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Volume 174

Second Order Elliptic Equations and Elliptic Systems

Ya-Zhe Chen
Lan-Cheng Wu

American Mathematical Society

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Second Order Elliptic Equations and Elliptic Systems

Ya-Zhe Chen
Lan-Cheng Wu

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二阶椭圆型方程与椭圆型方程组

(Second Order Elliptic Equations and Elliptic Systems)

Ya-Zhe Chen (陈亚浙) and Lan-Cheng Wu (吴兰成)

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ABSTRACT. This book is based on the authors' lecture notes at the Institute of Mathematics at Nankai University during the Partial Differential Equations Year in 1985, absorbing also the most recent materials from the lectures of experts.

There are two parts of the book. For the Dirichlet problem of second order elliptic partial differential equations, various kinds of a priori estimate methods, including the most recent techniques, are rather completely introduced in the first part. Linear, quasilinear and fully nonlinear equations are studied. In the second part, the existence and regularity theories of the Dirichlet problem for linear and nonlinear second order elliptic partial differential systems are introduced. This book chooses appropriate materials; it is a very good textbook for graduate students.

This book can also be used as a reference book for undergraduate mathematics majors, graduate students, professors and scientists.

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Preface to the English Translation

We are very pleased that our graduate textbook has been chosen by the American Mathematical Society for translation into English. In this English edition, some obvious typographic errors in the Chinese edition are corrected; some names are modified according to the current convention. We also adopted the translator's suggestion and added very brief bibliographic notes for each chapter, together with updated references.

We would like to express our wholehearted gratitude to Professors Alice Chang, Tsit-Yuen Lam and other AMS personnel for their vast amount of work on this translation project. We would also like to thank the translator, Dr. Bei Hu, for his valuable suggestions; he typed the LaTeX manuscript for this English edition.

Owing to the authors' limited knowledge, errors and inappropriateness are hard to avoid; we welcome corrections from readers.

Ya-Zhe Chen, Lan-Cheng Wu
July 9, 1997

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Preface

The theory of second order elliptic equations and systems is fundamental for studying partial differential equations, and therefore it was listed as a basic course for graduate students at the Institute of Mathematics at Nankai University during the Partial Differential Equations Year in 1985. The authors were invited to give lectures to graduate students for this course. At that time, the Institute at Nankai also invited many well-known experts from around the world to give lectures, which provided the course with the most recent results in the area. This book is based on the authors' lecture notes, but we have absorbed also the most recent materials from the lectures of the other invited experts.

There are many excellent books on second order elliptic partial differential equations and systems, such as [GT], [LU] and [GQ1], listed in the bibliography of this book. These books give a thorough introduction in this area. However, some of these books are too big to be suitable as textbooks for beginners. The purpose of this book is to provide a textbook for graduate students. This book includes both basic materials and the most recent results and methods, so as to bring graduate students to the frontier of this area.

There are two parts of the book. For the Dirichlet problem for second order elliptic partial differential equations, various kinds of a priori estimate methods are rather completely introduced in the first part. The Krylov-Safonov estimate and fully nonlinear elliptic equations, which appeared in the 80's, are introduced in detail, but concisely. In the second part, the existence and regularity theories of linear and nonlinear second order elliptic partial differential systems are introduced. Basic facts about Sobolev spaces are given in Appendix 1. In order to emphasize the main theme, the proofs of some theorems, such as the Stampacchia interpolation theorem and the reverse Hölder inequality, are included in the Appendix.

Owing to the authors' limited knowledge, errors are hard to avoid; we welcome suggestions from readers.

Under the leadership of Professor Li-Shang Jiang (姜礼尚), the partial differential equation seminar at Peking University played an important role in this book. We would like to express our deep gratitude to Professor Li-Shang Jiang and the seminar participants for their contributions. We would also like to express our wholehearted gratitude to Professor Guang-Lie Wang (王光烈) from Jilin University, who read the draft of this book and made valuable suggestions.

Ya-Zhe Chen, Lan-Cheng Wu
May 20, 1990

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Bibliographic Notes

Part I

Detailed bibliographic notes for the theory of second order elliptic equations can be found in [GT]. Here we give brief comments on the materials that we choose to present in our textbook.

Chapter 1. L^2 Theory

Since the work of K. O. Friedrichs [FR], there have been major developments in the theory of variational methods for elliptic equations. The Fredholm alternative theory for equations of divergence form is due primarily to Ladyženskaja and Ural′ceva [LU] and G. Stampacchia [ST1]. Here we use their assumptions on the coefficients and their proofs. The maximum principle for weak solutions is due to M. Chicco [CH] and N. S. Trudinger [TR1]; the proof given here is a combination of their methods. We have also benefited from the lecture notes of L. C. Evans [EV1].

