

Translations of
**MATHEMATICAL
MONOGRAPHS**

Volume 169

**Ordinary Differential
Equations with
Constant Coefficient**

S. K. Godunov



American Mathematical Society

Selected Titles in This Series

- 169 **S. K. Godunov**, Ordinary differential equations with constant coefficient, 1997
- 168 **Junjiro Noguchi**, Introduction to complex analysis, 1997
- 167 **Masaya Yamaguti, Masayoshi Hata, and Jun Kigami**, Mathematics of fractals, 1997
- 166 **Kenji Ueno**, An introduction to algebraic geometry, 1997
- 165 **V. V. Ishkhanov, B. B. Lur'e, and D. K. Faddeev**, The embedding problem in Galois theory, 1997
- 164 **E. I. Gordon**, Nonstandard methods in commutative harmonic analysis, 1997
- 163 **A. Ya. Dorogovtsev, D. S. Silvestrov, A. V. Skorokhod, and M. I. Yadrenko**, Probability theory: Collection of problems, 1997
- 162 **M. V. Boldin, G. I. Simonova, and Yu. N. Tyurin**, Sign-based methods in linear statistical models, 1997
- 161 **Michael Blank**, Discreteness and continuity in problems of chaotic dynamics, 1997
- 160 **V. G. Osmolovskii**, Linear and nonlinear perturbations of the operator div, 1997
- 159 **S. Ya. Khavinson**, Best approximation by linear superpositions (approximate nomography), 1997
- 158 **Hideki Omori**, Infinite-dimensional Lie groups, 1997
- 157 **V. B. Kolmanovskii and L. E. Shaikhet**, Control of systems with aftereffect, 1996
- 156 **V. N. Shevchenko**, Qualitative topics in integer linear programming, 1997
- 155 **Yu. Safarov and D. Vassiliev**, The asymptotic distribution of eigenvalues of partial differential operators, 1997
- 154 **V. V. Prasolov and A. B. Sossinsky**, Knots, links, braids and 3-manifolds. An introduction to the new invariants in low-dimensional topology, 1997
- 153 **S. Kh. Aranson, G. R. Belitsky, and E. V. Zhuzhoma**, Introduction to the qualitative theory of dynamical systems on surfaces, 1996
- 152 **R. S. Ismagilov**, Representations of infinite-dimensional groups, 1996
- 151 **S. Yu. Slavyanov**, Asymptotic solutions of the one-dimensional Schrödinger equation, 1996
- 150 **B. Ya. Levin**, Lectures on entire functions, 1996
- 149 **Takashi Sakai**, Riemannian geometry, 1996
- 148 **Vladimir I. Piterbarg**, Asymptotic methods in the theory of Gaussian processes and fields, 1996
- 147 **S. G. Gindikin and L. R. Volevich**, Mixed problem for partial differential equations with quasihomogeneous principal part, 1996
- 146 **L. Ya. Adrianova**, Introduction to linear systems of differential equations, 1995
- 145 **A. N. Andrianov and V. G. Zhuravlev**, Modular forms and Hecke operators, 1995
- 144 **O. V. Troshkin**, Nontraditional methods in mathematical hydrodynamics, 1995
- 143 **V. A. Malyshev and R. A. Minlos**, Linear infinite-particle operators, 1995
- 142 **N. V. Krylov**, Introduction to the theory of diffusion processes, 1995
- 141 **A. A. Davydov**, Qualitative theory of control systems, 1994
- 140 **Aizik I. Volpert, Vitaly A. Volpert, and Vladimir A. Volpert**, Traveling wave solutions of parabolic systems, 1994
- 139 **I. V. Skrypnik**, Methods for analysis of nonlinear elliptic boundary value problems, 1994
- 138 **Yu. P. Razmyslov**, Identities of algebras and their representations, 1994
- 137 **F. I. Karpelevich and A. Ya. Kreinin**, Heavy traffic limits for multiphase queues, 1994
- 136 **Masayoshi Miyanishi**, Algebraic geometry, 1994
- 135 **Masaru Takeuchi**, Modern spherical functions, 1994
- 134 **V. V. Prasolov**, Problems and theorems in linear algebra, 1994
- 133 **P. I. Naumkin and I. A. Shishmarev**, Nonlinear nonlocal equations in the theory of waves, 1994
- 132 **Hajime Urakawa**, Calculus of variations and harmonic maps, 1993
- 131 **V. V. Sharko**, Functions on manifolds: Algebraic and topological aspects, 1993

