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Satoru Igari

Real Analysis— With an Introduction to Wavelet Theory

American Mathematical Society

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Real Analysis—
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Satoru Igari

Real Analysis— With an Introduction to Wavelet Theory

American Mathematical Society
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実解析入門

JIKKAISEKI NYŪMON
(An Introduction to Real Analysis)

by Satoru Igari

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ABSTRACT. This introduction to real analysis is based on the author's lectures at Tohoku University. It begins with a thorough discussion of Lebesgue measure and Lebesgue integration, and continues with the basics of distribution theory and Fourier analysis. The final chapter is devoted to wavelet theory.

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Preface

The reader may be a little confused at the words "Real Analysis" in the title of the book. Indeed, real analysis, like harmonic analysis, has a lot of contents nowadays, and the author cannot definitively state what are the subjects that the real analysis deals with.

I suppose that real analysis meant, at first, the field which corresponded to complex analysis and aimed to analyse and synthesize sets and functions on the real axis or the plane. However, complex variable methods have been deeply incorporated into the Fourier analysis which has been one of the great backbones of real analysis. Also the problems, the methods, and the connections of real analysis to other fields have changed drastically. As a consequence, the objects dealt with by real analysis have been diversified and have not always been limited to sets or functions on the Euclidean plane.

Real analysis is based on the real numbers, and it is naturally involved with practical mathematics. On the other hand, it has taken on subject matter from set theory, harmonic analysis, integration theory, probability theory, complex analysis, the theory of partial differential equations, and so on, and has provided these theories with important ideas and basic concepts. Such relations continue even today.

This book, with these backgrounds as its setting, contains the basic matter of "real analysis".

It is natural, in view of the object of this book, to introduce the definition of the real numbers, which is done by just mentioning characteristic properties.

The statements about sets and plane topology will be kept to an irreducible minimum.

Measurability of sets is one of the fundamental notions of Lebesgue integration theory. To define measurability, we apply the Carathéodory condition, although this method is not always intuitive. One of the merits of the choice is that it enables us to unify the measures on the plane and on general sets. Moreover, the Carathéodory condition is used to simplify the proof of the theorem on representation of linear functionals by measures.

The existence of Haar measure and integration on topological groups are not covered. For them, the reader may refer to the bibliography in the back of the book.

The basic theory of distributions and Fourier analysis is covered in chapters 7 and 8.

The last chapter is allotted to an introduction to wavelet theory. The theory of wavelets was born at the beginning of the 1980's from a practical purpose and, coincidentaly, a purely mathematical motive. At present, the lucid theory is constructed by an effective application of Fourier analysis.

Wavelet theory is also important in Fourier analysis. Roughly speaking, it enables us to treat a function and its Fourier transform more easily at the same time. Wavelet theory is now a useful tool of real analysis, like distribution theory.

I would like to thank Professors Hitoshi Arai and Kazuya Tachizawa for useful remarks and making graphs.

I thank all the staff of Iwanami Shoten. Espcially I would like to thank Mr. Hideo Arai, Ms. Mamiko Hamamo and Mr. Uichi Yoshida for generous help in preparing the manuscript.

Satoru Igari
Sendai
1996

Preface to the English Edition

The first mathematical textbook which I read was "Measure Theory", by P. R. Halmos.

Even now I remember clearly the polished exposition, so neatly put in order.

I have benefited, since then, from many books published in Eurpopean languages, and I have had the opportunity to make deep friendships with excellent mathematicians. Especially, I have spent a delightful time with the professors and the students of the University of Wisconsin in Madison.

Integration theory, distribution theory, and Fourier analysis are still basic subjects for all students who study analysis. Also, the theory of wavelets is becoming a new basic. I hope that this book will give students good training in these basics.

I thank Professor Katsumi Nomizu, who gave useful advice in translation. I am full of gratitude to the kindness of Ralph Sizer, who read through the manuscript and gave me polite and pertinent comments, and correction of English.

Finally, I thank the American Mathematical Society for giving me the opportunity to publish this book in English, and the staff there, especially Christine M. Thivierge and Vickie Ancona, for their kind assistance.

