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Volume 160

Linear and Nonlinear Perturbations of the Operator div

V. G. Osmolovskiĭ

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(*Continued in the back of this publication*)

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Linear and Nonlinear Perturbations of the Operator div

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V. G. Osmolovskiĭ

American Mathematical Society
Providence, Rhode Island

В. Г. Осмоловский

ЛИНЕЙНЫЕ И НЕЛИНЕЙНЫЕ ВОЗМУЩЕНИЯ ОПЕРАТОРА div

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ABSTRACT. This book presents the theory of boundary value problems for the operator div and its linear and nonlinear perturbations. Applications to geometry, the calculus of variations, and continuum mechanics are described. The book can be used by research mathematicians and graduate students working in partial differential equations and applications to mathematical physics.

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Contents

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Preface

In various questions of analysis, it is often necessary to describe the set of those solutions of a nonlinear problem that lie in a sufficiently small neighborhood of a known solution. The present book is devoted to problems of this type in the case of a scalar first order differential equation for a vector-valued function. As a typical example, we mention the following problem:

$$\det \dot{y}(x) = 1, \quad x \in \omega \subset \mathbb{R}^m,$$
$$y(x) - x = 0, \quad x \in \Gamma \subset \partial\omega,$$

where $y(x)$ is an m-dimensional vector-valued function and $\dot{y}(x)$ is the matrix of the first derivatives of y. It is required to describe the set of those solutions of this problem that lie in a sufficiently small neighborhood of the known solution $y(x) \equiv x$.

A natural tool for studying such problems is perturbation theory. Applying it, we necessarily linearize the nonlinear equation near the known solution. In the above example, the operator div is a linearization and the operator det can be regarded as a small nonlinear perturbation of the operator div.

For the general nonlinear equation $F(\dot{y}, y, x) = 0$, a linearization is $Lu = C_{ij}u^i_{x_j} + C_i u^i$, a first order operator. Under some restrictions on the coefficients $C_{ij}(x)$ and $C_i(x)$, the operator L can be considered as a compact perturbation of the operator div.

The book discusses three topics.

1. Exposition of the theory of boundary-value problems for the operator L.
2. Exposition of the theory of nonlinear perturbations of the operator L, which allows us to give a local description of the set of all solutions to the boundary-value problem for the equation $F(\dot{y}, y, x) = 0$ for which L can be taken as a linearization.
3. Applications of the results.

Among scalar first order differential operators acting on vector-valued functions, of particular interest is the operator div which, among other properties, expresses the incompressibility condition in hydromechanics. We present linear and nonlinear perturbation theory for the operator div. The results are illustrated by various applications.

The book consists of two chapters and an appendix. In the first chapter, we consider the operator L that acts on n-dimensional vector-valued functions $u(x)$, $x \in \omega \subset \mathbb{R}^m$, by the formula

$$Lu(x) = \operatorname{div}\big(A^*(x)u(x)\big) + \big(a(x), u(x)\big)_n,$$

where $A(x)$ is an $(n \times m)$-matrix-valued function, $a(x)$ is an n-dimensional vector-valued function, ω is a bounded domain, and $(\cdot,\cdot)_n$ denotes the inner product in $\mathbb{R}^n$.

The entries of the matrix $A(x)$, components of the vector $a(x)$, and the boundary of ω are assumed to be sufficiently smooth; moreover, $\det A^*(x)A(x) \neq 0$ for $x \in \overline{\omega}$. In the case $n = m$, $A(x) \equiv I$, $a(x) \equiv 0$, the operator L becomes the operator div. The literature on the operator div is quite extensive. We mention only [**2, 3, 14, 16, 17**]. To study boundary-value problems for the operator div, we use a traditional method based on the identity $\operatorname{div}\operatorname{rot} = 0$. The general operator L is considered as a compact perturbation of the operator div. This allows us to avoid the application of Fredholm complexes reducing the problem to a form which admits the immediate use of the Fredholm alternatives. For the operator L we consider various boundary-valued problems and find necessary and sufficient solvability conditions in explicit form. In particular, the problem