(Continued in the back of this publication)

This page intentionally left blank



Ordinary Differential
Equations with
Constant Coefficient

This page intentionally left blank

Translations of
**MATHEMATICAL
MONOGRAPHS**

Volume 169

**Ordinary Differential
Equations with
Constant Coefficient**

S. K. Godunov



American Mathematical Society
Providence, Rhode Island

EDITORIAL COMMITTEE

AMS Subcommittee

Robert D. MacPherson

Grigorii A. Margulis

James D. Stasheff (Chair)

ASL Subcommittee Steffen Lempert (Chair)

IMS Subcommittee Mark I. Freidlin (Chair)

С. К. ГОДУНОВ

ОБЫКНОВЕННЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ С ПОСТОЯННЫМИ КОЭФФИЦИЕНТАМИ

ИЗ-ВО НОВОСИБИРСКОГО
УН-ТА, 1994

Translated from the Russian by Tamara Rozhkovskaya
with the participation of Scientific Books (RIMIBE NSU),
Novosibirsk, Russia

1991 *Mathematics Subject Classification*. Primary 34–01;
Secondary 34A30, 34Bxx, 34Dxx, 49J15, 49K15.

ABSTRACT. This book presents the theory of ordinary differential equations with constant coefficients. The exposition is based on the matrix calculus. Boundary-value problems, Green matrices, the Lopatinskiĭ condition, and the Lyapunov stability are considered. Some qualitative aspects connected with the use of computers for the analysis and solution of boundary-value problems are discussed. Control problems reduced to systems of ordinary equations with constant coefficients are also discussed. The book can be used by researchers and students working in the theory of ordinary differential equations, in particular in its computational aspects.

Library of Congress Cataloging-in-Publication Data

Godunov, S. K. (Sergeĭ Konstantinovich)

[Obyknovennye differentsial'nye uravneniia s postoiannymi koeffitsientami. English]

Ordinary differential equations with constant coefficient / S. K. Godunov ; [translated from the Russian by Tamara Rozhkovskaya with the participation of Scientific Books (RIMIBE NSU), Novosibirsk, Russia].

p. cm. — (Translations of mathematical monographs, ISSN 0065-9282 ; v. 169)

“This book is a translation of the published first volume [Novosibirsk State University, 1994] and a chapter (Chapter 4 in this book) from the unpublished second volume in Russian”—Pref.

Includes bibliographical references.

ISBN 0-8218-0656-4 (hardcover : alk. paper)

1. Differential equations. I. Title. II. Series.

QA372.G56713 1997

515'.352—dc21

97-20182

CIP

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication (including abstracts) is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Assistant to the Publisher, American Mathematical Society, P. O. Box 6248, Providence, Rhode Island 02940-6248. Requests can also be made by e-mail to reprint-permission@ams.org.

© 1997 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights
except those granted to the United States Government.

Printed in the United States of America.