Satoru Igari
Sendai
1997

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Solutions to Problems

Chapter 1

1.1. Condition (O1) is trivial. Let $A_\alpha^i = A_\alpha$, and $A = \bigcup A_\alpha$. If $x \in A$, then $x \in A_\alpha$ for some α. Thus we can choose $\varepsilon > 0$ so small that $B(x,\varepsilon) \subset A_\alpha$. Thus $B(x,\varepsilon) \subset A$, which implies (O2). We have $(A_\alpha \cap A_\beta)^i \subset A_\alpha^i, A_\beta^i$, from which it follows that $(A_\alpha \cap A_\beta)^i \subset A_\alpha^i \cap A_\beta^i$. On the other hand, if $x \in A_\alpha^i \cap A_\beta^i$, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset A_\alpha$ and $B(x,\varepsilon) \subset A_\beta$. This implies that x is an inner point of $A_\alpha \cap A_\beta$. Thus $A_\alpha^i \cap A_\beta^i \subset (A_\alpha \cap A_\beta)^i$.

1.2. Suppose that the unit disc $B(0,1)$ is a countable union of mutually disjoint open intervals $\{I_j\}$. Denote $\{(a_1,0) : -1 < a_1 < 1\} = (-1,1)$ for simplicity. Then we have $(-1,1) \cap \bigcup_{j=1}^\infty I_j = (-1,1)$. The right-hand side is the countable union of mutually disjoint open intervals in $\mathbb{R}$. Thus $(-1,1) \cap I_j = (-1,1)$ for some I_j, which is absurd. In fact, pick a rectangle I_j so that $(a_1,b_1) = (-1,1) \cap I_j \neq \emptyset$. Since $a = (a_1,0)$ is not contained in the open interval I_j, a is an inner point of I_k for some $k \neq j$. Then $I_j \cap I_k \neq \emptyset$.

1.3. (i) $y \in \varphi(\bigcup A_\alpha) \Leftrightarrow \exists x \in \bigcup A_\alpha; y = \varphi(x) \Leftrightarrow \exists x \& \alpha_0; x \in A_{\alpha_0}$ and $y = \varphi(x) \Leftrightarrow y \in \bigcup \varphi(A_\alpha)$. The other cases are proved similarly.

1.4. Assume that $A' = A \neq \emptyset$. Pick a point a in A. For every $\varepsilon > 0$ the set $A \cap B(a,\varepsilon)$ contains infinitely many points. Fix an $\varepsilon > 0$ and choose two distinct points a_0 and a_1 in $A \cap B(a,\varepsilon)$. Take $\varepsilon/2 > \varepsilon' > 0$, so that $B(a_0,\varepsilon')$ and $B(a_1,\varepsilon')$ are mutually disjoint. Next pick two distinct points $a_{0,0}, a_{0,1}$ in $A \cap B(a_0,\varepsilon')$ and two distinct points $a_{1,0}, a_{1,1}$ in $A \cap B(a_1,\varepsilon')$. In the same way we get sequences $\{a_{i_1,i_2\dots,i_k}; i_k = 0,1\}$. Let $\lim_{k\to\infty} a_{i_1,\dots,i_k} = a_{i_1,i_2,\dots}$. Then $a_{i_1,i_2,\dots}$ ($i_j = 0$ or 1) are distinct points in A. Remark that $a_{i_1,i_2,\dots} \mapsto \sum_{k=1}^\infty i_k/2^k$ is a mapping of a subset of A onto $[0,1]$.

1.5. We can assume that F_1 is a compact set. By Theorem 1.14 the intersection $\bigcap F_j$ is nonempty and does not contain any distinct points by assumption.

1.6. Let I be a bounded closed cube of $\mathbb{R}^d$ which contains $\{a_j\}$. Divide I into 2^d congruent closed cubes. One of these cubes, say I_1, contains infinitely many points of $\{a_j\}$, which we denote by $\{a_{1,k}\}$. Next, divide I_1 into 2^d congruent cubes. One of these cubes, say I_2, contains infinitely many points of $\{a_{1,j}\}$, which we denote by $\{a_{2,l}\}$. Continuing this process, we get a decreasing sequence of closed cubes I_k and an infinite subsequence $\{a_{k,n}\}$ of $\{a_j\}$. Observe that $a_{k,k} \in I_k$ and $\{a_{k,k} : k = 1,2,\dots\}$ is a convergent sequence.

