Translations of
MATHEMATICAL MONOGRAPHS

Volume 176

Four-Dimensional Integrable Hamiltonian Systems with Simple Singular Points (Topological Aspects)

L. M. Lerman
Ya. L. Umanskiy

American Mathematical Society

Selected Titles in This Series

176 **L. M. Lerman and Ya. L. Umanskiy,** Four-dimensional integrable Hamiltonian systems with simple singular points (topological aspects), 1998
175 **S. K. Godunov,** Modern aspects of linear algebra, 1998
174 **Ya-Zhe Chen and Lan-Cheng Wu,** Second order elliptic equations and elliptic systems, 1998
173 **Yu. A. Davydov, M. A. Lifshits, and N. V. Smorodina,** Local properties of distributions of stochastic functionals, 1998
172 **Ya. G. Berkovich and E. M. Zhmud′,** Characters of finite groups. Part 1, 1998
171 **E. M. Landis,** Second order equations of elliptic and parabolic type, 1998
170 **Viktor Prasolov and Yuri Solovyev,** Elliptic functions and elliptic integrals, 1997
169 **S. K. Godunov,** Ordinary differential equations with constant coefficient, 1997
168 **Junjiro Noguchi,** Introduction to complex analysis, 1998
167 **Masaya Yamaguti, Masayoshi Hata, and Jun Kigami,** Mathematics of fractals, 1997
166 **Kenji Ueno,** An introduction to algebraic geometry, 1997
165 **V. V. Ishkhanov, B. B. Lur′e, and D. K. Faddeev,** The embedding problem in Galois theory, 1997
164 **E. I. Gordon,** Nonstandard methods in commutative harmonic analysis, 1997
163 **A. Ya. Dorogovtsev, D. S. Silvestrov, A. V. Skorokhod, and M. I. Yadrenko,** Probability theory: Collection of problems, 1997
162 **M. V. Boldin, G. I. Simonova, and Yu. N. Tyurin,** Sign-based methods in linear statistical models, 1997
161 **Michael Blank,** Discreteness and continuity in problems of chaotic dynamics, 1997
160 **V. G. Osmolovskiĭ,** Linear and nonlinear perturbations of the operator div, 1997
159 **S. Ya. Khavinson,** Best approximation by linear superpositions (approximate nomography), 1997
158 **Hideki Omori,** Infinite-dimensional Lie groups, 1997
157 **V. B. Kolmanovskiĭ and L. E. Shaĭkhet,** Control of systems with aftereffect, 1996
156 **V. N. Shevchenko,** Qualitative topics in integer linear programming, 1997
155 **Yu. Safarov and D. Vassiliev,** The asymptotic distribution of eigenvalues of partial differential operators, 1997
154 **V. V. Prasolov and A. B. Sossinsky,** Knots, links, braids and 3-manifolds. An introduction to the new invariants in low-dimensional topology, 1997
153 **S. Kh. Aranson, G. R. Belitsky, and E. V. Zhuzhoma,** Introduction to the qualitative theory of dynamical systems on surfaces, 1996
152 **R. S. Ismagilov,** Representations of infinite-dimensional groups, 1996
151 **S. Yu. Slavyanov,** Asymptotic solutions of the one-dimensional Schrödinger equation, 1996
150 **B. Ya. Levin,** Lectures on entire functions, 1996
149 **Takashi Sakai,** Riemannian geometry, 1996
148 **Vladimir I. Piterbarg,** Asymptotic methods in the theory of Gaussian processes and fields, 1996
147 **S. G. Gindikin and L. R. Volevich,** Mixed problem for partial differential equations with quasihomogeneous principal part, 1996
146 **L. Ya. Adrianova,** Introduction to linear systems of differential equations, 1995
145 **A. N. Andrianov and V. G. Zhuravlev,** Modular forms and Hecke operators, 1995
144 **O. V. Troshkin,** Nontraditional methods in mathematical hydrodynamics, 1995
143 **V. A. Malyshev and R. A. Minlos,** Linear infinite-particle operators, 1995
142 **N. V. Krylov,** Introduction to the theory of diffusion processes, 1995

(Continued in the back of this publication)

This page intentionally left blank

Four-Dimensional Integrable Hamiltonian Systems with Simple Singular Points (Topological Aspects)

10.1090/mmono/176

Translations of

MATHEMATICAL MONOGRAPHS

Volume 176

Four-Dimensional Integrable Hamiltonian Systems with Simple Singular Points (Topological Aspects)

