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Volume 161

# Discreteness and Continuity in Problems of Chaotic Dynamics

Michael Blank

American Mathematical Society

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10.1090/mmono/161

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# Discreteness and Continuity in Problems of Chaotic Dynamics

Michael Blank

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Providence, Rhode Island

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ДИСКРЕТНОСТЬ И НЕПРЕРЫВНОСТЬ
В ЗАДАЧАХ ХАОТИЧЕСКОЙ ДИНАМИКИ

Translated by the author from an original Russian manuscript.

1991 *Mathematics Subject Classification.* Primary 28D05, 58F11, 58F13;
Secondary 58F10, 58F14, 58F15, 58F30.

ABSTRACT. The book is devoted to the study of ergodic properties of so called chaotic dynamical systems. One of the central topics is the interplay between deterministic and quasi-stochastic behavior in chaotic dynamics, as well as between properties of continuous dynamical systems and those of their discrete approximations. Using simple examples, the author describes the main phenomena known in chaotic dynamical systems, studying such topics as the operator approach in chaotic dynamics, stochastic stability and the so called coupled systems. Two last chapters are devoted to problems of numerical modelling of chaotic dynamics.

The book can be used by researchers and graduate students working in ergodic theory, dynamical systems, and applications.

The author was supported in part by RFFI and INTAS grants.

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**Library of Congress Cataloging-in-Publication Data**

Blank, Michael (Michael L.)
Discreteness and continuity in problems of chaotic dynamics / Michael Blank.
p. cm. — (Translations of mathematical monographs ; v. 161)
"Translated by the author from an original Russian manuscript"—T.p. verso.
Includes bibliographical references (p. – ) and index.
ISBN 0-8218-0370-0 (alk. paper)
1. Ergodic theory. 2. Differentiable dynamical systems. 3. Chaotic behavior in systems. I. Title. II. Series.
QA611.5.B57 1997
515′.42—dc21 97-551
CIP

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To Nadja, Inna and Zhenja,
Mella and Leva

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# Contents

# Preface

In the last thirty years there has been an increasing interest in so-called "chaotic dynamical systems." In spite of a great deal of early pioneering work in the field by Poincaré, Birkhoff, Krylov, and others, it was not until the late 1950s and 1960s that the field really gathered momentum. In this period the main results and ideas were those of Kolmogorov, Arnold, and Moser for Hamiltonian systems, and of Hénon and Smale for more general diffeomorphisms, and also, as the power of numerical computation increased, those of Lorenz for chaos in deterministic dissipative systems. It is worth mentioning the interplay between deterministic chaos and random noise and the development of methods to distinguish them in experimental data. Notice that the type of the time and space variables – discrete or continuous – plays an important role here. Other subjects studied in recent years include chaotic behavior in quantum systems and the features that characterize their behavior compared to classical systems. Surprisingly, this question brings up once more the problem of the discreteness of the phase space in numerical simulations, which we discuss in detail in this book.

An important development in the ergodic theory of dynamical systems in the 1930s was the proof that certain special flows are ergodic. The main example here is geodesic flow on a surface of constant negative curvature. A recognition of the basic instability of geodesics in this example goes back to Hadamard (1898). Eventually, proofs of the ergodicity for more general flows were obtained by Sinai for the hard sphere gas and by Anosov for geodesic flows on a large class of smooth manifolds.

One of the fundamental problems of chaotic dynamics is the question of how can a physical system described by deterministic evolution equations exhibit a quasi-stochastic behavior. The answer to this question is that a chaotic map is characterized by a very sensitive dependence on initial conditions. This means that a small uncertainty of the initial condition grows very fast (typically, exponentially fast) with time. For this reason, if the initial value is known only up to a certain precision, then the behavior of the system is unpredictable even after a relatively short time interval. In a sense this can be considered as a loss of information about the initial data.

As we mentioned, the evolution of a dynamical system is defined by a deterministic evolution equation. In this book we restrict ourselves to discrete time dynamical systems, or maps. This means that the evolution is described by a recurrence relation: $x_{n+1} = f(x_n)$, where $x_n, x_{n+1} \in X \subset \mathbb{R}^d$ and $f : X \to \mathbb{R}^d$. The set $X$ is called the phase space of the dynamical system $(f, X)$, and the index $n$ may be considered as a discrete time coordinate. This relation determines the time evolution $\bar{x} := \{x_0, x_1, \ldots, x_n, \ldots\}$ of an initial state $x_0$, obtained by iteration of the map $f$. The sequence $\bar{x}$ is called a trajectory. Notice that the trajectory $\bar{x}$ is uniquely defined by the initial point $x_0 \in X$. However, typically a long trajectory can be calculated only using a computer, and cannot be described by analytical means. We shall show in Chapter 5 that the effects of arbitrarily small round-offs, inevitable in computer simulations, can change the behavior of the system in a very drastic way.