1.7. Let $\varepsilon > 0$ be given. Take $R > 0$ so that $|f(x)| < \varepsilon/2$ if $|x| > R-1$. Then we have $|f(x) - f(y)| < \varepsilon$ if $|x| > R-1$. Since a continuous function on a compact set is uniformly continuous, there exists $1 > \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $|x-y| < \delta$ and $|x|, |y| \leq R$. We remark that if $|x-y| < \delta$, then both $|x|$ and $|y|$ are larger than $R-1$ or smaller than R.

1.8. Assume that f is nondecreasing in $[a,b]$. Let $\Delta = \{a = a_0 < a_1 < \cdots < a_N = b\}$ be a partition of $[a,b]$, and put $I_j = [a_{j-1}, a_j]$. Since $\overline{f}_j = f(a_j), \underline{f}_j = f(a_{j-1})$, we have

$$\sum_{j=1}^{N}(\overline{f}_j - \underline{f}_j) = \sum_j [f(b_j) - f(a_j)] = f(b) - f(a). \qquad (*)$$

We have $0 \le \overline{s}(f,\Delta) - \underline{s}(f,\Delta) = \sum_{j=1}^{N}[\overline{f}_j - \underline{f}_j]|I_j| \le \sum_j [\overline{f}_j - \underline{f}_j]\sup_j |I_j|$. The last term is not greater than $[f(b) - f(a)]\sup_j |I_j|$ by (*).

1.9. (i) It is obvious that $1 < m_{J*}(O)$. Let $J_1, \ldots, J_k$ be mutually disjoint right half-open intervals contained in O. Let $J_j' \subset J_j$ be an arbitrary closed subinterval and put $J' = \bigcup_{j=1}^{k} J_j'$. Since J' is compact, it is covered by a finite number of open intervals $I_1, \ldots, I_l$. Thus $\sum_{j=1}^{k}|J_j'| \le \sum_{i=1}^{l}|I_i| < \sum_{i=1}^{\infty} 2/2^i = 2$. Thus $m_{J*}(O) \le 2$.

(ii) Since $\{r_j\}$ is dense in $\mathbb{R}$, the set O is dense in $\mathbb{R}$. This implies that if a finite number of open intervals cover O, then these open intervals cover the whole axis.

Chapter 2

2.1. $\chi_{\limsup E_j}(x) = 1 \Leftrightarrow x \in \limsup E_j \Leftrightarrow$ for any k we have $x \in E_j$ for infinitely many $j \ge k \Leftrightarrow$ for any k we have $\chi_{E_j}(x) = 1$ for infinitely many $j \ge k \Leftrightarrow \limsup \chi_{E_j}(x) = 1$.

2.2. Since $F = \limsup_{j\to\infty} E_j \subset \bigcup_{j=k}^{\infty} E_j$, we have $m(F) \le \sum_{j=k}^{\infty} m(E_j) \to 0$ as $k \to 0$.

2.3. If E is an interval, (i) and (ii) are obvious. Since open sets are countable unions of mutually disjoint right half-open intervals, they hold good for open sets. When E is a measurable set, observe that $m(E) = \inf\{m(O) : E \subset O \in \mathcal{O}\}$.

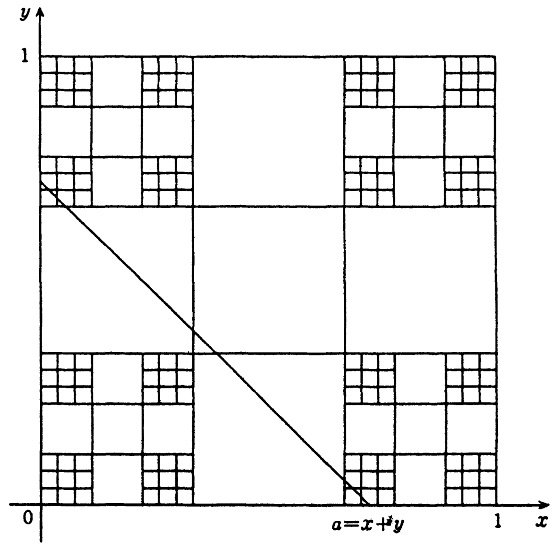
2.4. Note that $m(E) = m(E \cap F) + m(E - F)$ and $m(E - F) \le m([0,1] - F) = 1 - 1 = 0$.