L. M. Lerman
Ya. L. Umanskiy

American Mathematical Society
Providence, Rhode Island

Л. М. Лерман, Я. Л. Уманский

ЧЕТЫРЕХМЕРНЫЕ ИНТЕГРИРУЕМЫЕ ГАМИЛЬТОНОВЫ СИСТЕМЫ С ПРОСТЫМИ ОСОБЫМИ ТОЧКАМИ (ТОПОЛОГИЧЕСКИЙ ПОДХОД)

Translated from the original Russian manuscript
by A. Kononenko and A. Semenovich

1991 *Mathematics Subject Classification.* Primary 58F05, 70Hxx.

ABSTRACT. The main topic of this book is the isoenergetic structure of the Liouville foliation generated by an integrable system with two degrees of freedom and the topological structure of the corresponding Poisson action of the group $\mathbb{R}^2$. This is a first step towards understanding the global dynamics of Hamiltonian systems and applying perturbation methods. The main attention is paid to the topology of this foliation rather than to analytic representation. In contrast to books published before, the authors consistently use the dynamical properties of the action to achieve their results.

The book can be used by graduate students and researchers interested in studying dynamics of Hamiltonian systems. It can also be useful for people studying the geometric structure of symplectic manifolds.

Library of Congress Cataloging-in-Publication Data

Lerman, L. M. (Lev M.)
Four-dimensional integrable Hamiltonian systems with simple singular points (topological aspects) / L. M. Lerman, Ya. L. Umanskiy.
p. cm. — (Translations of mathematical monographs, ISSN 0065-3282 ; v. 176)
Translated from the original Russian manuscript.
Includes bibliographical references.
ISBN 0-8218-0375-1 (alk. paper)
1. Hamiltonian systems. 2. Four-manifolds (Topology) I. Umanskiy, Ya. L. (Yan L.) II. Title. III. Series.
QA614.83.L47 1998
514′.74—DC21 98-15030
CIP

♾ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at URL: `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 03 02 01 00 99 98

Contents

Introduction ix

Chapter 1. General Results of the Theory of Hamiltonian Systems 1
- 1.1. Hamiltonian systems on a symplectic manifold 1
- 1.2. Poisson brackets and first integrals 3
- 1.3. Integrable systems and Poisson actions 6
- 1.4. A local structure of the orbit foliation of an action 8
- 1.5. Formulation of the classification problem for actions and IHVFs 14

Chapter 2. Linear Theory and Classification of Singular Orbits 17
- 2.1. Linear Poisson actions 17
- 2.2. Simple singular points of an action and IHVF (local theory) 26
- 2.3. One-dimensional orbits of the action 28
- 2.4. Singular trajectories of IHVF 39

Chapter 3. IHVF and Poisson Actions of Morse Type 43
- 3.1. Limit set of an orbit of the action 43
- 3.2. Class of Poisson actions and IHVFs of Morse type 45
- 3.3. General properties of PA of Morse type 46

Chapter 4. Center-Center Type Singular Points of PA and Elliptic Singular Points of IHVF 51

Chapter 5. Saddle-Center Type Singular Points 57
- 5.1. Extended neighborhood of a singular point, and its topology 57
- 5.2. Isoenergetic equivalence of IHVFs 63

Chapter 6. Saddle Type Singular Points 71
- 6.1. The degeneracy set 71
- 6.2. The types of loops and SPTs 73
- 6.3. Construction of the extended neighborhood 82
- 6.4. The invariant and formulation of the equivalence theorem 84
- 6.5. The structure of the auxiliary system and its invariant foliations 86
- 6.6. The topology of ∂V_ϵ 99
- 6.7. Proof of the equivalence theorems 107
- 6.8. The topology of the extended neighborhood V 119

Chapter 7. Saddle-Focus Type Singular Points 129
- 7.1. The structure of a garland and its separatrix set 129

7.2. The structure of the auxiliary gradient system 131
7.3. The construction of the conjugating homeomorphism in U 135
7.4. Proof of the isoenergetic equivalence theorem 137
7.5. The equivalence of actions 145
7.6. The topology of V 145

Chapter 8. Realization 149
8.1. Elliptic points 149
8.2. Singular point of the saddle-center type 149
8.3. Singular point of the saddle type 150
8.4. Singular point of the saddle-focus type 156

