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Volume 178

Analytic Functionals on the Sphere

Mitsuo Morimoto

American Mathematical Society

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Volume 178

Analytic Functionals on the Sphere

Mitsuo Morimoto

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ABSTRACT. This book treats, first of all, spherical harmonic expansion of real-analytic functions and hyperfunctions on the (real) sphere. To study this, we construct a system of good complex neighborhoods of the sphere in the complex sphere by means of the Lie norm, and consider holomorphic functions and analytic functionals on the complex neighborhoods. The book then treats harmonic functions on the Euclidean ball and complex harmonic functions on the Lie ball. It also discusses the Fourier-Borel transformation of analytic functionals on the complex sphere. This English edition is a further development of the author's lecture notes circulated in Japanese.

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Preface

In his first paper [**35**] on the theory of hyperfunctions, M. Sato treated the Fourier expansion of hyperfunctions on the unit circle in connection with the Laurent expansion at the origin of the complex plane.

The characterization of spherical harmonic expansion of hyperfunctions on the sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1}; x^2 = x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$ was first obtained by Hashizume-Minemura-Okamoto [**7**]. Their methods rely on the characterization of real analytic functions by means of the Laplace-Beltrami operator (see Lions-Magenes [**16**] and Seeley [**37**]) and can be applied not only to the sphere but also to a general compact real analytic manifold. But their methods are far from the complex analysis that Sato employed in the case of the one-dimensional sphere, that is, the circle.

In this book, we shall construct a complex neighborhood $\tilde{\mathbb{S}}^n(r)$ of the sphere $\mathbb{S}^n$ by means of the Lie norm. The complex neighborhood $\tilde{\mathbb{S}}^n(r)$ is a direct generalization of the annular neighborhood $\{z \in \mathbb{C}; \frac{1}{r} < |z| < r\}$ of the circle and allows us a complex analysis approach to the theory of spherical harmonic expansion.

Let us overview the monograph chapter by chapter.

In Chapter 1, we recall, as a motivation for later chapters, some facts on Fourier expansion of real analytic functions, C^∞ functions, distributions, and hyperfunctions on the circle. These materials are well-known to a specialist but students may find them useful.

Our first tool for studying the higher-dimensional sphere is, of course, the classical theory of spherical harmonics. Following mainly Müller [**33**] and referring also to Vilenkin [**43**] and to Stein-Weiss [**41**], we present it in Chapter 2 with detailed calculation. (Recently another related book was published by Müller [**34**].) We also state the characterization of C^∞ functions and distributions on $\mathbb{S}^n$ by the growth condition of their spherical harmonic expansion. In §2.8 and §2.9 we study the Poisson formula which represents the unique harmonic function in the unit ball having a given continuous boundary value on the sphere $\mathbb{S}^n$.

Our second tool is the cross norm, which is presented in the first two sections of Chapter 3 according to Drużkowski [**1**]. The Lie norm $L(z)$ on $\mathbb{C}^{n+1}$ is the cross norm of the Euclidean norm on $\mathbb{R}^{n+1}$ and is powerful enough to estimate spherical harmonics.

In the later sections of Chapter 3, we introduce the Lie ball $\tilde{B}(r) = \{z \in \mathbb{C}^{n+1}; L(z) < r\}$, which turns out to be E. Cartan's classical bounded domain of type 4 (see Hua [**12**]). We study the space of holomorphic functions on $\tilde{B}(r)$ and their expansion by homogeneous polynomials. The Shilov boundary of $\tilde{B}(r)$ is called the Lie sphere. (See [**20**] for the spherical harmonic expansion of hyperfunctions on the Lie sphere.) The complex sphere $\tilde{\mathbb{S}}^n = \{z \in \mathbb{C}^{n+1}; z^2 = z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 1\}$ is the natural complexification of the sphere $\mathbb{S}^n$. We put $\tilde{\mathbb{S}}^n(r) = \tilde{\mathbb{S}}^n \cap \tilde{B}(r)$. The family $\{\tilde{\mathbb{S}}^n(r); r > 1\}$ is a fundamental system of complex neighborhoods of the sphere $\mathbb{S}^n$. We shall find a characterization of holomorphic functions on $\tilde{\mathbb{S}}^n(r)$ by the growth condition of their spherical harmonic expansion.