2.5. When E is bounded, the proof follows from Theorem 2.8 and the definition of the inner measure. For a general set E put $E_R = E \cap B(O,R)$. Then $m(E) \ge \sup\{m(C) : C \subset E\} \ge \sup\{m(C) : C \subset E_R\} = m(E_R) \to m(E)$ as $R \to \infty$.

2.6. From the square $[0,1] \times [0,1]$ remove a cross-type open set $\{(x,y) : 1/3 < x < 2/3\} \cup \{(x,y) : 1/3 < y < 2/3\}$, getting the closed set $\tilde{C}_1$. Thus $\tilde{C}_1$ consists of 4 closed squares. In the next step remove 1/3 cross-type open sets from each of the 4 squares. The remainder consists of 4^2 closed squares. Denote it by $\tilde{C}_2$. In the same way we get the closed set $\tilde{C}_n$ at the nth step.

Define $\tilde{C} = \bigcap_{n=1}^{\infty} \tilde{C}_n$. Then $(x,y) \in \tilde{C} \Leftrightarrow x \in C$ and $y \in C$, where C denotes the Cantor ternary set. If $0 \le a \le 2$, the line $a = x + y$ intersects $\tilde{C}_1, \tilde{C}_2, \ldots$. Thus by the finite intersection property it intersects $\tilde{C}$, which means that $a = x + y$ for some $(x,y) \in \tilde{C}$.

2.7. If f is continuous at a point a, then f is continuous at any point x, because $f(x+h) = f(x+a) - f(a-h)$ by the hypothesis. For positive integers n and m we have $f(n) = f(1 + \cdots + 1) = nf(1)$ and $f(1) = f(1/m + \cdots + 1/m)(n \text{ sum}) = mf(1/m)$. Thus $f(n/m) = f(1/m + \cdots + 1/m)(n \text{ sum}) = nf(1/m) = (n/m)f(1)$. If $x \ge 0$, choose positive integers m and n so that $n/m \to x$. Then we get $f(x) = xf(1)$ by the continuity. For a negative x use $f(-x) + f(x) = 0$.



Chapter 3

3.1. Let $f = u + iv$ and $g = u' + iv'$. (i) is obvious. (ii) Observe that $f/g = (uu' + vv')/(u'^2 + v'^2) + i(uv' + vu')/(u'^2 + v'^2)$ and apply the real valued function case. To prove (iii), apply (iii) for real valued functions to the function $f \cdot \overline{f}$.

3.2. Put $f_n(x) = (nx)/[1 + (nx)^2]$. Then $f_n \in C[0,1]$. For each $x \in [0,1]$, $f_n(x) \to 0$ as $n \to \infty$, but $f_n(1/n) = 1/2$.

3.3. Let E be a nonmeasurable set. Then $f = \chi_E - \chi_{E^c}$ is not measurable, but $|f| \equiv 1$.

3.4. $\{x : \sup f_\alpha(x) > a\} = \bigcup_\alpha \{x : f_\alpha(x) > a\}$. The right-hand side is an open set.

3.5. Let $E_n = \{x : f(x) > 1/n\}$. By Chebyshev's inequality

$$m(E_n) \le n \int f dm = 0, \quad n = 1, 2, \dots .$$

Now observe that $\{x : f(x) \neq 0\} = \bigcup_{n=1}^{\infty} E_n$.

3.6. Apply Fatou's lemma to the sequence $F_j = |f| - f_j$.

3.7. Consider the functions $-f_j$, where the f_j are the functions given in Remark 3.2.

3.8. (i) Put $f_j(x) = \chi_{E_j}(x)$ and apply Fatou's lemma. To prove (ii), apply (i) to $E - E_j$. A counterexample for (iii) is given by the set $E_j = (j, j+1)$. In fact, we have $\limsup E_j = \emptyset$.