Appendix A. Normal Forms of Quadratic Hamilton Functions and Their Centralizers in $sp(4, \mathbb{R})$ 163

Appendix B. The Gradient System on M Compatible with the Hamiltonian 167

Bibliography 175

Introduction

The purpose of this book is to study a smooth integrable Hamiltonian system on a smooth four-dimensional symplectic manifold M (a Hamiltonian system with two degrees of freedom in the terminology accepted in mechanics) in invariant domains containing singular points. The notion of integrability which appeared almost at the very beginning of the development of the theory of differential equations has undergone a significant evolution: from the attempts to obtain solutions of a differential equation in the form of elementary functions, their integrals, and inverse functions of integrals (integration in quadratures) in works of Johann, both Nicolai, and Daniel Bernoulli, Ricatti, Euler, Clairaut and other classics, via the discovery of the important particular cases of integrable systems in mechanics and geometry in the works of Euler, Lagrange, Jacobi, Neumann, Clebsch and many others, via the Liouville theorem, to the modern understanding of the integrability of a Hamiltonian system with n degrees of freedom as the existence of n almost everywhere independent integrals in the involution. Apparently, the Liouville theorem was the first general geometric result in the theory of integrable systems because this theorem depends not on a particular form of the system but only on the existence of n independent integrals in the involution, and gives a description of the behavior of all trajectories of the system in the considered neighborhood. In [**1**] Arnold stated this theorem in the modern form and globalized the theorem's statement.

The modern breakthrough in the theory of finite-dimensional integrable systems occurred after the discovery of infinite-dimensional integrable Hamiltonian systems of the types of Korteveg–de Vries equations, nonlinear Schrödinger equation, sin-Gordon equation, Landau–Lifshits equation, and the development of the inverse scattering problem method and algebraic integration methods (Gardner, Green, Kruskal, Miura, Lax, Zakharov, Faddeev, Hirota, Toda, Calogero, Moser, Novikov, Dubrovin, Shabat, Mischenko, Fomenko, Bogoyavlensky and many others), which led to the discovery of new and rediscovery of already known finite-dimensional integrable Hamiltonian systems. These powerful analytic methods allow us to establish integrability (to be more precise, not to establish integrability of particular systems but to construct classes of integrable equations) and obtain explicit solutions. But these methods are poorly suited for obtaining a global description of a particular system or its phase portrait, its structure, etc., in the terminology used in the theory of dynamical systems. The first step in the development of geometric theory was made by Smale [**2**] and Marsden and Weinstein [**3**]. Smale formulated the approach to the study of a Hamiltonian system invariant with respect to some Lie group action, and Marsden and Weinstein [**3**] developed the notion of the reduction for a Hamiltonian system with symmetries. These works were among the most important ones which led the authors of this book to the idea of applying the dynamical systems methods to the study of Hamiltonian systems [**4, 5, 6, 7, 8, 73**].

Before we start describing our approach it is natural to put the following question. It is known that integrable systems form a pretty "thin" [**13**] subset in the set of all Hamiltonian systems on a symplectic manifold (in the space of their Hamiltonians), so is it worthwhile to study their structure? In our opinion, the answer is positive and it is motivated by the following considerations. In the first place, "general position" arguments, although quite fruitful, are not absolute. For example, they do not explain such a phenomenon as frequent occurrence of integrable models in different physical systems. Apparently, the symmetry considerations are always tacitly present in the constructions of such models, which often leads to their integrability, or, in the multidimensional case, to the existence of a sufficiently rich group of symmetries. In the second place, integrable systems often appear in local problems when the dynamics of a nonintegrable system in neighborhoods of singular points and periodic trajectories are studied. Moreover, when a singular point is degenerate, it becomes necessary to study the properties of families of the Hamiltonian systems in a neighborhood of this point [**14**]. It often leads to integrable normal forms depending on parameters, i.e., to a peculiar bifurcation theory in the class of integrable systems. It is worth mentioning that although a normal form transformation usually diverges, it nevertheless conveys a lot of information about the behavior of trajectories in a neighborhood of a singular point or a periodic orbit. The reason is that nonintegrability effects (for example, splitting of separatrix surfaces of singular points and periodic trajectories) are usually exponentially small when we are engaged in the local study of unfolding of a Hamiltonian system with a degenerate singular point under the condition of integrability of its normal form. This leads to a good asymptotic approximation of the real solutions by the solutions obtained from the normal form [**15**].

