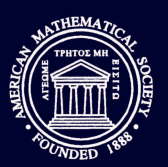
Translations of

MATHEMATICAL MONOGRAPHS

Volume 167

Mathematics of Fractals

Masaya Yamaguti
Masayoshi Hata
Jun Kigami



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Mathematics of Fractals

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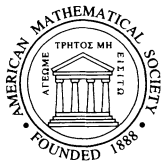

American Mathematical Society
Providence, Rhode Island

フラクタルの数理

(FURAKUTARU NO SŪRI)

(Mathematics of Fractals)

by Masaya Yamaguti, Masayoshi Hata, and Jun Kigami

Originally published in Japanese by Iwanami Shoten, Publishers, Tokyo, 1993

Translated from the Japanese by Kiki Hudson

1991 *Mathematics Subject Classification.* Primary 58F13; Secondary 70K50.

ABSTRACT. This book aims at providing a handy explanation of the notions behind the self-similar sets called "fractals" and "chaotic dynamical systems," intended for any level of readership at or above the undergraduate level, including non-mathematicians. We emphasize the beautiful relationship between fractal functions (such as Weierstrass's) and chaotic dynamical systems; these nowhere-differentiable functions are generating functions of chaotic dynamical systems. We can show that these functions are in a sense unique solutions of certain boundary problems. We can generalize to multiple dimensions. The last chapter of this book treats harmonic functions on fractal sets.

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Preface

Studies of fractals in Japan began in the 1980's. In Chapters One and Two Masayoshi Hata discusses Mandelbrot's fractals (or definite fractals), using his own work on self-similarities. Masaya Yamaguchi explains, in Chapter Three, one method of computing fractal functions in place of Newtonian differentiation, together with wavelet analysis. This is a joint work with Masayoshi Hata. In the final chapter Jun Kigami explains a tentative development of fractal analysis.

Although the reader may find much of the material covered in Chapters One and Two in textbooks elsewhere, the way we explained them is original, reflecting our research philosophy. The reader will not easily find what we offer in Chapters Three and Four anywhere else, for the contents of these chapters, both materials and methods, are our own.

We are very happy to have an opportunity to ask a wider audience to judge our book. We are deeply indebted to the members of the editorial board of this lecture series and the editors at the Iwanami Shoten Publishing Company.

Masaya Yamaguti
Masayoshi Hata
Jun Kigami

January, 1993

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Preface to the English Translation

It is a great honor for the authors that their book "Mathematics of Fractals" was selected for translation into English by the American Mathematical Society. We hope that this will help a new field of mathematics to be accepted by a broader audience.

This book aims at providing a handy explanation of the notions behind the self-similar sets called "fractals" and "chaotic dynamical systems," intended for any level of readership at or above the undergraduate level, including non-mathematicians. The novel contributions presented here include the consideration of fractal functions as unique solutions of some boundary-value problems. We emphasize the beautiful relationship between some fractal functions (such as Weierstrass's) and chaotic dynamical systems; these nowhere-differentiable functions are generating functions of some chaotic dynamical systems. We can show that these functions are in some sense unique solutions of some boundary value problems. We can generalize to multiple dimensions. The last chapter of this book treats harmonic functions on some fractal sets.

After the publication of the original version of this book in 1993, the authors have moved to different fields: M. Hata to number theory, J. Kigami to analysis on fractal sets, and M. Yamaguti to the discretization of ordinary differential equations.

We take the opportunity here to thank the translator, K. Hudson, who helped us to make some improvements from the original Japanese-language version.

Masaya Yamaguti
Masayoshi Hata
Jun Kigami

February 1997

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Recommended Reading

[1] Yamaguti, M., *An Introduction to Chaos and Fractals*, a text for the University of the Air, The Society for the Promotion of the University of the Air, 1992.

When we wrote *Mathematics of Fractals* in Japanese we had this elementary textbook in mind to complement the first three chapters. Unfortunately the book has not been translated into English, and we cannot think of anything written in English comparable to it.

Chapters One and Two.

[2] Falconer, K. J., *The Geometry of Fractal Sets*, Cambridge Univ. Press, 1985.

This is the best book with which to pursue further study of the Hausdorff measure in geometric measure theory. It has a compact survey of classical results as well, together with an extensive bibliography.

[3] Falconer, K. J., *Fractal Geometry – Mathematical Foundations and Applications*, Wiley, 1990.

Falconer presents various results concerning non-integer dimensions, at the same time putting an emphasis on applications. The book has a rich collection of topics in dynamical systems, including complex cases and random fractals.

[4] *Fractal Geometry and Analysis*, (Edited by Bélair, J. and Dubuc, S.), Proc. of NATO ASI Series C Vol. 346, Kluwer Academic Publishers, 1991.

This proceedings (of a meeting held in Montreal in 1989) contains the invited talks, including the most recent results on subjects in measure theory, dimension theory, structures of fractals, complex dynamical systems, multi-fractals, *etc.*

Chapter Three.

[5] Nishida, T., Mimura, M., Fujii, H., eds., *Patterns and Waves, Studies in Mathematics and its Application*, Kinokuniya & North-Holland, 1986: Uchiki, S., Chaotic Phenomena and Fractal Object, 221–258. Hata, M., Fractals in Mathematics, 259–278.

[6] Hata, M., Yamaguti, M., Weierstrass's function and chaos, Hokkaido J. Math., 12(1983), 333–343.

[7] Meyer, Y., translated by Ryan, R. D., *Wavelets: Algorithms and Applications*, SIAM, 1993.

[8] Sekiguchi, T., Shiota, Y., Hausdorff dimension of graphs of some Rademacher series, Japan J. Appl. Math., 7(1990), 121–129.

[9] Sekiguchi, T., Shiota, Y., A generalization of Hata–Yamaguti's results on the Takagi function, Japan J. Indust. Appl. Math., 8(1991), 203–219.

Chapter Four.

The fractal analysis we studied in Chapter Four is still at its early developing stage and there is no comprehensive reference in the literature (to the authors' knowledge). In the following the reader will find discussion of the current research activities and their future development in this field.

[10] Kigami, J., Laplacians on self-similar sets – Analysis on fractals, Amer. Math. Soc. Transl. (2), Vol. 161(1994), 75–93.

The reader wishing to study the subject further might consult the references in [10].

[11] Kigami, J., Laplacians on self-similar sets and their spectral distributions, in *Fractal Geometry and Stochastics*, eds. Bandt *et al.*, **Progress in Probability 37**, Birkhauser, 1995, 221–238.

There are many other interesting articles in *Fractal Geometry and Stochastics.*

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