3.9. Observe that

$$\sum_{k=-\infty}^{\infty} 2^k \chi_{E_k}(x) \le \sum_{k=-\infty}^{\infty} |f(x)| \chi_{E_k}(x) \le \sum_{k=-\infty}^{\infty} 2^{k+1} \chi_{E_k}(x).$$

3.10. (i) For a given $\varepsilon > 0$ choose a simple function $s = \sum_{n=1}^{N} c_n \chi_{F_n}$ so that $\int |f - s| dm < \varepsilon$. Thus $\left| \int_{E_j} (f - s) dm \right| < \varepsilon$ for all j. On the other hand, $\int_{E_j} s dm = \sum_{n=1}^{N} c_n m(F_n \cap E_j) \to 0 \, (j \to \infty)$.

(ii) Suppose $t_j \to t$. Let $E_j = (\min(t, t_j), \max(t, t_j))$. Then $|F(t_j) - F(t)| \le \int_{E_j} |f| dm \to 0$. A proof for the case where $t \to -\infty$ is similar.

3.11. For a given $\varepsilon > 0$ there exists a simple function $s(x) = \sum_{j=1}^{N} a_j \chi_{E_j}(x)$ such that $\int |f - s| dm < \varepsilon$, where the I_j are bounded intervals. Note that if $|y|$ is large enough, $\int |s(x+y) - s(x)| dm(x) = \int |s(x+y)| dm(x) + \int |s(x)| dm(x)$.

3.12. Apply Lebesgue's dominated convergence theorem to $F_n(x) = x^n f(x)$.

3.13. We have $|f(y)\varphi(x-y)| \le |f(y)| \|\varphi\|_\infty \in L^1(\mathbb{R})$. Thus

$$|F(x)| \le \int |f(y)| dm \|\varphi\|_\infty.$$

By the dominated convergence theorem and by the continuity of φ we have

$$\lim_{x \to a} \int f(y)\varphi(x-y) dm(y) = \int \lim_{x \to a} f(y)\varphi(x-y) dm(y) = \int f(y)\varphi(a-y) dm(y).$$

3.14. We have

$$[F(x+h) - F(x)]/h = \int_{-\infty}^{\infty} f(y)[\varphi(x+h-y) - \varphi(x-y)]/h dm(y). \qquad (*)$$

By the mean value theorem $[\varphi(x+h-y) - \varphi(x-y)]/h = \partial\varphi(x+\theta h - y)/\partial x$, where $0 < \theta < 1$. Since $d\varphi(x)/dx$ is bounded and continuous, the dominated convergence theorem can be applied to (*).

3.15. We have

$$\int f(x) \frac{e^{-2\pi i \eta x} - 1}{\eta} e^{2\pi i \xi x} dm(x) = -2\pi i \int f(x) \frac{\sin \pi \eta x}{\pi \eta} e^{-\pi i \eta x} e^{-2\pi i \xi x} dm(x).$$

Observe that the absolute value of the integrand of the last integral is $\le |xf(x)|$, which is integrable, and apply the dominated convergence theorem.

3.16. By the Taylor expansion, $x^p/(1-x)\log(1/x) = \sum_{n=0}^{\infty} x^{n+p} \log(1/x)$. Since each term on the right-hand side is positive in $(0,1)$, we can integrate the sum term by term by B. Levi's theorem. Furthermore, integration by parts yields

$$\int_0^1 x^{p+n} \log(1/x) dx = [x^{p+n+1} \log(1/x)/(p+n+1)]_0^1 + \int_0^1 x^{p+n} dx/(p+n+1).$$

3.17. $\int_0^{\infty} u_n dm = 0$, and $\int_0^{\infty} [e^{-x}/(1+e^{-x})] dx = \log 2$.

Chapter 4

4.1. Let E be a nonmeasurable set. Put $f(x) = 1\ (x \in E)$, and $= -1\ (x \notin E)$. If $\varphi(x) = |x|$, then $\varphi(f) \equiv 1$.