In our opinion, there is a third reason for the interest in integrable systems. It is known that the gradient systems played a significant role in the differential topology for the studies of the topology of smooth manifolds [**16**], in particular, for the solution of the famous Poincaré problem for $n > 4$. We believe that the study of integrable Hamiltonian systems may help in the study of the topology of symplectic manifolds. The future will show whether this is true or not.

In addition to the above reasons, one can mention another important argument in favor of the study of integrable systems: this is one of few classes of Hamiltonian systems for which it is possible, in principle, to carry out a complete study of the structure of the flow. Another well-known example of this type is the opposite case of complete nonintegrability—the case of the geodesic flows on manifolds of negative curvature (see [**17**] and the references therein). Intuition is sharpened on such classes.

The authors' interest in these problems was stimulated to a large extent by the desire to understand the structure of some model systems which appeared in the theory of the domain walls propagation in magnetic media [**18, 19**]. It is known that the main phenomenological equation of this theory is the Landau–Lifshits equation, which for one-dimensional medium (plane wave) has the form:

$$m_t = m \times m_{xx} + m \times Jm,$$

where $m(x,t)$ is a three-dimensional vector of magnetic momentum, $m^2 = 1$, $J = \operatorname{diag}(J_1, J_2, J_3)$, $x \in \mathbb{R}$. Stationary waves of the form $m(x,t) = v(\xi)$, $\xi = x - ut$ satisfy the equation which reduces, after a substitution, to an integrable Hamiltonian system whose phase space is the cotangent bundle T^*S^2 to the sphere

$S^2 : v_1^2 + v_2^2 + v_3^2 = 1$. This system depends on two parameters u and $\epsilon = (J_3 - J_2)/(J_2 - J_1)$ and can be integrated in terms of Prim's θ-functions [**20**]. However, it is still quite difficult to "see" the dynamics of all solutions from this fact. This led us to the problem of developing some version of a "qualitative theory of integrable Hamiltonian systems" [**4, 5**]. In the base of this theory we put the study of the orbit structure for the induced action of the group $\mathbb{R}^2$ generated by a pair of commuting Hamiltonian vector fields X_H, X_K, where H is the Hamilton function of the Hamiltonian vector field, and K is its additional integral. Such an approach was quite natural for us since it is typical for "Andronov's school of oscillation theory" to which the authors belong. From this point of view the first problem was to study the action in a neighborhood of the singular set of the action, i.e., in the union of orbits whose dimension is less than two. The theorem of Liouville–Arnold, which describes the structure of orbits in a neighborhood of a two-dimensional compact orbit, the torus, does not work here. We note that studying the structure of orbits in a neighborhood of a singular point of the action we are led to the necessity of the global study of their behavior, since usually there are orbits containing a singular point in their closure but deviating far from this singular point. Therefore, we are led to the notion of extended neighborhood of a singular point and to the study of the behavior of action orbits and Hamiltonian system trajectories in this neighborhood, which is invariant with respect to the action. This is the main topic of our book.

It should be noted that such a study is conducted in a four-dimensional extended neighborhood and cannot be reduced to the study of the dynamics on a smooth three-dimensional level set of the Hamilton function H. The constructed equivalence invariants are determined both by the behavior of the system on the level set of H containing the singular point and by its behavior on nearby level sets. Clearly, for a global study of a system on a symplectic manifold one needs to know the behavior of the system on those level sets of the Hamilton function H that do not contain singular points. This was done in the works of Fomenko and coauthors [**9, 10, 11**]. Since these results are presented in many publications and books, we will not repeat them here, referring the interested reader to the original works.

For the reader's convenience we list some nonstandard abbreviations used throughout the book:

IHVF – integrable Hamiltonian vector field,

PA – Poisson action,

SPT – singular periodic trajectory,

CSP – curve of singular points.

The exact meaning of this terms is explained in the appropriate sections of this book. Also we would like to mention the numeration of the formulas, definitions, statements (lemmas, propositions, theorems, corollaries) and remarks accepted in the book. Every chapter, except for small Chapter 4, is divided into sections, which have double numeration. For example, Section 3.2 denotes Section 2 of Chapter 3. Formulas are numbered consecutively within each chapter. Definitions, remarks and figures are numbered consecutively within each chapter, and their numbers are of the form m.n, where m is the number of the chapter and n is the number of the corresponding definition, remark, or figure. The statements are numbered consecutively within each section, so they have triple numbers p.q.r., where p is the number of the chapter, q, the number of the section, and r, the number of the corresponding statement.

