral projections of the operators $\mathbf{A}$, $\mathbf{A}^+$, 1.6.3
$\mathbf{E}_\nu(z')$, $\mathbf{E}^+_\nu(z')$	spectral projections of the operators $\mathbf{A}(z')$, $\mathbf{A}^+(z')$, 5.4.2
$\mathcal{E}$	$=C^\infty$
$\mathcal{E}'$	space of distributions with compact support (dual of $\mathcal{E}$)
F_+	class of monotone functions, Section B.1
$\mathcal{F}_{\lambda\to t}[f(\lambda)]$	$=\hat{f}(t)=\int e^{-it\lambda}f(\lambda)\,d\lambda$ one-dimensional Fourier transform
$\mathcal{F}^{-1}_{t\to\lambda}[\hat{f}(t)]$	$=f(\lambda)=(2\pi)^{-1}\int e^{it\lambda}\hat{f}(t)\,dt$ inverse one-dimensional Fourier transform
H^s	Sobolev space W_2^s
$\mathcal{I}_{\varphi,a}$	oscillatory integral with phase function φ and amplitude a, 2.1.1, 2.7.1
J_y	1.8.4

Meas	Riemannian $(n-1)$-dimensional volume of the boundary
$\overset{\circ}{M}$	interior of the manifold M
$N(\lambda)$	counting function of the boundary value problem, 1.2.1
$\mathbf{N}^+(\nu)$	counting function of the auxiliary one-dimensional problem, 1.6.3
$\mathbb{N}$	$=\{1,2,\dots\}$ the set of natural numbers
O	(often with indices) various open conic subsets of $T'\overset{\circ}{M}$
O_T	set of T-admissible points, 1.3.3
O_∞	set of admissible points, 1.3.3
$\mathcal{O}$	(often with indices) various open conic subsets of $(T_-,T_+)\times M\times O$
$\mathbf{O}$	conically compact conic subset of O, 2.7.1
$\mathbf{O}^\pm(d)$	2.10.2
P_1	principal symbol of a (pseudo)differential operator P of order $\mathbf{l}$
P_{sub}	subprincipal symbol of a (pseudo)differential operator P
$\mathcal{P}^{\mathrm{d}}$	1.3.2, 1.3.3
$\mathcal{P}^{\mathrm{g}}$	1.3.2, 1.3.3
$\mathbf{Q}(\lambda)$	1.7.2
$\mathbf{Q}(y,\lambda)$	1.8.4
$\mathbf{Q}_P(y,\lambda)$	1.8.4
$R(\nu)$	reflection matrix, Section 1.4
$\mathbf{R}_\mu$, $\mathbf{R}_\mu^+$	resolvents of the operators $\mathbf{A}$, $\mathbf{A}^+$, 1.6.3
$\mathbf{R}_\mu(z')$, $\mathbf{R}_\mu^+(z')$	resolvents of the operators $\mathbf{A}(z')$, $\mathbf{A}^+(z')$, 5.4.2
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^n_+$	$=\{x\in\mathbb{R}^n : x_n\geqslant 0\}$
$S(\nu)$	scattering matrix, 1.6.3
S^l	class of amplitudes, 2.1.1
$S^{-\infty}$	$=\bigcap_{l\in\mathbb{R}} S^l$, or $\bigcap_{l\in\mathbb{R}} S^l(1;1,\varepsilon_1)$ in 2.10.6
$S^l(0,0;1,\varepsilon_1)^\pm$	class of parameter-dependent amplitudes, 2.10.2
$S^{-\infty,\pm}$	$=\bigcap_{l\in\mathbb{R}} S^l(0,0;1,\varepsilon_1)^\pm$
$S^l(1;1,\varepsilon_1)$	class of parameter-dependent amplitudes, 2.10.6
S^*M	cosphere bundle, 1.1.10
$\mathcal{S}(\mathbb{R})$	Schwartz space of rapidly decreasing test functions