4.2. $f^{-1}(A)$ consists of countable middle third intervals and a subset of the Cantor ternary set which is null.

4.3. First check that the series coincides with the Cantor singular function in each middle third interval, and then observe that both the sum and the Cantor singular function are nondecreasing.

4.4. For $\delta > 0$ the function $F_\delta(x) = \sup_{0<h<\delta}[f(x+h) - f(x)]/h$ is lower semicontinuous. Thus $\{x : D^+ f(x) \le a\} = \bigcap_{n=1}^{\infty} \{x : F_{1/n}(x) \le a\}$ is measurable.

4.5. For any partition of $[a,b]$ with points of division $\{a_j\}$ we have

$$\sum |f(a_j) - f(a_{j-1})| \le C \sum |a_j - a_{j-1}| = C(b-a).$$

Thus f is of bounded variation. Furthermore, every function of bounded variation is a linear combination of four nondecreasing functions (see Corollary 4.1).

Chapter 5

5.1. See Problem 1.2 for (i) and (ii). (iii) Let $\mathcal{X} = \{\emptyset, [-1,1), [-1,0), [0,1)\}$ and $f(x) = x^2$. Then $f(\mathcal{X})$ is not a ring of sets.

5.2. Let $E \in \mathcal{M}(\mathbb{R})$. Put $B = \{y \in Y : g(y) \in E\}$. Then $B \in \mathcal{Y}$. Therefore $\{x \in X : g(T(x)) \in E\} = \{x \in X : T(x) \in B\} \in \mathcal{X}$.

5.3. Check the definition of measure.

5.4. Observe that $\rho(E, F) = \int_X |\chi_E(x) - \chi_F(x)| d\mu(x)$.

5.5. When f is a simple function on $\mathbb{N}$, the result is a direct consequence of the definition. The general case is obtained by passing to the limit.

5.6. f_k is uniquely given by $f_k(x) = 2^k \int_{I_j} f(t)dt \quad (x \in I_j)$.

5.7. The set of points x such that $\mu(\{x\}) \neq 0$ is countable.

5.8. We can assume that μ_j $(j = 1, 2, 3)$ are nonnegative without losing generality. If N is a μ_3-null set, then it is μ_2-null, so also μ_1-null. Put $\frac{d\mu_1}{d\mu_2} = f$ and $\frac{d\mu_2}{d\mu_3} = g$. Then $f, g \geq 0$. For a simple function $s = \sum a_j \chi_{E_j}$, we have $\int_E s d\mu_2 = \sum_j a_j \mu_2(E \cap E_j) = \sum_j a_j \int_{E \cap E_j} g d\mu_3 = \int_E (\sum_j a_j \chi_{E_j}) g d\mu_3$. Replace s by simple functions s_n such that $0 \leq s_1 \leq s_2 \leq \cdots \to f$ and let $n \to \infty$. Then $\int_E f d\mu_2 = \lim \int_E s_n d\mu_2 = \lim \int_E s_n g d\mu_3 = \int_E f g d\mu_3$. Since $\mu_1(E) = \int_E f d\mu_2$, it follows that $\mu_1(E) = \int_E f g d\mu_3$.

5.9. Observe that $\mathrm{diam} f(B(a, r)) \leq C(2r)^\alpha$ and apply the definition of the Hausdorff dimension.

5.10. Cf. Example 5.11. The Hausdorff dimension of D is 2.

Chapter 6

6.1. Apply Hölder's inequality with the indices $1/(q/p) + 1/[q/(q-p)] = 1$. Then $\int |f|^p d\mu \leq \left(\int |f|^q d\mu\right)^{p/q} \left(\int 1 d\mu\right)^{(q-p)/q}$. Put $f(x) = x^{-1/p}$ for $x > 1, = 0$ otherwise.

6.2. Let $p \neq 2$. Check the parallelogram law for the functions $f = \chi_{[0,1]}$ and $g = \chi_{[2,3]}$.

6.3. $\left|\int (f - f_n) g d\mu\right| \leq \left(\int |f - f_n|^p d\mu\right)^{1/p} \left(\int |g|^{p'} d\mu\right)^{1/p'}$.