⊗ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

Visit the AMS homepage at URL: <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 02 01 00 99 98 97

Contents

Preface	ix
Chapter 1. Matrix Exponentials, Green Matrices, and the Lopatinskii Condition	1
§1. Linear systems of equations with constant coefficients. The uniqueness and existence of a solution to the Cauchy problem for homogeneous equations	1
§2. Fundamental matrices and matrix exponentials	9
§3. Estimates for the matrix exponential. Polynomial representation of the matrix exponential	14
§4. Fundamental systems of solutions to a linear equation of higher order	21
§5. The continuation of the study of fundamental systems of solutions to a linear equation of higher order	32
§6. Computation of the matrix exponential by means of reducing matrices to the Jordan form	43
§7. The definition of a Green matrix. Existence and uniqueness	49
§8. The polynomial representation of the Green matrix. Green functions	53
§9. Nonhomogeneous linear equations	62
§10. The representation of the matrix exponential and of the Green matrix by contour integrals	76
§11. Boundary-value problem on a segment	82
§12. The closeness estimates for solutions to the boundary-value problems with close coefficients and close right-hand sides	90
§13. The Lopatinskii condition	96
§14. The Green matrices for boundary-value problems on a half-line	105
Chapter 2. Quadratic Lyapunov Functions	111
§1. Sufficient conditions for the global existence of a solution to a vector differential equation	111
§2. The Lyapunov stability	118
§3. The matrix Lyapunov equation	123
§4. The Lyapunov functions	128
§5. The theorems on stability from the first approximation	135
§6. The Hermite theorem	141
§7. The Routh–Hurwitz criterion for stability	150

Chapter 3. Qualitative Properties of Problems and Algorithmic Aspects	159
§1. The computation of the matrix exponential and solution of the Cauchy problem	159
§2. Computational difficulties in the Hurwitz problem and their solutions	165
§3. Solving of boundary-value problems by the orthogonal sweep method	174
§4. The two-way sweep for the computation of the Green matrices	186
§5. Integral and local estimates for the Green matrix	191
§6. Computation of the Green matrix on an infinite line and the determination of the dichotomy parameter	197
Chapter 4. Linear Control Systems	209
§1. Controllability and observability	209
§2. The simplest variational control problem	214
§3. Stabilizability	231
§4. Variational approach to the construction of a stabilizing control	237
§5. Hamiltonian systems of equations and their use in the study of stabilizability	242
§6. Further study of variational problems on a half-line. The notion of detectability	250
§7. The Lur'e–Riccati matrix equation	263
§8. The use of variational problems in the study of control systems	271
References	279
Index	281

Preface

This book presents the theory of ordinary differential equations with constant coefficients and follows lectures given by the author at Novosibirsk State University. The first version of the book was based on the ideas in the article [14] which develops the Gelfand–Shilov theorem [6] on the polynomial representation of a matrix exponential. The course was constantly improved by V. I. Kostin, V. M. Gordienko, G. V. Demidenko, S. I. Fadeev, E. V. Zolotareva, and G. A. Chumakov. Remarks by Yu. F. Borisov were also very useful.

Chapters 1 and 2 can be regarded as an extension of the books by the author [8, 9] published by Novosibirsk State University in 1982–1983. It turned out that the exposition of material suggested there is convenient for practical study of ordinary differential equations using computers. Therefore Chapter 3 is devoted to algorithmic and qualitative questions and contains computational procedures for solving boundary-value problems. In particular, the orthogonal sweep method is described in detail. Based on this method, some algorithms that allow us to clarify the dichotomy of the spectrum of the matrix of coefficients are also explained here. S. I. Fadeev provided substantial help in preparing this chapter.

Chapter 4 deals with stationary optimal control systems that are described by systems of differential equations with constant coefficients. In this chapter, the notions of controllability, observability, and stabilizability are analyzed. Some results of the analysis were published in [3, 10, 11], where some questions on the matrix Lur'e–Riccati equations were studied.

This book is a translation of the manuscript “Ordinary differential equations”, submitted for publication to the Novosibirsk State University. The manuscript consists of two volumes. The publication of both volumes was approved by Novosibirsk State University, and they were scheduled to be published in 1989. However, Novosibirsk State University was able to publish only the first volume (Chapters 1–3), and only in 1994. The second volume remains unpublished.

This book is a translation of the published first volume and a chapter (Chapter 4 in this book) from the unpublished second volume in Russian. In the English version of the book some misprints and inaccuracies are corrected. Some style changes are also made in order to facilitate the reader's understanding.