$\mathcal{S}'(\mathbb{R})$	Schwartz space of tempered distributions (dual of $\mathcal{S}(\mathbb{R})$)
$\mathbb{S}^k$	k-dimensional sphere (in $\mathbb{R}^{k+1}$)
$T^\pm_\pm(d)$	2.10.2
T_δ	3.4.1
$\mathbf{T}$	period function, 1.7.2
$\mathbf{T}_y$	1.8.4
$T'M$, $T'\overset{\circ}{M}$	cotangent bundles T^*M, $T^*\overset{\circ}{M}$ with the zero section $\{\xi=0\}$ excluded
$T'\partial M$	cotangent bundle $T^*\partial M$ with the zero section $\{\xi'=0\}$ excluded
Tr	trace of an operator
U_y, $U_{y,\lambda}$	1.8.4
$\mathbf{U}(t)$	wave group, Introduction to Chapter 3
$\mathbf{U}_P(t)$	$=\mathbf{U}(t)P$
$\mathbf{U}_{P,Q}(t)$	$=Q^*\mathbf{U}(t)P$
Vol	Riemannian n-dimensional volume
WF	wave front set, 2.1.3
$\arg_0$	a fixed branch of the argument

c_0, c_1	Weyl coefficients for the counting function
$c_0(y)$	first Weyl coefficient for $e(\lambda, y, y)$, 1.8.3
$c_1(x')$	5.4.1
$c_{0;P}(y)$	$= c_{0;P,P}(y)$
$c_{0;P,Q}(y)$	1.8.3
$c_{1;P}(y)$	1.8.3
cone hull	conic hull, Section 2.2
cone supp	conic support, 2.1.1
d	parameter characterizing distance to ∂M, Introduction to Section 2.10
d_φ	Section 2.2 and 2.7.1
d^∞	2.10.5–2.10.7
div	divergence
$đ\xi$	$= (2\pi)^{-n}\, d\xi$
$đ\xi'$	$= (2\pi)^{1-n}\, d\xi'$
$e(\lambda, x, y)$	spectral function (integral kernel of E_λ), 1.8.1
$e_{P,Q}(\lambda, x, y)$	integral kernel of $Q^* E_\lambda P$, 1.8.1
$\mathbf{e}$, $\mathbf{e}^+$	integral kernels of the operators $\mathbf{E}$, $\mathbf{E}^+$, 1.6.3
$\mathbf{f}^+(\mu)$	regularized trace of $\mathbf{R}^+_\mu$, 1.6.3
grad	gradient
$\mathbf{k}(x', \xi')$	Hamiltonian curvature, 1.3.2
λ_k	eigenvalues of the operator $\mathcal{A}^{1/(2m)}$
meas	canonical measure on the cosphere bundle S^*M, 1.1.10
meas_x	canonical measure on the unit cosphere $S^*_x M$, 1.1.10
$\text{meas}_{\partial M}$	measure on $S^*M\|_{\partial M}$, Introduction to Appendix D
$\mathbf{q}$	1.7.2
$\mathbf{q}_y$	1.8.4
$\mathbf{r}_\nu$, $\mathbf{r}^+_\nu$	integral kernels of the operators $\mathbf{R}$, $\mathbf{R}^+$, 1.6.3
shift^+	spectral shift of the auxiliary one-dimensional problem, 1.6.1
sing supp	singular support of a distribution, 2.1.1
$t_\star$	4.1.1
$\widetilde{t}_\star$	4.1.2
t^*_k	moments of reflection
tr	trace of a matrix
$\mathbf{u}(t, x, y)$	Schwartz kernel of the operator $\mathbf{U}(t)$
$\mathbf{u}_P(t, x, y)$	Schwartz kernel of the operator $\mathbf{U}_P(t)$
$\mathbf{u}_{P,Q}(t, x, y)$	Schwartz kernel of the operator $\mathbf{U}_{P,Q}(t)$
v_k	eigenfunctions (half-densities) of the operator $\mathcal{A}^{1/(2m)}$
vol	symplectic volume on T^*M, 1.1.10
vol$'$	symplectic volume on $T^*\partial M$, 1.1.10