6.4. The first part follows from the inequality $|\|f\|_p - \|g\|_p| \leq \|f - g\|_p$.

To prove the second part, notice that $\int_F |f|^p d\mu \to 0$ if $\mu(F) \to 0$. Put $E(n) = \{x : 2|f(x)| \geq |f_n(x)|\}$ and $g_n = f_n \chi_{E(n)}$. Then $g_n \to f$ a.e. and $|g_n| \leq 2|f|$. Thus $\int |g_n - f|^p \to 0$, and $\int |f_n - g_n|^p = \int |f_n|^p - \int |f_n|^p \chi_{E(n)} = \int |f_n|^p - \int |g_n|^p \to \int |f|^p - \int |f|^p = 0$.

6.5. Put $E = \{x : |f(x)| > M\}$. By the assumption that $\|f\|_q < \infty$, $\mu(E) < \infty$. If $\mu(E) > 0$, then $\|f\|_p \geq \left(\int_E M^p d\mu\right)^{1/p} \to M$ as $p \to \infty$. On the other hand, $\|f\|_p \leq \left(\int_X |f(x)|^q \|f\|_\infty^{p-q} d\mu\right)^{1/p} \to \|f\|_\infty$.

6.6. We observe that

$$\int_Y \left|\int_X f(x, y) d\mu(x)\right|^p d\nu(y) \leq \int_Y \int_X \left(\int_X |f(x', y)| d\mu(x')\right)^{p-1} |f(x, y)| d\mu(x) d\nu(y).$$

Change the order of integration and apply Hörder's inequality to the integral over Y with indices $1/[p/(p-1)] + 1/p = 1$.

6.7. The case $n = 3$. We may suppose that $f_1, f_2, f_3 \geq 0$. If

$$1/q_1 = 1/[p_1(1 - 1/p_3)] \text{ and } 1/q_2 = 1/[p_2(1 - 1/p_3)],$$

then $1/q_1 + 1/q_2 = 1$. Apply Hölder's inequality with indices $1/q_1 + 1/q_2 = 1$ to

$$\int f_1 f_2 f_3 d\mu = \int (f_1 f_3^{1/q_1})(f_2 f_3^{1/q_2}) d\mu,$$

and then, with indices $1/p_3 + 1/(p_j/q_j) = 1$, to $\int f_3 f_j^{q_j} d\mu\,(j = 1,2)$.

6.8. We may suppose that $f, g \geq 0$. Observe that

$$f * g(x) = \int f(y)^{p/r} g(x-y)^{q/r} f(y)^{p[(1/p)-(1/r)]} g(x-y)^{q[(1/q)-(1/r)]} dy.$$

If $1/s = 1/p - 1/r$ and $1/t = 1/q - 1/r$, then $1/r + 1/s + 1/t = 1$. Apply Hölder's inequality.

6.9. (i) Consider the family of simple functions $s = \sum_{j=1}^n a_j \chi_{I_j}$, where the real and imaginary parts of a_j and the end points of the interval I_j are all rational. The family of such simple functions is dense in L^p and countable. (ii) Let D be a dense subset of $L^\infty(0,1)$. Put $f_t(x) = \chi_{[0,t]}(x)$ for $0 < t < 1$. For each $0 < t < 1$ pick a function $u^t \in L^\infty(0,1)$ such that $\|f_t - u^t\|_\infty < 1/2$. Since $u^s \neq u^t$ if $s \neq t$, D is not countable.

Chapter 7

7.1. We have $|\langle T_\mu, u\rangle| \leq p_{0,K}(u)|\mu|(K)$, where $K = \operatorname{supp} u$.

7.2. See Theorem 6.13.

7.3. $\langle \Delta U, u\rangle = \langle T, \langle -(4\pi)^{-1}\Delta |x|^{-1}, \tau_{(-\cdot)} u\rangle\rangle$.

7.4. (i) δ. (ii) $-|\sin t| + 2\sum_{n=-\infty}^{\infty} \delta_{2\pi n}$.