$$Lu = f, \qquad u\big|_{\partial\omega} = \varphi$$

is solvable for all pairs of functions f, φ of a certain smoothness, while for the operator $L = \operatorname{div}$ the well-known consistency condition serves as a solvability criterion. Particular attention has been given to the kernel $N = \{u : Lu = 0\}$ of the operator L. We prove an analog of the Weyl decomposition and show its stability. We emphasize that N is a natural generalization of the space of solenoidal vector fields. To study the operator L, we essentially use the description of the set of solutions to the problem

$$L^*p \equiv -A\nabla p + ap = 0, \qquad p \in C^1(\overline{\omega}).$$

Due to the very simple form of this problem, the description can be obtained without using the technique of overdetermined systems.

Using the results of the first chapter, we can classify the sets of all solutions to various boundary-value problems for the scalar nonlinear equation

$$F[y] = F(\dot{y}, y, x) = 0, \qquad x \in \omega,$$

where a mapping $y(x)$ of $\omega \subset \mathbb{R}^m$ into $\mathbb{R}^n$ with the Jacobi matrix $\dot{y}(x)$ belongs to a sufficiently small neighborhood of the known solution $z(x)$; this can be done provided that the linearization of the operator $F[y]$ on $z(x)$ is given by the operator L whose coefficients satisfy the conditions formulated above. The classification and applications in geometry, the calculus of variations, and incompressible continua are discussed in Chapter 2.

To describe the set of all those solutions to the problem $F[y] = 0$ that lie in a small neighborhood of the known solution $y = z$, it is natural to use the implicit function theorem or the implicit function theorem together with the Lyapunov–Schmidt splitting procedure. We can manage with the implicit function theorem alone if the problem $L^*p = 0$ has only the zero solution; otherwise, it is necessary to use, in addition, the Lyapunov–Schmidt splitting procedure. If $L^*p = 0$ has only the zero solution, then the set of solutions has the structure of a surface in a small neighborhood of the point z in some function space; moreover, N is the tangent space to this surface at the point z. Otherwise, there are variants depending on properties of the bifurcation equation and the boundary conditions imposed on $y(x)$. In particular, two extreme cases can occur. In the first case, the set of solutions looks like a piece of surface in some function space and N is the tangent space to this surface at the point z. In the second case, $y(x) \equiv z(x)$ is an isolated solution. The boundary conditions or some other additional conditions imposed on $y(x)$ are called rigid if the set of solutions to the problem $F[y] = 0$ consists of an isolated solution $y(x) = z(x)$. This term is introduced by analogy with the terminology in the problem about isometric deformations (cf. the historical survey in [**1**]). Particular emphasis is placed on studying the rigidity of various conditions.

As examples of $F[y]$ we consider operators such that $F(\dot{y}, y, x)$ coincides with an invariant of the metric tensor $g_y = \dot{y}^*\dot{y}$ of the mapping $y(x)$. The most popular such problem is that of describing the set of all those solutions to the equation $\det g_y - 1 = 0$, $x \in \omega$, $y(x) - x = 0$, $x \in \partial\omega$, that lie in a sufficiently small neighborhood of the solution $z(x) \equiv x$. Such a problem often occurs in continuum mechanics.

Themain part of the second chapter is devoted to variational problems for the functional with constraint

$$J[y] = \int\limits_{\omega} H(\dot{y}, y, x)\,dx + \int\limits_{\partial\omega} h(y, x)\,dS, \qquad F(\dot{y}, y, x) = 0.$$

For the existence conditions for a global minimum of this functional we refer to [**4, 22**]. We will touch only on necessary conditions for an extremum. A general approach to the search for necessary conditions for an extremum in problems with constraints is discussed, e.g., in [**13, 20**]. Information about the structure of the set of solutions to the problem $F[y] = 0$ in a neighborhood of the extremal point z allows us, following the general approach, to derive the Euler–Lagrange equation, compute the second variation of the functional, obtain the Legendre–Hadamard conditions, and justify the method of Lagrange multipliers. We note that under the incompressibility condition, Lagrange multipliers were applied for solving variational problems in [**10, 11**]. To obtain sufficient conditions for a local minimum over sufficiently small smooth perturbations, we use the multiplicative inequalities (cf. the Appendix).