We express our gratitude to V. M. Eleonsky, who brought to our attention integrable models in physics and discussed with us many questions related to integrable systems, and to L. P. Shilnikov whose lively interest has been of help to us in our work.

We acknowledge the partial financial support of the International Science Foundation (grant R97000), which allowed us to complete this book.

Nizhni Novgorod

September 1994

Bibliography

[1] V. I. Arnold, *A theorem of Liouville concerning integrable problems of dynamics*, Sibirsk. Mat. Zh. **4** (1963), no. 2, 471–474; English transl. in Siberian Math. J. **4** (1963).

[2] S. Smale, *Topology and mechanics*, 1, 2, Invent. Math. **10** (1970), 305–331; **11** (1970) 45–64.

[3] J. R. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Reports Math. Phys. **5** (1974), no. 1, 121–130.

[4] L. M. Lerman and Ya. L. Umanskii, *The structure of a Poisson action of $\mathbb{R}^2$ on a four-dimensional symplectic manifold*, Selecta Math. Sovietica **6** (1987), no. 4, 365–396.

[5] ———, *The structure of a Poisson action of $\mathbb{R}^2$ on a four-dimensional symplectic manifold*. 2, Selecta Math. Sovietica **7** (1988), no. 1, 39–48.

[6] ———, *On the classification of four-dimensional Hamiltonian systems in extended neighborhoods of simple singular points*, Methods of the Qualitative Theory of Differential Equations (E. A. Leontovich-Adronova, ed.), Gorkov. Gos. Univ., Gorki, 1988, pp. 67–75. (Russian)

[7] ———, *Classification of four-dimensional Hamiltonian systems and Poisson actions of $\mathbb{R}^2$ in extended neighborhoods of simple singular points*, 1, Mat. Sb. **183** (1992), no. 12, 141–176; English transl., Russian Acad. Sci. Sb. Math. **77** (1993), 511–542.

[8] ———, *Classification of four-dimensional Hamiltonian systems and Poisson actions of $\mathbb{R}^2$ in extended neighborhoods of simple singular points*, 2, Mat. Sb. **184** (1993), no. 4, 105–138; English transl., Russian Acad. Sci. Sb. Math. **78** (1994), 479–506.

[9] A. T. Fomenko, *The topology of surfaces of constant energy of integrable Hamiltonian systems and obstructions to integrability*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 6, 1276–1307; English transl., Math USSR-Izv. **29** (1987), 629–658.

[10] ———, *Topological invariants of Hamiltonian systems that are integrable in the sense of Liouville*, Funktsional. Anal. i Prilozhen. **22** (1988), no. 4, 38–51; English transl. in Functional Anal. Appl. **22** (1988).

[11] A. T. Fomenko and H. Zieschang, *A criteria for topological equivalence of integrable Hamiltonian systems with two degrees of freedom*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 3, 546–575; English transl. in Math USSR-Izv. **36** (1991).

[12] A. T. Fomenko, *Integrability and nonintegrability in geometry and mechanics*, Kluwer, Dordrecht, 1988.

[13] L. Marcus and K. S. Meyer, *Generic Hamiltonian dynamical systems are neither integrable nor ergodic*, Mem. Amer. Math. Soc., no. 144 (1974).

[14] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Mathematical aspects of classical and celestial mechanics*, Encyclopedia of Mathematical Sciences, vol. 3, Springer-Verlag, Berlin–Heidelberg–New York, 1988.

[15] A. I. Neishtadt, *The separation of motions in systems with rapidly rotating phase*, Prikl. Mat. Mekh. **48** (1982) no. 2, 197–204; English transl. in J. Appl. Math. Mech. **48** (1982).

[16] J. Milnor, *Lectures on the h-cobordism theorem*, Princeton Univ. Press, Princeton, NJ, 1965.

[17] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Trudy Mat. Inst. Steklov. **90** (1967); English transl. in Proc. Steklov Math. Inst. **1969**.

[18] V. M. Eleonskii, N. N. Kirova, and N. E. Kulagin, *New conservation law for the Landau–Lifshitz equations*, Zh. Eksp. Teoret. Fiz. **77** (1979), 409–419; English transl. in Soviet Physics JETP **50** (1979).