S. K. Godunov
Novosibirsk, March 1997

This page intentionally left blank

This page intentionally left blank

References

1. A. Ya. Bulgakov, *An effectively calculable parameter for the stability property of a system of linear differential equations with constant coefficients*, *Sibirsk. Mat. Zh.* **21** (1980), no. 3, 32–41; English transl., *Siberian Math. J.* **21** (1980), 339–346.
2. ———, *The basis of guaranteed accuracy in the problem of separation of invariant subspaces for non-self-adjoint matrices*, *Trudy Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR* **15** (1989), 12–92; English transl., *Siberian Advances in Math.* **1** (1991), no. 1,2, 64–108, 1–56.
3. ———, *An algorithm for solving the Lur' e–Riccati matrix equation with guaranteed accuracy*, *Trudy Inst. Mat. Sibirsk. Otdel. Rus. Akad. Nauk* **24** (1994), 63–104; English transl., *Siberian Advances in Math.* **5** (1995), no. 4, 1–49.
4. F. R. Gantmacher, *The theory of matrices*, vol. I, II, Chelsea Publishing Company, New York, 1959.
5. I. M. Gelfand and S. V. Fomin, *Calculus of variations*, Fizmatgiz, Moscow, 1961; English transl., Prentice-Hall, Englewood Cliffs, NJ, 1963.
6. I. M. Gelfand and G. E. Shilov, *Some questions of the theory of differential equations*, Fizmatgiz, Moscow, 1958; English transl., Academic Press, New York and London, 1967.
7. S. K. Godunov, *Problem of the dichotomy of the spectrum of a matrix*, *Sibirsk. Mat. Zh.* **27** (1986), no. 5, 24–37; English transl., *Siberian Math. J.* **27** (1986), no. 5, 649–660.
8. ———, *Quadratic Lyapunov functions*, Novosibirsk State University, Novosibirsk, 1982.
9. ———, *Matrix exponential, the Green matrix, and the Lopatinskiĭ conditions*, Novosibirsk State University, Novosibirsk, 1983.
10. ———, *An estimate for Green's matrix of a Hamiltonian system in the optimal control problem*, *Sibirsk. Mat. Zh.* **34** (1993), no. 4, 70–80; English transl., *Siberian Math. J.* **34** (1993), no. 4, 653–662.
11. ———, *Norms of solutions to the Lur' e–Riccati matrix equation as a criterion for quality of stabilizability and detectability*, *Trudy Inst. Mat. Sibirsk. Otdel. Rus. Akad. Nauk* **22** (1992), 3–22; English transl., *Siberian Advances in Math.* **2** (1992), no. 3, 135–158.
12. ———, *Current aspects of linear algebra*, “Nauchnaya kniga”, Novosibirsk, 1997.
13. S. K. Godunov, A. G. Antonov, O. P. Kirilyuk, and V. I. Kostin, *Guaranteed accuracy for solutions of system of linear equations in Euclidean spaces*, “Nauka”, Novosibirsk, 1988; English transl., Kluwer, Dordrecht, 1993.
14. S. K. Godunov and V. M. Gordienko, *The Green matrix of the boundary-value problem for ordinary differential equations*, *Uspekhi Mat. Nauk* **39** (1984), no. 1, 39–76; English transl. in *Russian Math. Surveys* **39** (1984).
15. A. N. Kolmogorov and S. V. Fomin, *Introductory real analysis*, “Nauka”, Moscow, 1981; English transl. of the 2nd edition, Dover, New York, 1975.
16. A. G. Kurosh, *Higher algebra*, “Nauka”, Moscow, 1971. (Russian)
17. S. V. Kuznetsov, *Solution of boundary-value problems for ordinary differential equations*, *Trudy Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR* **6** (1985), 85–110.
18. L. A. Lyusternik and V. I. Sobolev, *A short course in functional analysis*, “Vyssh. Shkola”, Moscow, 1982; English transl. of the 1st edition, Halsted Press, New York, 1974.
19. A. N. Malyshev, *Guaranteed accuracy in spectral problems of linear algebra*, *Trudy Inst. Mat. Sibirsk. Otdel. Rus. Akad. Nauk* **17** (1991), 19–103; English transl., *Siberian Advances in Math.* **2** (1992), no. 1, 144–197.
20. ———, *Introduction in linear algebra*, “Nauka”, Novosibirsk, 1991. (Russian)
21. G. I. Petrovskii, *Lectures on the theory of ordinary differential equations*, “Nauka”, Moscow, 1964; English transl., Prentice-Hall, Englewood Cliffs, NJ, 1966.
22. J. H. Wilkinson, *The algebraic eigenvalue problem*, Clarendon Press, Oxford, 1965.