In Chapter 4, we introduce hyperfunctions on $\mathbb{S}^n$ and more generally analytic functionals on the complex sphere $\tilde{\mathbb{S}}^n$. Then in §4.10, we show that a harmonic function in the unit ball is in one-to-one correspondence with its hyperfunction boundary value on $\mathbb{S}^n$. This fact motivated the introduction of hyperfunctions.

A special case of the complex sphere is the complex light cone $\tilde{\mathbb{S}}_0 = \{z \in \mathbb{C}^{n+1}; z^2 = z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 0\}$. Because of the cone structure, we can develop the theory of expansion holomorphic functions and analytic functionals on $\tilde{\mathbb{S}}_0$ into homogeneous components. It was our starting point (see Morimoto-Fujita [**27**]) but in this book we try to state the theory in the complex sphere of complex radius in general.

Now suppose a hyperfunction T on the sphere $\mathbb{S}^n$ is given. If we define the function $F(\xi)$ on $\mathbb{R}^{n+1}$ by

$$F(\xi) = \langle T_\omega, \exp(i\lambda\xi \cdot \omega)\rangle \tag{0.1}$$

the function $F(\xi)$ satisfies the differential equation

$$(\Delta_\xi + \lambda^2)F(\xi) = 0. \tag{0.2}$$

But a solution of the differential equation (0.2) is not always represented in the form of (0.1) with a hyperfunction T. In order to represent all the solutions of (0.2), T in the formula (0.1) should be something more general than hyperfunctions (see Hashizume-Kowata-Minemura-Okamoto [**6**]). In the case of $n = 1$, Helgason [**8**] showed that this "something" is an analytic functional of a certain kind (entire functional). We obtained some detailed results in the case of $n = 1$ in [**19**], and then, in [**22**], proved that this "something" is an entire functional for general n; we treat this topic in Chapter 5 (see also Helgason [**10**]).

In Chapter 6, we introduce several spaces of entire functions, which will be used to describe the image of the Fourier-Borel transformation in subsequent chapters.

Chapter 7 is based on Morimoto-Fujita [**31**], where we introduce the Fourier-Borel transformation for analytic functionals on the complex sphere $\tilde{\mathbb{S}}_\lambda$. The image turns out to be the space of λ-harmonic entire functions.

Chapter 8 is based on Morimoto-Fujita [**30**], where we introduce the spherical Fourier-Borel transformation of γ-harmonic functionals on the Lie ball. The image turns out to be the space of entire functions on the complex sphere $\tilde{\mathbb{S}}_\gamma$.

The present volume is an enlarged version of my lecture note [**21**] on spherical harmonic expansion of hyperfunctions on the sphere, which was intended to complement the book [**17**] on hyperfunctions and microfunctions and the lecture note [**18**] on the Fourier transformation of hyperfunctions.

Chapter 1 was read at Rikkyo University from April to July, 1978. From September 1978 for one year of my sabbatical leave of absence, I stayed in Europe and attended seminars on harmonic analysis at the Universities of Nancy, of Strasbourg, and of Lyon, where I read some part of this lecture note.

In May, 1979, I was invited by Stephan Banach International Mathematical Center as a lecturer at the Semester on Complex Analysis and read [**22**], which was published 4 years later. The discussion with Professor J. Siciak at the semester was stimulating; I learned the Lie norm and could improve my results considerably.

The aim of [**21**] was to present a self-contained exposition of the theory of holomorphic functions and analytic functions on the complex sphere using the Lie norm. The lecture was read at Sophia University in 1980 and in 1987.

Meanwhile, many interesting results of the theory have been obtained by R. Wada [**45**], [**48**], and [**46**], especially in connection with the complex light cone. I tried to present them in a unified manner in this enlarged version, which was prepared while visiting the University of Maryland at College Park from November 1992 to March 1993 (see [**23**]).

After coming back to Japan, I was able to obtain several results jointly with K. Fujita [**3**], [**26**], [**27**], [**2**], [**28**], [**29**], [**30**], [**31**], [**4**], [**5**]. I try to incorporate them into this book as much as possible.

March 19, 1998
Mitsuo Morimoto
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