[19] ———, *On random degeneration of the self-localized solutions of the Landau-Lifshitz equations*, Zh. Eksp. Teoret. Fiz. **75** (1978), no. 6, 2210–2219; English transl. in Soviet Physics JETP **48** (1978).

[20] A. P. Veselov, *The Landau–Lifshits equation and integrable systems of classical mechanics*, Dokl. Akad. Nauk SSSR **270** (1983), no. 5, 1094-1097; English transl. in Soviet Math. Dokl. **27** (1983).

[21] V. I. Arnold, *Mathematical methods of classical mechanics*, Springer-Verlag, Berlin–Heidelberg, 1978.

[22] R. Abraham and J. Marsden, *Foundations of mechanics*, Benjamin, New York–Amstredam, 1978.

[23] A. T. Fomenko, *Symplectic geometry. Methods and applications*, Moskov. Gos. Univ., Moscow, 1988. (Russian)

[24] V. I. Arnold and A. B. Givental, *Symplectic geometry*, Encyclopedia of Mathematical Sciences, vol. 4, Springer-Verlag, Berlin–Heidelberg, 1988.

[25] A. Weinstein, *Lectures on symplectic manifolds*, CBMS Conference Ser., vol. 29, Amer. Math. Soc., Providence, RI, 1977.

[26] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York and London, 1962.

[27] B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, *Modern geometry*, vol. 1, "Nauka", Moscow, 1979; English transl., Springer-Verlag, New York, Berlin, 1984.

[28] S. P. Novikov, *Hamiltonian formalism and multivalued analog of Morse theory*, Uspekhi Mat. Nauk **37** (1982), no. 5, 3–49; English transl. in Russian Math. Surveys **37** (1982).

[29] B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, *Integrable systems*. 1, Encyclopedia of Mathematical Sciences, vol. 3, Springer-Verlag, Berlin–Heidelberg–New York, 1988.

[30] N. N. Nekhoroshev, *The action-angle variables and their generalizations*, Trudy Moskov. Mat. Obshch. **26** (1972), 181–198; English transl. in Moscow Math. Soc. Trans. **26** (1974).

[31] V. I. Arnold, *Lectures on bifurcations and versal families*, Uspekhi Mat. Nauk **27** (1972), no. 5, 119–184; English transl. in Russian Math. Surveys **27** (1972).

[32] N. K. Gavrilov, *On some bifurcations of the equilibrium states of codimension two for divergence-free fields*, Methods of the Qualitative Theory of Differential Equations (E. A. Leontovich-Adronova, ed.), Gorkov. Gos. Univ., Gorki, 1985, pp. 46–55. (Russian)

[33] V. V. Nemytskii and V. V. Stepanov, *Qualitative theory of differential equations*, Gostehizdat, Moscow–Leningrad, 1947; English transl., Princeton Univ. Press, Princeton, NJ, 1960.

[34] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maier, *Qualitative theory of dynamical systems of the second order*, "Nauka", Moscow, 1966. (Russian)

[35] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

[36] W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc., Providence, RI, 1955.

[37] V. V. Kalashnikov, *Bott property and general position property of integrable Hamiltonian systems*, Uspekhi Mat. Nauk **48** (1993), no. 6, 151–152; English transl. in Russian Math. Surveys **48** (1993).

[38] ———, *On the generic character of Bott integrable Hamiltonian systems*, Mat. Sb. **185** (1994), no. 1, 107–120; English transl. in Russian Acad. Sci. Sb. Math. **79** (1995).

[39] F. R. Gantmakher, *Theory of matrices*, "Nauka", Moscow, 1988.

[40] J. Williamson, *On an algebraic problem concerning the normal forms of linear dynamical systems*, Amer. J. Math. **58** (1936), 141–163.

[41] D. Husemoller, *Fibre bundles*, McGraw-Hill, New York, 1966.

[42] A. D. Bruno, *Local methods in nonlinear differential equations*, Springer-Verlag, Berlin, 1989.

[43] D. Birkhoff, *Dynamical systems*, Amer. Math. Soc., New York, 1927.

[44] K. L. Siegel, *On the integrals of canonical systems*, Ann. Math. **42** (1941), 806–822.

[45] J. Moser, *On generalization on a theorem of Liapounoff*, Comm. Pure Appl. Math. **11** (1958), 257–271.

[46] H. Russmann, *Über das Verhalten analytisher Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtslösung*, Math. Ann. **154** (1964), 285–300.