This page intentionally left blank

Index

- QR*-decomposition, 185
- Cauchy criterion, 71, 73
Cauchy formula, 71, 72, 231
Cauchy problem, 9, 12, 22, 62, 77, 113, 133
Cauchy–Schwarz–Bunyakovskii inequality, 74, 217
Cayley–Hamilton theorem, 18, 32, 211
Chetaev theorem, 130
condition number, 91, 188, 204
Conditions H1–H3, 130
Conditions L1, L2, L3', 113, 114
control admissible, 215
control optimal, 223
control problem, 209, 212
controllability criterion, 213
- dichotomy of spectrum, 194, 272
dichotomy parameter, 195, 197, 272
domain of attraction, 117, 136
dynamic compensator, 277
- equilibrium point, 119, 129, 140
- feedback, 275
Fourier transform, 81
- Gelfand–Shilov estimate, 20
Gram matrix, 24, 32, 222
Gram–Schmidt orthogonalization, 185
Gram–Schmidt orthogonalization process, 178
Green function, 58
Green matrix, 49, 78, 81, 87, 90, 109, 186, 197
- Hamiltonian matrix, 243
Hamiltonian system, 243, 276
Hermite theorem, 143, 146
Hermitian matrix, 125, 132, 143
Hurwitz inequalities, 151
Hurwitz matrix, 134
Hurwitz polynomial, 151
- Jordan canonical form, 45
- Lagrange multiplier, 276
Liouville formula, 10, 23
Lopatinskii condition, 96, 207, 248
Lopatinskii constant, external, 97, 99, 104
Lopatinskii constant, internal, 104
Lopatinskii constants, 207
Lopatinskii determinant, 99
Lopatinskii estimate, 107
Lur'e equation, 266
Lur'e–Riccati equation, 266
Lyapunov equation, 125, 128, 134, 166, 193, 269
Lyapunov function, 117, 129, 130, 136
Lyapunov function quadratic, 117
Lyapunov theorem, 129, 138, 148
- matrix exponential, 13, 15, 31, 44, 122
matrix fundamental of solutions, 11, 22, 83
matrix norm Euclidean, 4
matrix norm Frobenius, 4
matrix norm spectral, 3
- optimality condition, 221
Ostrowski theorem, 166
- pair controllable, 210, 212
pair detectable, 255
pair observable, 213
pair stabilizable, 234, 248, 249, 263
Parseval equality, 82
Plancherel theorem, 81
Poincaré–Lyapunov theorem, 244, 261
polynomial generating, 148
projection, 103
- Riccati equation, 266
Routh algorithm, 150
Routh–Hurwitz conditions, 157, 165
- singular value decomposition, 199, 273
solution, generalized, 75, 218
solution, asymptotically stable, 119, 129, 135, 139, 149
solution, Lyapunov stable, 119, 121, 129
solution, Lyapunov unstable, 119, 130, 135, 139
- space control, 212
sweep backward, 185
sweep direct, 185
sweep method, 185
sweep orthogonal, 185
sweep two-way, 190

- symbol, 25
- system fundamental of solutions, 22, 26, 84
- system, autonomous, 113, 118
- theorem on continuous dependence, 65
- Wronski determinant, 23
- Wronskian, 23