7.5. By the definition of differentiation, $\langle D^\alpha(\varphi T), u\rangle = (-1)^{\|\alpha\|}\langle T, \varphi D^\alpha u\rangle$ and $\left\langle \sum_{\beta\leq\alpha} \binom{\alpha}{\beta}(D^{\alpha-\beta}\varphi)(D^\beta T), u\right\rangle = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(-1)^{\|\beta\|}\left\langle T, D^\beta((D^{\alpha-\beta}\varphi)u)\right\rangle$. Observe that $\sum_{\beta\leq\alpha}(-1)^{\|\beta\|}\binom{\alpha}{\beta}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}(D^{\alpha-\gamma}\varphi)D^\gamma u = \varphi D_u^\alpha$.

7.6. Apply the formula (7.3).

Chapter 8

8.1. The residue of $e^{-itz}/(1+z^2)$ at the point $z = -i$ is $e^{-t}/2i$. Integrate $e^{-itz}/(1+z^2)$ on the curve $\{x : -R \leq x \leq R\} \cup \{Re^{i\theta} : \pi \leq \theta < 2\pi\}$ to get $ie^{-t}/2i = (2\pi i)^{-1}\int_R^{-R} e^{-itx}/(1+x^2)dx + (2\pi i)^{-1}\int_\pi^{2\pi} e^{-itR[\cos\theta + i\sin\theta]} \times iRe^{i\theta}/(1+R^2e^{2i\theta})d\theta$. The second term on the right-hand side converges to 0 as $R \to \infty$.

8.2. (i) $\widehat{x^\alpha} = (2\pi\xi)^{-\|\alpha\|}D^\alpha\delta$. (ii) $\widehat{Y}(\xi) = (1/2)\delta + (1/2\pi i)\text{P.V.}(1/\xi)$. (iii) $1/(2\pi i\xi + 1)$.

8.3. A direct computation yields

$$\begin{aligned}\sum_{j,k} c_j\overline{c_k}\widehat{\mu}(\xi_j - \xi_k) &= \int_{\mathbb{R}}\sum_{j,k} c_j\overline{c_k}e^{-2\pi i\xi_j x + 2\pi i\xi_k x}d\mu(x)\\ &= \int_{\mathbb{R}}|\sum_j c_j e^{-2\pi i\xi_j x}|^2 d\mu(x) \geq 0.\end{aligned}$$

8.4. Suppose that there exists a unit e in L^1. Then $\widehat{e * P_t}(\xi) = \widehat{e}(\xi)\widehat{P_t}(\xi) = \widehat{P_t}(\xi)$. Thus $\widehat{e}(\xi) \equiv 1$. On the other hand, by the Riemann-Lebesgue theorem we have $\widehat{e}(\xi) \to 0\,(\xi \to \infty)$.

8.5. We have $\tilde{x} = m$. Thus by translation we can assume $m = 0$. We have $\int_{-\infty}^\infty f(x)^2 dx = c^2(\pi/2a)^{1/2}$ and $\int_{-\infty}^\infty x^2 f(x)^2 dx = c^2(1/2)(d/da)\int e^{-2ax^2}dx = c^2(\pi/2a)^{1/2} \times (1/4a)$. Thus $\Delta_f^2 = 1/4a$. Since $\widehat{f}(\xi) = c^2\sqrt{\pi/a}e^{-(\xi-b)^2\pi^2/a}$, we get $\Delta_{\widehat{f}}^2 = a/4\pi^2$.

8.6. Let $0 < t < 1$ and $|x| < 1/2$. (s1): Integrate w_t term by term. (s2)$'$: Let $n \neq 0$. Since $(x+n)^2/4t > n^2/16t$, we have $e^{-(x+n)^2/4t} < 16t/n^2$, and thus $w_t(x) \leq (4\pi t)^{-1/2}(1 + \sum_{n\neq 0} t/n^2) < (\text{constant}) \times t^{-1/2}$. (s3)$'$: If $x^2/t > 1$, then

$e^{-x^2/4t} < 4t/x^2$. Thus $w_t(x) \leq (4\pi t)^{-1/2}[(4t/x^2) + \sum_{n\neq 0} t/n^2] <$ (constant) $\times$ $t^{1/2}/x^2$. Therefore we get (s2)$'$ and (s3)$'$ with $t^{1/2}$ in place of t.

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