Due to the unpredictability, it is more natural to describe the behavior of the system as a whole by statistical means, rather then the behavior of individual trajectories. Notice that this is exactly the case with random processes. We shall show that even the statistical properties of chaotic dynamics may depend very sensitively on arbitrarily small perturbations (noise). We shall consider various types of these perturbations, starting from deterministic (perturbations due to weak coupling, round-off errors, quasi-stochastic perturbations), and finishing with pure random ones (random Markov additive noise). It will be shown that both stability and instability can occur under the action of perturbations. Stability here means that the statistical properties of the perturbed systems converge to that of the genuine ones when the perturbation tends to zero, while instability means the opposite type of behavior. Notice that the chaoticity actually corresponds to the situation when typical trajectories are "filling" a large region of the phase space. From this point of view the instability we are discussing leads to the fact that the trajectories of the perturbed system remain confined to a small region, whose volume vanishes when the perturbation tends to zero. Thus some sort of a "localization" phenomenon takes place. On the other hand, this situation can be considered also as a stabilization under the action of some sort of noise of an unstable invariant set (a fixed point, for example). It is worth noticing that in spite of the discontinuous character of the perturbations we are considering here, some very deep results about smooth and even analytical perturbations of dynamical systems may still be generalized to this case. For example, in Chapter 5 we elaborate a generalization of the famous Kolmogorov–Arnold–Moser theory for rotation-like maps under the action of round-off errors.

Eventually the localization phenomenon breaks the phase space into discrete invariant components. This can happen also due to the fact that the perturbation is discrete (as in the case of round-off errors), or the map that we consider is only an approximation of the dynamics on the discrete phase space. An example of this type is the quantum description of nature, which claims that the phase space needs to be discrete, rather than continuous. The interplay between the discreteness and continuity on the one hand, and stability and localization on the other hand, are the main subjects of this book.

It should be emphasized that the book does not intend in any way to be complete or exhaustive. The subject is in constant and active progress, and a wealth of results, both numerical and analytical, is accumulating rapidly. Our main aim is to acquaint the reader with some recently discovered and (at first sight) rather unusual properties of chaotic dynamical systems and their small perturbations. We also would like to mention that we consider the study of piecewise expanding maps (we shall deal mainly with this class of dynamical systems in our examples) as much more than just a mathematical curiosity. One reason to study these maps is their simplicity. We shall see that the statistical description of these maps is fairly well understood. On the other hand, practically all phenomena known for chaotic dynamical systems can be found in the analysis of piecewise expanding maps. Another advantage of this class is that there exists a direct operator description (see Chapter 1) of the dynamics of measures and densities in this case. We shall use this operator approach very much in our studies. It it worth noting that this class is practically the only class of chaotic systems with singularities for which we have a good control over their statistical properties. A number of results presented in the monograth are published here for the first time, while some others are described in a much more general way than was done in the original journal publications.

The book is divided into six chapters. In the introductory Chapter 1 we present a short and elementary discussion of some basic notions of chaotic dynamics. Of course, this part is not meant as a complete introduction to this field: there are many excellent and more detailed textbooks (see, for example, [**31**], [**73**]). Rather we restrict ourselves to those subjects that we shall deal with later.

In Chapter 2, "Operator approach in chaotic dynamics", we describe in detail the operator approach in a sufficiently abstract form and apply it to the study of piecewise expanding maps. The approach is based on the analysis of spectral properties of the so-called Perron–Frobenius operator of a dynamical system, which determines the dynamics of densities under the action of the map. In this chapter we also introduce and analyze in detail the space of functions of generalized bounded variation, which we use very often throughout the book.

Chapter 3, "Random perturbations of dynamical systems", analyzes the problem of stochastic stability of chaotic dynamical systems. In Chapter 4 we deal with so-called "extended" or "coupled" systems (coupled map lattices). We show that the effect of weak coupling is very similar to the effect of pure random noise. The main results of these chapters are sufficient conditions for the convergence of statistical properties of randomly perturbed systems and coupled map lattices to those of the original systems. On the other hand, conditions under which the localization phenomenon takes place under both random perturbations and weak coupling are also considered.

Finally, in Chapters 5 and 6 we discuss the problem of numerical modeling of chaotic dynamics. In the first of these two chapters we describe the influence of arbitrarily fine phase space discretizations (of round-off errors type) to various dynamical systems. Once again the statistical description (in a sense that we consider the statistics along all perturbations from the given family) makes it possible for us to answer a lot of interesting questions. In the last chapter we discuss mathematical foundations of several numerical methods and prove their convergence for some sufficiently broad classes of chaotic maps.

This book took a long time to write, and I gratefully acknowledge research support from the French Ministry of Higher Education and the kind hospitality of Observatoire de la Côte D'Azur (France), where a part of this work was done. I also acknowledge the support by the Russian Foundation of Fundamental Research. It is a pleasure to thank L. Bunimovich, N. Chernov, R. Dobrushin, U. Frisch, B. Gurevich, M. Hénon, G. Keller, Yu. Kifer, V. Oseledets and Ya. Sinai for stimulating discussions and helpful comments and suggestions.

Michael L. Blank

Erlangen
November, 1996

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