Chapter 2. Schauder Theory

The Schauder theory for the Dirichlet problem of second order elliptic equations was established as early as the 1930s by J. Schauder [SC1], [SC2]. Around the same time, E. Hopf [HO] and R. Caccioppoli [CA] established some similar results. The method here is due to N. S. Trudinger [TR4], who lectured in 1985 at the Institute of Mathematics at Nankai University, Tianjin, China. He established the equivalent Hölder norm of a solution using its mollified function; his proof avoids the tedious computations of potentials. In fact, this equivalent norm can be simplified further; another equivalent Hölder norm was introduced in [CY2] without using the derivatives with respect to the parameter of the mollified function.

Chapter 3. L^p Theory

Based on the Calderón-Zygmund decomposition lemma [CZ] and the singular integral operator theory, the L^p theory for the Dirichlet problem of second order elliptic equations was first established by D. Greco [GR] and A. I. Koselev [KO]. The results were generalized later to higher order elliptic equations as well as more general boundary conditions by S. Agmon, A. Douglis and L. Nirenberg [ADN]. Here, we use the special technique in [GT] on potential operators, which avoids the Fourier transform.

Chapter 4. De Giorgi-Nash-Moser Estimates

The Hölder estimates for solutions of elliptic equations with bounded measurable coefficients are fundamental in studying quasilinear elliptic equations. For divergence equations, such estimates were obtained by E. De Giorgi [DG1]. J. Nash independently obtained similar estimates for elliptic and parabolic equations. A new proof was given by Moser [MJ1], [MJ2], who also established Harnack's inequality for nonnegative solutions. We use Moser's proof. For the proof of the weak Harnack inequality, we have also consulted lecture notes of R. Schoen.

Chapter 5. Quasilinear Equations of Divergence Form

After the establishment of the De Giorgi-Nash-Moser estimates, there were rapid developments for the theory of quasilinear equations in divergence form. The major works are: Ladyženskaja and Ural′ceva [LU], Serrin [SE], Trudinger [TR1], etc. Ladyženskaja and Ural′ceva also reduced general quasilinear equations to divergence form so that the De Giorgi-Nash-Moser estimates are applicable. This chapter is based on [GT]. For gradient estimates of the solutions, we employ a technique from [CY1], which simplifies the proof in [GT].

Chapter 6. Krylov-Safonov Estimates

The maximum principle for strong solutions was obtained by A.D. Alexandroff [AL1], [AL2], I. Ya. Bakelman [BA], and C. Pucci [PU] in different forms. Applying this maximum principle and L^p-estimates, the existence of a $W^{2,p}$-strong solution can be established. Based on Pucci's paper and [GT], we give a clear, rigorous and detailed proof for the maximum principle.

The Ladyženskaja-Ural′ceva techniques for solving general quasilinear elliptic equations cannot be generalized to fully nonlinear elliptic equations, as these equations cannot be reduced to equations of divergence form. To study fully nonlinear elliptic equations, the estimates of De Giorgi-Nash-Moser's type for equations in nondivergence form are indispensable. Based on Alexandroff-Bakelman-Pucci's maximum principle and techniques similar to the Calderón-Zygmund decomposition, N. V. Krylov and M. V. Safonov [KS] established Harnack's inequality for nonnegative solutions of uniformly elliptic equations in nondivergence form with bounded measurable coefficients (in fact, they gave a proof for more complicated parabolic equations). Using their ideas, Trudinger[TR2] gave a different proof; the proof here is his ([GT]).

Chapter 7. Fully Nonlinear Elliptic Equations

Since this is a textbook, we discuss only fully nonlinear uniformly elliptic equations under the natural structure conditions and the concavity condition. A typical example is the Bellman equation from control theory. The Monge-Ampère equation, which is a fully nonlinear nonuniformly elliptic equation, is not discussed in this book. For the Bellman equation, partial differential equation techniques were first used by H. Brezis and L. C. Evans [BE], L. C. Evans and A. Friedman [EF],

P. L. Lions [LP], and L. C. Evans and P. L. Lions [EL]. In the last two papers, $C^{1,1}$–strong solutions were established. Afterwards, L. C. Evans established the interior Hölder estimates for second order derivatives of the solution and obtained the classical solution. These results were generalized by N. S. Trudinger [TR3] to fully nonlinear uniformly elliptic equations under the natural structure conditions. In his lectures at Nankai University in 1985, Trudinger [TR4] uses the mollified function and freezing coefficients techniques to derive the Schauder estimate; for the Bellman equation, an a priori estimate for the lower order derivatives is no longer necessary. This chapter is primarily based on Trudinger [TR3]. For the gradient estimate, our technique simplifies the Bernstein method. The Hölder estimate for the gradient near the boundary was first obtained by N. V. Krylov; the simplified proof given here is due to L. A. Caffarelli. The results for the estimate of Schauder type for fully nonlinear elliptic equations are chosen from [TR4].

Part II

Detailed bibliographic notes for the theory of second order elliptic systems can be found in [GQ1]. Here we give a brief commentary on the materials that we chose for our textbook.