[47] J. Vey, *Sur certains systèmes dynamiques séparables*, Amer. J. Math. **100** (1978), 591–614.

[48] H. Ito, *Convergence of Birkhoff normal forms for integrable systems*, Comment. Math. Helv. **64** (1989), 412–461.

[49] L. H. Eliasson, *Hamiltonian systems with Poisson commuting integrals*, Thesis, Univ. of Stockholm, 1984.

[50] ———, *Normal forms for Hamiltonian systems with Poisson commuting integrals*, Comment. Math. Helv. **65** (1990), 4–35.

[51] A. G. Kushnirenko, *Analytic action of a semisimple Lie group near fixed point is equivalent to a linear action*, Funktsional. Anal. i Prilozhen. **1** (1967), no. 1, 89–90; English transl. in Functional Anal. Appl. **1** (1967).

[52] T. Poston and I. Stewart, *Catastrophe theory and its applications*, Pitman, London–San Francisco–Melbourn, 1978.

[53] G. Seifert and W. Trelfall, *Lehrbuch der Topologie*, Teubner, Leipzig, 1934.

[54] V. S. Medvedev and Ya. L. Umanskii, *On decomposition of n-dimensional manifolds into simple manifolds*, Izv. Vyssh. Uchebn. Zav. Mat. **1979**, no. 1, 46–50; English transl. in Soviet Math. (Iz. Vuz.) **23** (1979).

[55] G. M. Goluzin, *Geometric theory of functions of a complex variable*, "Nauka", Moscow, 1966; English transl., Amer. Math. Soc., Providence, RI, 1969.

[56] A. G. Sokolskii, *On stability of autonomous Hamiltonian systems with two degrees of freedom in the case of equal frequencies*, Prikl. Mat. Mekh. **38** (1974), no. 5, 791–799; English transl. in J. Appl. Math. Mech. **38** (1974).

[57] ______, *On stability of autonomous Hamiltonian systems with two degrees of freedom in the case of first order resonance*, Prikl. Mat. Mekh. **41** (1977), no. 1, 24–33; English transl. in J. Appl. Math. Mech. **41** (1977).

[58] J.-C. Van der Meer, *The Hamiltonian Hopf bifurcation*, Lecture Notes in Math., vol. 1160, Springer-Verlag, Berlin, 1985.

[59] J. T. Schwartz, *Differential geometry and topology*, New York, 1968.

[60] S. Smale, *On gradient dynamical systems*, Ann. Math. **74** (1961), 199–206.

[61] M. W. Hirsch, C. C. Pugh, and M.Shub, *Invariant manifolds*, Lecture Notes in Math., vol. 583, Springer-Verlag, Berlin, 1977.

[62] A. N. Shoshitaishvili, *Bifurcations of the topological type of vector field near a singular point*, Trudy Sem. Petrovsk., Vyp. 1 (1975), 279–309. (Russian)

[63] C. Neumann, *De problemate quodam mechanico, quod ad primam integralium ultraelliptico-rum classem revocatur*, J. Reine. Angew. Math. **56** (1858), 46–63.

[64] M. P. Kharlamov, *Topological analysis of integrable problems in dynamics of solid body*, Leningrad. Gos. Univ., Leningrad, 1988. (Russian)

[65] R. Devaney, *Transversal homoclinic orbits in an integrable system*, Amer. J. Math. **100** (1979), 632–642.

[66] L. M. Lerman, *Once again on the structure of integrable stationary waves for the Landau–Lifshitz equation*, Selecta Math. Sovietica **12** (1993), 333–351.

[67] A. I. Bobenko, *Explicit integration of nonlinear equations by the inverse problem method with the parameter on an elliptic curve*, Ph. D. Thesis, Leningrad, 1985.

[68] P. J. Hilton and S. Wylie, *Homology theory. An introduction to algebraic topology*, Cambridge Univ. Press, Cambridge, 1960.

[69] A. V. Bolsinov, A. T. Fomenko, and S. V. Matveev, *Topological classification of the integrable Hamiltonian systems with two degrees of freedom. The list of systems of little complexity*, Uspekhi Mat. Nauk **45** (1990), no. 2, 49–77; English transl. in Russian Math. Surveys **45** (1990)

[70] V. F. Lazutkin, *KAM-theory and semi-classical approximation to eigenfunctions*, Springer-Verlag, New York, 1993.