(x^*, ξ^*)	Hamiltonian or billiard trajectory, 1.3.1–1.3.3
x^*_η	matrix with elements $(x^*_j)_{\eta_i}$ (j being the number of the row and i that of the column)
$\mathfrak{C}_i$	2.4.1
$\mathfrak{C}'_i$	2.4.1
$\mathfrak{C}^{(l)}$	3.4.3
$\mathfrak{F}^0$	class of phase functions corresponding to pseudodifferential operators, 2.7.5
$\mathfrak{F}_i$	class of standard phase functions, 2.4.1

$\mathfrak{F}'_i$	class of boundary phase functions, Section 2.5
$\mathfrak{F}_i^{\mathrm{bl}}$	class of boundary layer phase functions, 2.6.4
$\mathfrak{L}_{-r}$	2.7.3
$\mathfrak{N}(O)$	set of types of billiard trajectories originating from O, 3.4.2
$\mathfrak{O}_i$	2.4.1
$\mathfrak{O}^{(l)}$	3.4.3
$\hat{\mathfrak{O}}_j$	2.7.2
$\check{\mathfrak{O}}_j$	1.5.1
$\mathfrak{S}$, $\mathfrak{S}_{-r}$	2.7.3
$\mathfrak{S}'$, $\mathfrak{S}'_{-r}$	Section 2.9
$\mathfrak{T}_i(y,\eta)$	2.3.2
$\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$	(full) symbols of Lagrangian distributions
$\mathfrak{a}_k$, $\mathfrak{b}_k$, $\mathfrak{c}_k$	homogeneous terms of (full) symbols
$\mathfrak{f}$	total phase shift, 1.7.2
$\mathfrak{f}_c$	phase shift generated by caustics, 1.7.2
$\mathfrak{f}_r$	phase shift generated by reflections, 1.7.2
$\mathfrak{f}_s$	phase shift generated by the subprincipal symbol, 1.7.2
$\mathfrak{m}$	$= \mathfrak{m}_1 \vert \ldots \vert \mathfrak{m}_{\mathbf{r}}$, 1.3.3
Δ	Laplace operator, Example 1.2.4
Π_T	1.3.1–1.3.3
Π	$= \bigcup_{T>0} \Pi_T$
Π_T^a	1.3.1–1.3.3
Π^a	$= \bigcup_{T>0} \Pi_T^a$
$\Pi_{y,T}$	1.8.2
Π_y	$= \bigcup_{T>0} \Pi_{y,T}$
$\Pi_{y,T}^a$	1.8.2
$\Pi_{y,T,i}$	4.1.2
Π_y^a	$= \bigcup_{T>0} \Pi_{y,T}^a$
$\Pi_{y,T,i}^a$	4.1.2
Φ_y	1.8.4
Φ_{xx}, $\Phi_{\eta\eta}$, $\Phi_{x\eta}$	2.4.1
Φ^t	Hamiltonian or billiard flow (see 1.3.1 and 1.3.2)
Ψ^l, Ψ_0^l	classes of pseudodifferential operators, 2.1.3
α_Γ	Maslov index of the trajectory Γ, 1.5.1
δ_l^k	Kronecker symbol
$\zeta_j^{\mp}$	Section 1.4
ν^{st}	threshold, Section 1.4
ξ_η^*	matrix with elements $(\xi_j^*)_{\eta_i}$ (j being the number of the row and i that of the column)
$\vert\xi\vert_x$	length of the covector $\xi \in T^*M$ (when M is a Riemannian manifold)
$\hat{\rho}$	function of the class $C_0^\infty(\mathbb{R})$, Introduction to Chapter 4
ρ_T, $\hat{\rho}_T$	Introductions to Section 4.2 and Chapter 5
ς	cut-off function in the oscillatory integral, Section 2.2
χ	(often with indices) various cut-off functions
$\chi_\pm$, χ_0	special cut-off functions on $\mathbb{R}_+$, 5.1.2
ω_n	$= \pi^{n/2}/\Gamma(1+n/2)$ the volume of the unit ball in $\mathbb{R}^n$

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