Chapter 8. L^2 Theory for Linear Elliptic Systems of Divergence Form

The Hilbert space method for linear elliptic systems dates back to 1900 (D. Hilbert [HI]) and 1907 (H. Lebesgue [LE]). A brief introduction to the developments thereafter can be found in the bibliographical notes for Chapter 1. In this chapter, we study the existence and differentiability of weak solutions of linear divergence elliptic systems; only those systems without lower order terms are studied.

Chapter 9. Schauder Theory for Linear Elliptic Systems of Divergence Form

In the 1930s, the Schauder theory was established using delicate estimates of the $C^{2,\alpha}$ norm of the potential integrals. In Chapter 2, we introduced the method from [TR4] to establish an equivalent C^α norm using the mollified function, which is then used in establishing the Schauder theory without tedious computations of the potential integrals. In this chapter, we use the methods developed in the 1950s and 1960s by C. B. Morrey [MR3] and S. Campanato [CP] to establish the Schauder theory for elliptic systems in divergence form. They introduced an equivalent C^α norm by integral representations, which is very convenient in studying regularity of weak solutions; this gives another method for establishing the Schauder theory without tedious computations of the potential integrals. M. Giaquinta systematically introduced this method in his books [GQ1], [GQ2]. We introduce this method in a way similar to that in [GQ1].

Chapter 10. L^p Theory for Linear Elliptic Systems of Divergence Form

The L^p theory can be established by using the Calderón-Zygmund decomposition lemma and singular integral operator theory. In this chapter, however, we use the method given by S. Campanato and G. Stampacchia [CS] to establish the L^p theory; this method uses the Stampacchia interpolation theorem [ST2] rather than potential theory. We introduce this method in a way similar to that in [GQ2].

Chapter 11. Existence of Weak Solutions of Nonlinear Elliptic Systems

In this chapter, we introduce the variational method, which has a long history. Using this method, B. Riemann obtained many interesting results on geometric function theory in the nineteenth century. However, he assumed the Dirichlet principle to be valid; i.e., there exists a unique $u \in K$ such that

$$D[u] = \min_{v \in K} D[v],$$

where $D[v] = \int_\Omega |Dv|^2 dx$, $K = \{v \in C^1(\Omega) \cap C^0(\overline{\Omega}),\ v|_{\partial\Omega} = \varphi\}$, φ is a given function, and u is harmonic in Ω. With the development of rigorous analysis, Riemann's work was criticized. K. Weierstrass pointed out that $D[v]$ being bounded below does not imply the existence of a minimum point; furthermore, a minimum point, if it exists, need not be harmonic. Probably this criticism is why the variational method was not taken seriously until the 1900s, when D. Hilbert [HI] and H. Lebesgue [LE] rigorously established the Dirichlet principle for some special cases; the more general case was later obtained by L. Tonelli. Around 1940, the Sobolev spaces were employed by C. B. Morrey [MR1], and the existence of a minimum point was established for a large class of integral functionals. Nowadays, the variational method has become an effective method for studying the existence of weak solutions for elliptic equations and systems. We adopted the treatment in [GQ2], assuming that $f(x, u, p)$ is convex in p. This convexity condition is necessary when $N = 1$ and should be replaced by the quasi-convexity condition when $N > 1$. For further discussion, we refer readers to [MR2], [MR4], [GQ1], [GQ2] and the references therein.

Chapter 12. Regularity for Weak Solutions of Nonlinear Elliptic Systems

In 1968, De Giorgi [DG2] gave an example illustrating that the De Giorgi-Nash-Moser theorem is in fact not valid for linear elliptic systems, therefore the regularity of a weak solution of a nonlinear elliptic system cannot be established in the same way as in the single equation case. But the question whether a weak solution of a nonlinear elliptic system can be regular was still open. Later on, E. Giusti and M. Miranda [GD1] and other mathematicians gave many examples (cf. [GQ1]) showing that, in general, a weak solution of a nonlinear elliptic system does not have regularity everywhere. After that, Morrey [MR5], Giusti and M. Miranda [GD2], Giusti [GI], and Pepe [PE] showed that weak solutions of some nonlinear elliptic systems have partial regularity (i.e., regularity in some open subset $\Omega_0 \subset \Omega$, and $\text{meas}(\Omega \setminus \Omega_0) = 0$); they used the indirect method. For more general nonlinear elliptic systems, partial regularity of weak solutions was established by M. Giaquinta and E. Giusti [GG], and by M. Giaquinta and G. Modica [GM1], [GM2]; they used the direct method based on the reverse Hölder inequality. Here, our main purpose is to introduce the methods, and we do not pursue the most general results. Therefore, we study only two special classes of quasilinear elliptic systems. We introduce the indirect and direct methods in a way similar to that in [GQ1]. For more general results, we refer readers to [GQ1] and the references therein, and [AF2], [EV2], [EG], [FU], [FH], [GM3], [HM], etc.

For nonlinear elliptic systems of some special structures, such as nonlinear elliptic systems of diagonal form, a weak solution has regularity everywhere. We do not present such results in this book, but refer interested readers to [HW1], [HW2], [WI1], [WI2], [HB], [GH], etc.

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