[71] A. T. Fomenko and D. B. Fuks, *A course in homotopic topology*, "Nauka", Moscow, 1989. (Russian)

[72] D. M. Galin, *Versal deformations of linear Hamiltonian systems*, Trudy Sem. Petrovsk., Vyp. 1 (1975), 63–74. (Russian)

[73] L. M. Lerman and Ya. L. Umanskii, *Classification of four-dimensional Hamiltonian systems and Poisson action of* $\mathbb{R}^2$ *in extended neighborhoods of simple singular points.* 3. *Realization*, Mat. Sb. **186** (1995), no. 10, 89–102; English transl. in Russian Acad. Sci. Sb. Math. **80** (1996).

This page intentionally left blank

Selected Titles in This Series

(Continued from the front of this publication)

141 **A. A. Davydov,** Qualitative theory of control systems, 1994
140 **Aizik I. Volpert, Vitaly A. Volpert, and Vladimir A. Volpert,** Traveling wave solutions of parabolic systems, 1994
139 **I. V. Skrypnik,** Methods for analysis of nonlinear elliptic boundary value problems, 1994
138 **Yu. P. Razmyslov,** Identities of algebras and their representations, 1994
137 **F. I. Karpelevich and A. Ya. Kreinin,** Heavy traffic limits for multiphase queues, 1994
136 **Masayoshi Miyanishi,** Algebraic geometry, 1994
135 **Masaru Takeuchi,** Modern spherical functions, 1994
134 **V. V. Prasolov,** Problems and theorems in linear algebra, 1994
133 **P. I. Naumkin and I. A. Shishmarev,** Nonlinear nonlocal equations in the theory of waves, 1994
132 **Hajime Urakawa,** Calculus of variations and harmonic maps, 1993
131 **V. V. Sharko,** Functions on manifolds: Algebraic and topological aspects, 1993
130 **V. V. Vershinin,** Cobordisms and spectral sequences, 1993
129 **Mitsuo Morimoto,** An introduction to Sato's hyperfunctions, 1993
128 **V. P. Orevkov,** Complexity of proofs and their transformations in axiomatic theories, 1993
127 **F. L. Zak,** Tangents and secants of algebraic varieties, 1993
126 **M. L. Agranovskiĭ,** Invariant function spaces on homogeneous manifolds of Lie groups and applications, 1993
125 **Masayoshi Nagata,** Theory of commutative fields, 1993
124 **Masahisa Adachi,** Embeddings and immersions, 1993
123 **M. A. Akivis and B. A. Rosenfeld,** Élie Cartan (1869–1951), 1993
122 **Zhang Guan-Hou,** Theory of entire and meromorphic functions: deficient and asymptotic values and singular directions, 1993
121 **I. B. Fesenko and S. V. Vostokov,** Local fields and their extensions: A constructive approach, 1993
120 **Takeyuki Hida and Masuyuki Hitsuda,** Gaussian processes, 1993
119 **M. V. Karasev and V. P. Maslov,** Nonlinear Poisson brackets. Geometry and quantization, 1993
118 **Kenkichi Iwasawa,** Algebraic functions, 1993
117 **Boris Zilber,** Uncountably categorical theories, 1993
116 **G. M. Fel′dman,** Arithmetic of probability distributions, and characterization problems on abelian groups, 1993
115 **Nikolai V. Ivanov,** Subgroups of Teichmüller modular groups, 1992
114 **Seizô Itô,** Diffusion equations, 1992
113 **Michail Zhitomirskiĭ,** Typical singularities of differential 1-forms and Pfaffian equations, 1992
112 **S. A. Lomov,** Introduction to the general theory of singular perturbations, 1992
111 **Simon Gindikin,** Tube domains and the Cauchy problem, 1992
110 **B. V. Shabat,** Introduction to complex analysis Part II. Functions of several variables, 1992
109 **Isao Miyadera,** Nonlinear semigroups, 1992
108 **Takeo Yokonuma,** Tensor spaces and exterior algebra, 1992
107 **B. M. Makarov, M. G. Goluzina, A. A. Lodkin, and A. N. Podkorytov,** Selected problems in real analysis, 1992
106 **G.-C. Wen,** Conformal mappings and boundary value problems, 1992
105 **D. R. Yafaev,** Mathematical scattering theory: General theory, 1992

(See the AMS catalog for earlier titles)

ISBN 0-8218-0375-1
9 780821 803752