

Translations of  
**MATHEMATICAL  
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Volume 159

**Best Approximation  
by Linear Superpositions  
(Approximate Nomography)**

S. Ya. Khavinson




**American Mathematical Society**

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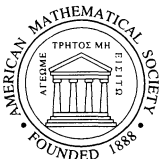
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**American Mathematical Society**  
Providence, Rhode Island

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## НАИЛУЧШЕЕ ПРИБЛИЖЕНИЕ ЛИНЕЙНЫМИ СУПЕРПОЗИЦИЯМИ (АППРОКСИМАТИВНАЯ НОМОГРАФИЯ)

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ABSTRACT. Problems concerning the approximation of functions of several variables by superpositions of functions of fewer variables originated from Kolmogorov's celebrated theorem. This book presents the current state of the theory of approximation by superpositions in various function spaces.

The book is intended for research mathematicians and graduate students working in approximation theory, constructive function theory, and functional analysis.

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