For statements of variational problems in continuum mechanics we refer to [**19**]. In particular, if $J[y]$ is the energy functional of deformations of an elastic body, then for $F[y] = \det \dot{y}(x) - 1$ the problem of minimizing the functional $J[y]$ with the constraint $F[y] = 0$ is a problem about the equilibrium state of an elastic incompressible medium.

As for prerequisites, the reader is expected to be familiar with basic notions and facts from functional analysis and the theory of elliptic differential equations (cf., e.g., [**5, 7, 8, 12, 18, 20, 21, 23, 35–37**]). For the convenience of the reader, in Chapter 2 we present the necessary facts from the differential calculus in normed spaces and from bifurcation theory, following [**5, 12, 23, 35**]. Necessary information about function spaces, extension theorems, and multiplicative inequalities is contained in the Appendix.

The main results presented in the book are based on the author's papers [**24–34**]. We note that the choice of the material included in the book was conditioned by the author's interests. It is not our purpose to present all current results on boundary-value problems for scalar first order equations. In particular, we do not discuss such questions as nonstationary problems, quasilinear first order equations, equations with nonsmooth coefficients, and domains with nonsmooth boundaries.

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Notation

Euclidean space

The summation convention over repeated indices is sometimes assumed.

$\mathbb{R}^m$ is m-dimensional Euclidean space, with inner product $(\cdot,\cdot)_m$ and modulus (absolute value) $|\cdot|$.

$\mathbb{R}^m_+ = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : x_m \geqslant 0\}$.

$B_\rho(x_0) = \{x \in \mathbb{R}^m : |x - x_0| < \rho\}$.

Domains in Euclidean space

ω is a bounded domain in $\mathbb{R}^m$ with closure $\overline{\omega}$ and boundary $\partial\omega$.

Γ is an open subset of $\partial\omega$ with closure $\overline{\Gamma}$.

$S = \partial\omega/\overline{\Gamma}$.

$\rho(x)$ is the distance from x to $\partial\omega$.

$d(x)$ is the oriented distance from x to $\partial\omega$ (cf. the Appendix).

$\nu(x)$ is the unit outward normal to $\partial\omega$.

h_ν is the normal component of a vector-valued function h defined on $\partial\omega$.

Function spaces

$L_q(\omega)$, $q \in [1,\infty)$, is the space of q-integrable functions in ω, equipped with norm $\|\cdot\|_q$.

$W^l_q(\omega)$, $l \geqslant 0$, $q \in [1,\infty)$ (the Sobolev–Slobodetskiĭ space), is the space of functions that, together with their Sobolev derivatives of order up to l (fractional l is possible), are q-integrable in w, equipped with the norm $\|\cdot\|_{l,q}$.

$C(\overline{\omega})$ is the space of continuous function defined in $\overline{\omega}$, with the norm $|\cdot|_0$.

$C^k(\overline{\omega})$ is the space of k-times continuously differentiable functions defined in $\overline{\omega}$, with the norm $|\cdot|_k$.

$C^{k,\varepsilon}(\overline{\omega})$ (the Hölder space) is the space of functions from $C^k(\overline{\omega})$ whose kth derivatives satisfy the Hölder condition with exponent $\varepsilon \in [0,1]$, equipped with the norm $|\cdot|_{k,\varepsilon}$.

We introduce the spaces of functions defined on $\partial\omega$, Γ, or S in a smiilar way. We preserve the above notation for the norms in these spaces. If it is not clear from the context over which set the norm is taken, we indicate the set in the notation (e.g., $\|\cdot\|(\Gamma)$). For spaces of vector-valued functions we indicate the space to which the values of the functions belong in the notation (e.g., $C^{k,\varepsilon}(\overline{\omega},\mathbb{R}^n)$, $L_q(\partial\omega,\mathbb{R}^n)$). The norms in the spaces of vector-valued functions are denoted as above.

Necessary information about function spaces used in the book can be found in the Appendix. Throughout the book only real numbers are used.

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