Selected Titles in This Series

(Continued from the front of this publication)

- 130 **V. V. Vershinin**, Cobordisms and spectral sequences, 1993
- 129 **Mitsuo Morimoto**, An introduction to Sato's hyperfunctions, 1993
- 128 **V. P. Orevkov**, Complexity of proofs and their transformations in axiomatic theories, 1993
- 127 **F. L. Zak**, Tangents and secants of algebraic varieties, 1993
- 126 **M. L. Agranovskiĭ**, Invariant function spaces on homogeneous manifolds of Lie groups and applications, 1993
- 125 **Masayoshi Nagata**, Theory of commutative fields, 1993
- 124 **Masahisa Adachi**, Embeddings and immersions, 1993
- 123 **M. A. Akivis and B. A. Rosenfeld**, Élie Cartan (1869–1951), 1993
- 122 **Zhang Guan-Hou**, Theory of entire and meromorphic functions: deficient and asymptotic values and singular directions, 1993
- 121 **I. B. Fesenko and S. V. Vostokov**, Local fields and their extensions: A constructive approach, 1993
- 120 **Takeyuki Hida and Masuyuki Hitsuda**, Gaussian processes, 1993
- 119 **M. V. Karasev and V. P. Maslov**, Nonlinear Poisson brackets. Geometry and quantization, 1993
- 118 **Kenkichi Iwasawa**, Algebraic functions, 1993
- 117 **Boris Zilber**, Uncountably categorical theories, 1993
- 116 **G. M. Fel'dman**, Arithmetic of probability distributions, and characterization problems on abelian groups, 1993
- 115 **Nikolai V. Ivanov**, Subgroups of Teichmüller modular groups, 1992
- 114 **Seizō Itô**, Diffusion equations, 1992
- 113 **Michail Zhitomirskiĭ**, Typical singularities of differential 1-forms and Pfaffian equations, 1992
- 112 **S. A. Lomov**, Introduction to the general theory of singular perturbations, 1992
- 111 **Simon Gindikin**, Tube domains and the Cauchy problem, 1992
- 110 **B. V. Shabat**, Introduction to complex analysis Part II. Functions of several variables, 1992
- 109 **Isao Miyadera**, Nonlinear semigroups, 1992
- 108 **Takeo Yokonuma**, Tensor spaces and exterior algebra, 1992
- 107 **B. M. Makarov, M. G. Goluzina, A. A. Lodkin, and A. N. Podkorytov**, Selected problems in real analysis, 1992
- 106 **G.-C. Wen**, Conformal mappings and boundary value problems, 1992
- 105 **D. R. Yafaev**, Mathematical scattering theory: General theory, 1992
- 104 **R. L. Dobrushin, R. Kotecký, and S. Shlosman**, Wulff construction: A global shape from local interaction, 1992
- 103 **A. K. Tsikh**, Multidimensional residues and their applications, 1992
- 102 **A. M. Il'in**, Matching of asymptotic expansions of solutions of boundary value problems, 1992
- 101 **Zhang Zhi-fen, Ding Tong-ren, Huang Wen-zao, and Dong Zhen-xi**, Qualitative theory of differential equations, 1992
- 100 **V. L. Popov**, Groups, generators, syzygies, and orbits in invariant theory, 1992
- 99 **Norio Shimakura**, Partial differential operators of elliptic type, 1992
- 98 **V. A. Vassiliev**, Complements of discriminants of smooth maps: Topology and applications, 1992 (revised edition, 1994)
- 97 **Itiro Tamura**, Topology of foliations: An introduction, 1992
- 96 **A. I. Markushevich**, Introduction to the classical theory of Abelian functions, 1992
- 95 **Guangchang Dong**, Nonlinear partial differential equations of second order, 1991
- 94 **Yu. S. Il'yashenko**, Finiteness theorems for limit cycles, 1991

(See the AMS catalog for earlier titles)

ISBN 0-8218-0656-4



9 780821 806562